# 1 Setting

$$\begin{array}{ccccccc} L & & \mathcal{O}(1) & & & M & = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1) \\ \downarrow & & \downarrow & & & \downarrow & \\ X & \xrightarrow{f} & \mathbf{CP}^\infty & \underset{\pi_2}{\overset{\pi_1}{\longleftarrow\!\!\!\longleftarrow}} & \mathbf{CP}^\infty & \times & \mathbf{CP}^\infty \end{array}$$

Objects:

$X$ : a "nice space" — "nice" means paracompact and Hausdorff

$L$ : a complex line bundle over $X$

$\mathcal{O}(1)$ : the universal line bundle

$f$ : a continuous map

**Def:** Given a cohomology theory $E$, the *first Chern class* of $L$ with respect to $E$ is $c_1^E(L) = f^*t$, with $t \in E^2(\mathbf{CP}^\infty)$.

Let's simplify some things.

**Def:** A *complex-oriented* cohomology theory $E$ is one whose Atiyah–Hirzebruch spectral sequence $H^p(X, E^q(\cdot)) \Rightarrow E^{p+q}(X)$ degenerates at the second page.

From here on out $E$ is complex-oriented multiplicative, so $E^*(X \times Y) \cong E^*(X) \times E^*(Y)$.

For such $E$, $E^*(\mathbf{CP}^\infty) \cong E^*(\mathrm{pt})[[t]]$ and $t \in E^2(\mathbf{CP}^\infty)$. Same $t$ as above

We talk about the first Chern class because that's easier to compute than other Chern classes.

**Q:** What is $c_1^E(L \otimes L')$?
**A:** When $E$ is singular cohomology, it is $c_1^E(L) + c_1^E(L')$. Not clear in general.

The bundle $L \otimes L'$ is over $X$. It suffices to consider it over $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$. Set $M = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$.

$$E^*(\mathbf{CP}^\infty) \cong E^*(\mathrm{pt})[[t]]$$
$$E^*(\mathbf{CP}^\infty \times \mathbf{CP}^\infty) \cong E^*(\mathrm{pt})[[u,v]]$$
$$u = \pi_1^* t \qquad v = \pi_2^* t$$

So $c_1^E(M) = c_1^E(L \otimes L') = f(u,v) \in E^*(\mathrm{pt})[[u,v]]$.

**Q (refined):** What is $f$?
**A (refined):** When $E$ is singular cohomology, $f(u,v) = u + v$. Not clear in general.

What do we know about $f$? Related to the tensor product: so it has similar properties

$$f(u,v) = f(v,u),$$
$$f(u,0) = u,$$
$$f(u, f(v,w)) = f(f(u,v), w).$$

**Def:** Let $R$ be a commutative ring. Any $f \in R[[x,y]]$ that satisfies the properties above is called a *formal group law* (or *fgl*).

So every multiplicative complex-oriented cohomology theory determines a fgl over the ring $E^{\text{even}}(\cdot)$.

$$\begin{pmatrix} \text{mult.} \\ \text{cpx-ori} \\ \text{cohomology} \\ \text{theory } E \end{pmatrix} \underset{\text{somtimes}}{\overset{\text{always}}{\rightleftarrows}} \text{FGL}$$

Doesn't always work the other way around.

**Example:** If $f(u,v) = u + v + uv$, then the associated cohomology theory is complex $K$-theory.

How do we come up with fgls? Take a step back to a larger ring.

**Thm/Def:** There exist:
· a commutative ring $L$, called the *Lazard ring*,
· a fgl $f \in L[[u,v]]$, called the *universal fgl*, such that
for any fgl over a commutative ring $R$, there is a ring homomorphism $L \to R$ that takes $f$ to the given fgl.

