Sheaf cohomology may be used to calculate invariants of groups and dimensions (relevant to geometric group theory).

1 Setting

Def: Let X be a topological space. A presheaf \mathcal{F} on X is:

1. an assignment of an abelian group $\mathcal{F}(U)$ to every open $U \subset X$ **2.** maps $\rho_{UV} : U \to V$ for all $V \subset U$

such that:

1. $\mathcal{F}(\emptyset) = 0$,

2. $\rho_{UU} = id$,

3. $\rho_{VW}\rho_{UV} = \rho_{UW}$ whenever $W \subset V \subset U$.

Eg: Some examples of presheaves:

- **1.** $\mathcal{O}(U) = \{ \text{differentiable functions } U \to \mathbb{R} \}. \rho \text{ is function restriction}$
- **2.** $\underline{A}(U) = \{(\text{locally}) \text{ constant } A \text{-valued functions } U \to \mathbb{R}\}. \rho \text{ is identity on connected components}$
- **3.** skyscraper sheaf $\mathcal{S}(U) = \begin{cases} A & \text{if } x \in U, \\ \emptyset & \text{if } x \notin U. \end{cases}$ ρ is identity or 0.

Def: A sheaf \mathcal{F} on X is a presheaf on X such that for any family $\{U_{\alpha}\}_{\alpha \in A}$ of open sets $U_{\alpha} \subset X$ and any family $\{s_{\alpha}\}_{\alpha \in A}$ of sections $s_{\alpha} \in \mathcal{F}(U_{\alpha})$, if

$$\rho_{U_{\alpha},U_{\alpha}\cap U_{\beta}}(s_{\alpha}) = \rho_{U_{\beta},U_{\alpha}\cap U_{\beta}}(s_{\beta})$$

for all $\alpha, \beta \in A$, then there exists a unique $s \in \mathcal{F}(U = \bigcup_{\alpha \in A} U_{\alpha})$ such that $\rho_{UU_{\alpha}}(s) = s_{\alpha}$. gluing axiom

Rem: $\mathcal{O}(X)$ and $\mathcal{S}(X)$ are sheaves, but the constant sheaf $\underline{A}(U)$ is not. Consider $A = \mathbb{Z}$, $X = \mathbb{R}$, $U_0 = (0, 1)$ and $U_1 = (2, 3)$, with $s_0 = 0 \in \mathbb{Z}(U_0)$ and $s_1 = 1 \in \mathbb{Z}(U_1)$. Since $U_0 \cap U_1 = \emptyset$, the restriction map condition is satisfied. However, there exists no $s \in \mathbb{Z}(U)$ that restricts to both 0 and 1.

Def: A *stalk* of a sheaf \mathcal{F} at $x \in X$ is the direct limit

$$\mathcal{F}_x := \lim_{x \in U} \mathcal{F}(U) = \bigoplus_{x \in U} \mathcal{F}(U) \middle/ \begin{array}{c} \iota_V \rho_{UV}(s) \sim \iota_U(s) \\ \forall \ s \in \mathcal{F}(U), V \subset U, \end{array}$$

where $i_U: U \to \bigoplus_{x \in U} \mathcal{F}(U)$ is the natural inclusion map. A germ of a stalk is the equivalence class of a section under the quotient above.

Def: Given a presheaf \mathcal{F} , there is a sheaf \mathcal{F}^+ associated to it, called the *sheafification* of \mathcal{F} . Set

$$\mathcal{F}^{+}(U) \coloneqq \left\{ s \in \bigoplus_{t \in U} \mathcal{F}_{t} : \forall q \in U, \exists \stackrel{\cdot}{a \text{ nbhd } V \subset U \text{ of } q}_{\cdot a \text{ section } \widetilde{s} \in \mathcal{F}(V)} \text{ such that } s \mid_{\mathcal{F}_{v}} = \widetilde{s} \mid_{\mathcal{F}_{v}} \forall v \in V \right\}$$

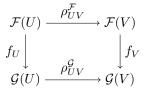
for every open set U. The restriction maps are induced naturally from $\bigoplus_{t \in U} \mathcal{F}(U) \to \bigoplus_{t \in V} \mathcal{F}(V)$.

Now we may always talk about sheaves only satisfying the three conditions given first above.

can be in any cat, not just Top
can be in any cat, not just Ab
elements of F(U) are sections
elements ρ_{UV} are restriction maps

2 Cohomology

Given sheaves \mathcal{F}, \mathcal{G} both on X, we may define a morphism $f : \mathcal{F} \to \mathcal{G}$ by maps $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the diagram



commutes for every $V \subset U$.

This gives structure of category on sheaves

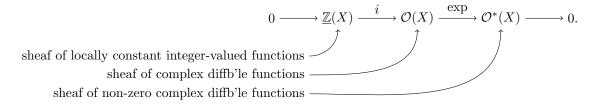
Given a sheaf \mathcal{F} on X, we may calculate cohomology groups $H^i(X, \mathcal{F})$ of X with values in \mathcal{F} by \cdot cocycles modulo coboundaries

· derived functors of global section functor $\Gamma(X, -) : Sh(X) \to Ab$ Need exact sequences of sheaves.

Thm: A sequence of sheaves is exact iff it is exact on the stalks.

Calculating cohomology is difficult. Try to create *flabby* (flasque) sheaves (ρ_{XU} is surjective for all $U \subset X$) for which $H^i = 0$ for all i > 0.

Eg: Consider the following exact sequence of sheaves on a complex manifold X:



Check it is exact by checking exactness on the stalks. Let $X = S^2 = \mathbb{C}P^1$ and take cohomology. Interesting fact - connecting map δ below takes line bundle $(H^1(X; \mathcal{O}^*(X))$ classifies line bundles on X) to its first Chern class:

$$\cdots \longrightarrow H^1(X; \mathcal{O}(X)) \longrightarrow H^1(X; \mathcal{O}^*(X) \xrightarrow{\delta} H^2(X; \underline{\mathbb{Z}}(X)) \longrightarrow \cdots$$