The Poincare homology sphere P may be described by removing and gluing back neighborhoods of S^3, another way of describing an S^1 Seifert bundle over S^2.

0.1 Seifert bundles and Seifert manifolds

A *Seifert manifold* is a 3-manifold that is "made up" of simple closed curves whose neighborhood (a solid torus) has a certain property. We may instead think about *Seifert bundles*, a special case of which are equivalent to Seifert manifolds (Section 5.2 in [Orl72]).

Definition 0.1.1. A *Seifert manifold* M is a 3-manifold for which

1. each $x \in M$ lies on a unique simple closed curve C, and

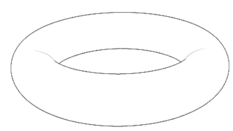
2. C has a neighborhood $V \cong (D^2 \times S^1)/(\mathbf{Z}/n\mathbf{Z})$ for some n, depending on C.

The action $\mathbf{Z}/n\mathbf{Z} \curvearrowright (D^2 \times S^1)$ is in fact more of an action just on D^2, rotating the disk an integer multiple of $2\pi/n$ (Sections 1.6 and 1.7 in [Orl72]). We will only consider the integer multiple being 1, and the n will be called the *index* of the curve. Then

$$(D^2 \times S^1)/(\mathbf{Z}/n\mathbf{Z}) := (D^2 \times I)/\sim,$$

where $(r, \theta, 0) \sim (r, \theta + 2\pi/n, 1)$ for all r and all θ (introductory remarks in Chapter IV of [Jac80]).

Example 0.1.2. An open torus is a Seifert manifold. We may also cut it, twist it, glue it back together, and still have a Seifert manifold.



Definition 0.1.3. Let U be a topological space and F some object such that a group G acts on both F and U (we will think of F as a subgroup of G, with action left multiplication). A *Seifert product bundle* with fiber F is the bottom line of the commutative diagram

$$\begin{tikzcd} F \times U \arrow[r, "\text{proj}"] \arrow[d, "q"'] & U \arrow[d, "q"] \\ (F \times U)/G \arrow[r, "\pi_s"] & U/G, \end{tikzcd}$$

where π_s is defined so as to make the diagram commute. Now, let X, Y be topological spaces with $Y = \bigcup_i V_i$ an open cover, with G_i acting on F and topological spaces U_i. A *Seifert bundle* $\pi : X \to Y$ with structure group G (we have $G_i \leqslant G$ for all i) with fiber F is a fiber bundle that is locally a Seifert product bundle. That is, for each V_i there is a bundle isomorphism making the diagram below commute:

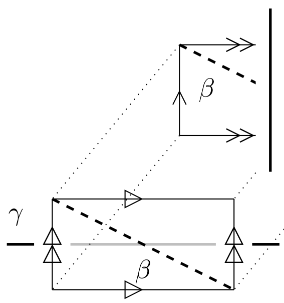
$$\begin{tikzcd} F \times U_i \arrow[r, "\text{proj}"] \arrow[d, "q"'] & U_i \arrow[d, "q"] \\ (F \times U_i)/G_i \arrow[r, "\pi_s"] \arrow[d, "h_i"'] & U_i/G_i, \arrow[d, "t_i"] \\ \pi^{-1}(V_i) \arrow[r, "\pi"] & V_i \end{tikzcd}$$

Example 0.1.4. Any group G may always act trivially on any other object, in which case $U_i = V_i$ and $F = \pi^{-1}(y) \times V_i$ for $y \in Y$ define a Seifert bundle, from any vector bundle $\pi : X \to Y$.

Example 0.1.5. Every Seifert manifold is the total space of a Seifert bundle (but not the other way around). The structure group is $\mathbf{Z}$, the fiber F is S^1, and the base is covered by disks D^2.

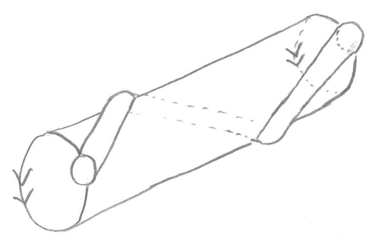
0.2 Construction of P

Recall the 3-sphere S^3 may be described as two solid tori with their boundaries identified. We will be interested in three curves in S^3, one inside each torus, and one on their boundary.



- Glue the single arrows together first
- Glue one torus on top of the other
- Glue together each pair of double arrows

The space P is constructed by removing a neighborhood of each curve α, β, γ, and gluing them back in with some twists - a neighborhood of α with 2 twists, a neighborhood of β with 3 twists, and a neighborhood of γ with 5 twists. The neighborhood of β sits on each of the neighborhoods of α and γ as below.