The map $\varphi$ induces a map on formal power series that takes $f$ to the given fgl

We may write $f(x,y) = \sum_{i,j} c_{ij} x^i y^j$. Tensor product conditions give:

$$\begin{aligned} c_{ij} &= c_{ji} \\ c_{i0} &= \delta_{i1} \\ p_k(c_{ij}) &= 0 \end{aligned}$$

**Def:** The *Lazard ring* is by $L := \mathbf{Z}[c_{ij}]/(p_k(c_{ij}))$ where the $p_k$ include all the polynomial relations above

$L$ has a grading: For $c_{ij} \in L$, set $\deg(c_{ij}) = 2(i+j-1)$. This is chosen so $f$ has degree $-2$.

What have we learned so far?

$$\begin{aligned} \text{choosing a fgl over } R \quad & \textit{is equiv to} \quad \text{choosing } \{c_{i,j}\} \subset R \text{ satisfying } p_k(c_{i,j}) = 0 \\ & \textit{is equiv to} \quad \text{choosing } \varphi \in \operatorname{Hom}(L, R) \end{aligned}$$

This concludes the introduction. Let's get to the next part.

## 2 Lazard's theorem

**Thm:** (Lazard, 1955) $L \cong \mathbf{Z}[t_1, t_2, \ldots]$, where $\deg(t_i) = 2i$.

To prove this, we need some tools and some lemmas.

**Remark:** Let $f \in R[[x, y]]$ be a fgl and $g = z + b_1 z^2 + b_2 z^3 + \cdots \in R[[z]]$. Then $(g \circ f)(g^{-1}(x), g^{-1}(y))$ is a fgl over $R$.

Tensor laws (first two) satisfied easily. Inverse exists because constant term is 0 and linear coeff is 1.

Then $\tilde{g}(x, y) = g(g^{-1}(x) + g^{-1}(y))$ is an FGL over $\mathbf{Z}[b_i]$.
Let $\varphi : L \to \mathbf{Z}[b_i]$ be the associated ring homomorphism.

$I(\subset L)$ = the ideal consisting of elements of positive degree of $L$
$J(\subset \mathbf{Z}[b_i])$ = the ideal generated by the $b_i$.

Lazard's isomorphism will be made on graded parts. Consider two new spaces:

$$I/I^2 \qquad\qquad \left(J/J^2\right)_{2n} \cong \mathbf{Z}$$

Both have an induced grading from their rings.

To prove the theorem, we need to prove some lemmas.

**Lemma 1:** For $n > 0$, there is an injection $(I/I^2)_{2n} \overset{i}{\hookrightarrow} \mathbf{Z}$. Moreover, $i((I/I^2)_{2n}) = \begin{cases} p\mathbf{Z} & \text{if } n+1=p^r, \\ \mathbf{Z} & \text{else.} \end{cases}$

Think of showing $(I/I^2)_{2n} \cong \mathbf{Z}$ as showing $\mathrm{Hom}((I/I^2)_{2n}, M) \cong M$ for an abeliean group $M$.

Compare functor corepresented by $(I/I^2)_{2n}$ to functor corepresented by $L$.

*MORE*

**Lemma 2:** There is a surjective map $\theta : \mathbf{Z}[t_i] \to L$, where $t_i \in L_{2n}$ is the lift of a generator of $(I/I^2)_{2n}$.

Done by showing $\theta_n : \mathbf{Z}[t_n] \to L_{2n}$ is surjective, by induction.

*HOW INDUCTION WORKS*

**Lemma 3:** The map $\theta$ is injective.

Instead show the composition

$$\begin{array}{ccccc} \mathbf{Z}[t_i] & \longrightarrow & L & \longrightarrow & \mathbf{Z}[b_i] \\ \big\downarrow & & & & \big\downarrow \\ \mathbf{Q}[t_i] & & \xrightarrow{\quad\cong\quad} & & \mathbf{Q}[b_i] \end{array}$$

is injective.

Use above lemmas for rational part.

Since $\theta$ is a bijective ring homomorphism, it is a ring isomorphism.

*WHY $\theta$ HOMOMORPHISM*