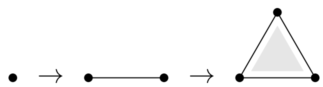
Let R be a simplicial ring and $|R|$ the geometric realization of R. want to understand this

Def: Define two categories:
· Ring: objects are rings, morphisms are ring homomorphisms
· Δ: objects are ordered n-tuples $(a_1 < a_2 < \cdots < a_n)$ $\forall\, n$, morphisms are order-preserving maps
objects may be considered only as tuples of consecutive numbers $[n] = (1, \ldots, n)$

Example:

$$\mathrm{Hom}([1],[2]) = \{(1) \;\to\; (1,2)\}$$
$$\begin{array}{c} 1 \to 1 \\ 1 \to 2 \\ 1 \leftarrow 2,\; 1 \leftarrow 1 \end{array}$$



$$\mathrm{Hom}([2],[3]) = \{(1,2) \;\to\; (1,2,3)\}$$
$$\begin{array}{c} 1 \to 1,\; 2 \to 1 \\ 1 \to 1,\; 2 \to 2 \\ 1 \to 1,\; 2 \to 3 \\ 1 \to 2,\; 2 \to 2 \\ 1 \to 2,\; 2 \to 3 \\ 1 \to 3,\; 2 \to 3 \\ 1 \leftarrow 3,\; 1 \leftarrow 2,\; 1 \leftarrow 1 \\ 2 \leftarrow 3,\; 1 \leftarrow 2,\; 1 \leftarrow 1 \\ 2 \leftarrow 3,\; 2 \leftarrow 2,\; 1 \leftarrow 1 \\ 2 \leftarrow 3,\; 2 \leftarrow 2,\; 2 \leftarrow 1 \end{array}$$

$\mathrm{Hom}(n,m)$ may be viewed as maps from an n-simplex into an m-simplex respecting edges and vertices

Remark: There are special morphisms that may be viewed as a generating set:

face maps: $\varphi_i \;:\; [n-1] \;\to\; [n]$
$$\begin{array}{ll} k \mapsto k & k < i \\ k \mapsto k+1 & k \geqslant i \end{array}$$
maps to a face

degeneracy maps: $\delta_i \;:\; [n+1] \;\to\; [n]$
$$\begin{array}{ll} k \mapsto k & k \leqslant i \\ k \mapsto k-1 & k > i \end{array}$$
maps two faces together

Let $R \in \mathrm{Fun}(\Delta, \mathrm{Ring})$:
$$\begin{array}{ccc} [n] & \overset{R}{\rightsquigarrow} & R_n \\ [n] \to [m] & \rightsquigarrow & R_n \to R_m \text{ ring homomorphsim} \end{array}$$

Let $R \in \mathrm{Fun}(\Delta^{op}, \mathrm{Ring})$:
$$\begin{array}{ccc} [n] & \overset{R}{\rightsquigarrow} & R_n \\ [n] \to [m] & \rightsquigarrow & R_m \to R_n \text{ ring homomorphsim} \end{array}$$

nezīmē no jauna, tikai izdzēs iepriekšējo

Why Δ^{op}? Generalizes chain maps of singular complexes $d_n = \sum_{i=0}^{n} (-1)^i (1, \ldots, \widehat{i}, \ldots, n)$ from C_n to C_{n-1}

Notation: $R_n = R([n]) \in \mathrm{Ob}(\mathrm{Ring})$, the elements of which are called *n-simpleces.*
s_n is the standard n-simplex (convex hull of $(1,0,\ldots,0), \ldots, (0,\ldots,0,1) \in \mathbb{R}^{n+1}$)
$R \in \mathrm{Fun}(\Delta^{op}, \mathrm{Ring})$ is called a *simplicial ring*

Def: Let R be a simplicial ring. The *geometric realization* of R is the space

$$|R| = \coprod_{n\geqslant 0} R_n \times s_n \Big/ \sim$$

where

$$(r, \varphi_i^*(t)) \sim (R(\varphi_i)(r), t) \qquad \forall\, r \in R_n,\ t \in s_{n-1},\ i$$
$$(r, \delta_i^*(t)) \sim (R(\delta_i)(r), t) \qquad \forall\, r \in R_n,\ t \in s_{n+1},\ i$$

Elements are equivalence classes $|r, t|$. The maps φ_i^*, δ_i^* are induced from φ_i, δ_i as follows:

$$\begin{array}{rcl} \varphi_i^* : s_{n-1} & \to & s_n, \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) \end{array} \qquad \begin{array}{rcl} \delta_i^* : s_{n+1} & \to & s_n, \\ (t_0, \dots, t_{n+1}) & \mapsto & (t_0, \dots, t_i + t_{i+1}, \dots, t_{n-1}) \end{array}$$

Def: An n-simplex $r \in R_n$ is *degenerate* if $r = R(\varphi_i)(r)$ for some i. A class $|r, t| \in |R|$ is called *degenerate* if r is degenerate.

· Degenerate elements may be viewed as "in the boundary" of R_n.
· Every $|r, t| \in |R|$ has a representative that is non-degenerate.

Applications:

· Instead of Ring, use Ab. Let $X_n = C_n(X)$, the free abelian group of the n-simpleces of X. Then $|C_*(X)| = X$.

· Instead of Ring, use Set. Then for K a simplicial set, $|K|$ is a CW-complex with exactly one n-cell for each non-degenerate n-simplex of K. model for homotopy theory of spaces

· Calculate $\pi_*(R) := \pi_*(|R|)$ topological properties now given to arbitrary objects

· Let $R \in$ CRing be simplicial. Then $\pi_0(R)$ is a commutative ring, and $\pi_i(R)$ is a $\pi_0(R)$-module, for all i. homotopy groups much easier to decribe, usually hard

· CRing^{op} is the category of affine schemes (algebraic geometry)