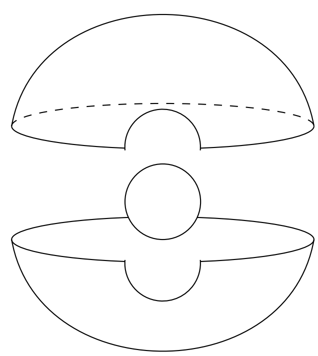


Note that all points in the interior of two tori are clearly Seifert, and on boundary are as well (almost clearly). On the boundary we take a neighborhood of β and remove that from the neighborhoods of α and β. Hence the space may be described as a Seifert bundle with fiber S^1 over S^2. Begin by describing $S^2 = V_1 \cup V_2 \cup V_3$, as in the diagram below (with slight enlargements of the given sets):

$G_1 = \mathbf{Z}_2 \qquad V_1 \cong D^2$

$G_2 = \mathbf{Z}_3 \qquad V_2 \cong D^2$

$G_3 = \mathbf{Z}_5 \qquad V_3 \cong D^2$

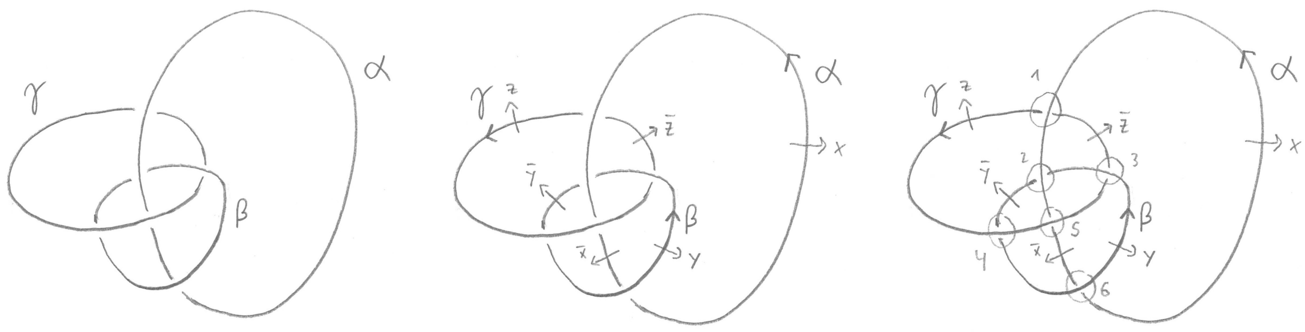


$$\begin{tikzcd}
S^1 \times D^2 \arrow[r, "\text{proj}"] \arrow[d, "n_i \text{ twists}"'] & D^2 \arrow[d, "\text{id}"] \\
(S^1 \times D^2)/\mathbf{Z}_{n_i} \arrow[r, "\pi_s"] \arrow[d, "\text{id}"'] & D^2/\mathbf{Z}_{n_i}, \arrow[d, "\text{id}"] \\
(S^1 \times D^2)/\mathbf{Z}_{n_i} \arrow[r, "\pi"] & D^2
\end{tikzcd}$$

The Seifert bundle structure is as on the right, with $U_i = V_i = D^2$ and $G_i = \mathbf{Z}_{n_i}$ acting by the twists on the disks as described previously. Twisting a disk by itself does nothing, so the maps on the right are trivial.

0.3 Calculation of π_1

The Seifert bundle structure described may also be viewed as surgery on the link L formed by the three circles α, β, γ in S^3. We name the arcs and find the Wirtinger presentation of the fundamental group as below.



Then going around each intersection counter-clockwise in the order given above, we get

$$\pi_1(S^3 - L) = \langle x, \bar{x}, y, \bar{y}, z, \bar{z} \mid z^{-1}x^{-1}\bar{z}x = \bar{y}^{-1}x^{-1}yx = y^{-1}zy\bar{z}^{-1} = zy^{-1}z^{-1}\bar{y} = z\bar{x}^{-1}z^{-1}x = yx^{-1}y^{-1}\bar{x} = e \rangle,$$

for e the identity element. Solving for some elements, we find

$$\bar{y} = x^{-1}yx, \qquad \bar{z} = y^{-1}zy, \qquad \bar{x} = yxy^{-1},$$

hence our group becomes

$$\pi_1(S^3 - L) = \langle x, y, z \mid z = x^{-1}y^{-1}zyx,\ y = z^{-1}x^{-1}yxz,\ x = zyxy^{-1}z^{-1} \rangle.$$

As mentioned, the Poincare homology sphere is surgery performed on this link with the indicated "surgery coefficients" for each link component (2 for α, 3 for β, 5 for γ). Surgery along each component kills curves that wrap a certain number of times *around* and *along* those curves, giving us new relations

$$x = y^{-1}z^{-1}, \qquad y^2 = xz, \qquad z^4 = yx.$$

The powers are decreased by 1 from the surgery coefficients because wrapping around n times also involves winding along once. The factors on the right side of each relation come from following each curve around in the marked direction (as in the diagrams above). Combining these three relations with the three above, we get

$$\pi_1(P) = \langle x, y, z \mid x = y^{-1}z^{-1},\ x^2 = y^3 = z^5 = e \rangle.$$

References

[Hem76] John Hempel. 3-*Manifolds*. Ann. of Math. Studies, No. 86. Princeton University Press, Princeton N. J.; University of Tokyo Press, Tokyo, 1976, pp. xii+195.

[Jac80] William Jaco. *Lectures on three-manifold topology*. Vol. 43. CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, R.I., 1980, pp. xii+251. ISBN: 0-8218-1693-4.

[KS79] R. C. Kirby and M. G. Scharlemann. "Eight faces of the Poincaré homology 3-sphere". In: *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*. Academic Press, New York-London, 1979, pp. 113–146.

[Orl72] Peter Orlik. *Seifert manifolds*. Lecture Notes in Mathematics, Vol. 291. Springer-Verlag, Berlin-New York, 1972, pp. viii+155.