The Grassmannian and group actions

Def: The *Grassmannian* $G_k(\mathbb{C}^n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$.
- · Can use other vector spaces, this is most common
- · $G_k(\mathbb{C}^n)$ is an $k(n-k)$-dimensional compact manifold.

Eg: $G_1(\mathbb{C}^n) \cong \mathbb{CP}^{n-1}$

There is an association $V \in G_k(\mathbb{C}^n) \to A_V \in M^*_{k\times n}$ (full-rank matrices)
- · Not one-to-one. To make one-to-one, need to get rid of repetitions
  - · Repetitions are same subspaces with different bases, associated by change of basis matrices
- · Since $V \subseteq \mathbb{C}^n$ and $GL_n(\mathbb{C})$ acts transitively on $\mathbb{C}^n$, GL_n also acts on $G_k(\mathbb{C}^n)$.
- · More precisely, some part of $GL_n(\mathbb{C})$ acts transitively on V. Which part?

To see which part, need to use Schubert cell decomposition

Def: A *flag* of a vector space V is an ordered collection of subspaces $V_0, V_1, \ldots, V_n$ such that $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$. The flag is *complete* if $n = \dim(V)$.

Eg: For $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{C}^n$ and $V_i = \text{span}\{e_1, \ldots, e_i\}$, the collection $0, V_1, \ldots, V_n$ is a complete flag for $\mathbb{C}^n$.

Def: For every n-tuple σ of non-increasing integers bounded by 0 and k, define the *Schubert cell*

$$e(\sigma) = \{X \in G_k(\mathbb{C}^n) \ : \ \dim(X \cap V_1) = \sigma_1, \ldots, \dim(X \cap V_n) = \sigma_n\}.$$

Note that:
- · $\bigcup_\sigma e(\sigma) = G_k(\mathbb{C}^n)$
- · Every element of $e(\sigma)$ may be written in a nice way so that it is clear how GL_n acts on it.

Eg: For $G_2(\mathbb{C}^4)$, we have

$$\sigma^1 = (1,2,2,2) \qquad \sigma^2 = (1,1,2,2) \qquad \sigma^3 = (1,1,1,2)$$
$$\sigma^4 = (0,1,2,2) \qquad \sigma^5 = (0,1,1,2) \qquad \sigma^6 = (0,1,2,2)$$

Each $X \in \sigma^i$ may be written in a special way so that it is clear how GL_n acts on it

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 1 \end{bmatrix}$$