1 Part 1 (March 15)

def of presheaf: functor from $Op(X)^{op}$ to Set. inclusion becomes restriction
def of sheaf: gluing+locality axiom / limit condition

locally constant sheaf

poset-stratification
concical strat
refinement power set

artin gluing

2 Part 2 (March 22)

Recall from last time: poset stratification, conical stratification, contructible sheaf.
$f : X \to (A, \leqslant)$ is a poset-stratified space.
$\Delta^n_{alg} = \mathbf{P}(\{0, 1, \ldots, n\})$ is the *algebraic* n-simplex
$\Delta^n_{top} = \{t_0e_0 + t_1e_1 + \cdots + t_ne_n \ : \ t_i \in [0, 1], e_i = (0, \ldots, 0, 1, 0, \ldots, 0), \sum t_i = 1\}$ is the *topological* n-simplex
Geometric realization links the two: $|\Delta^n_{alg}| \cong \Delta^n_{top}$

2.1 Exit paths

Cat of top spaces has pathologies, not nice. Consider spaces homotopy equiv to simplicial complexes.
Use $\mathrm{Sing}(X) = \{\text{continuous maps } \Delta^n_{top} \to X\}$, with $|\mathrm{Sing}(X)| \cong X$ when X is a "simp top space"

exit path (1-version): A path $\gamma : I \to X$ is an *exit path* if $f(\gamma(0)) = a \leqslant b = f(\gamma(t > 0))$.

exit path (∞-version): A map $\gamma : \Delta^n_{top} \to X$ is an *exit path* if there exists a chain $a_0 \leqslant \cdots \leqslant a_n$ in A such that $f(\gamma(t_0, \ldots, t_i, 0, \ldots, 0)) = a_i$ for $t_i \neq 0$.

The ∞-version paths form a subcategory $\mathrm{Sing}^A(X) \subseteq \mathrm{Sing}(X)$.

Lurie describes the following maps:

$$\mathrm{Fun}(\mathrm{Sing}^A(X), \mathcal{S}) \left(\xrightarrow{\cong} N(\mathrm{Fib}(\mathrm{Fun}(\mathrm{Sing}^A(X), \mathrm{sSet}))) \xrightarrow{\cong} N(\mathrm{Fib}(\mathrm{sSet}_{/\mathrm{Sing}^A(X)})) \right) \longrightarrow \mathrm{PreShv}(X)$$

<u>On left:</u> A is very simple "space," natural functor $\mathrm{Sing}^A(X) \to \mathcal{S}$ with $\gamma \mapsto (a_0 \to a_1 \to \cdots \to a_n)$
What is associated presheaf?

<u>On right:</u> when $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, have natural sheaf
Does it come from a functor $\mathrm{Sing}^A(X) \to \mathcal{S}$?

When $f : X \to A$ is a conical stratification (and X is paracompact), last map is isomorphism onto subcategory of constructible ($\mathcal{S}$-valued) sheaves on X.

2.2 The Riemann–Hilbert correspondence

Hilbert's 21st problem asks for: *Proof of the existence of linear differential equations having a prescribed monodromic group.*

The **monodromy group** of differential equations on $S \subseteq \mathbf{C}$ open, connected, is (the image of) a homomorphism $\pi_1(S) \to GL_n(\mathbf{C})$ (a linear representation of $\pi_1(S)$).

H21Q answered in negative. Answered in positive when adjusted and extended to higer dims as: For M a compact manifold, there is an equivalence between:

vector bundles with flat connections on M	and	representations of $\pi_1(M)$
(regular holonomic) $\mathcal{D}$-modules on M	and	$\mathbf{C}$-constructible sheaves on M

A *connection* on a vector bundle $E \to M$ is a map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$. $\mathcal{D}$ is the sheaf of differential operators on M (derivations of the sheaf of differentiable functions on M)