Universal persistent homology / Poset persistent homology

Jānis Lazovskis

UIC Graduate Geometry and Topology Seminar
2018-10-10 and 10-17

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1 Talk 1: Universal persistent homology

1.1 Simplicial complexes

Def: A *simplicial complex* C is a pair of sets $(V(C), S(C))$, where $S(C) \subseteq V(C)$ is closed under taking faces. A *simplicial map* $C \to D$ of simplicial complexes is a set map $V(C) \to V(D)$ so that its natural extension $S(C) \to S(D)$ is a function. Write SC for the category of simplicial complexes and simplicial maps.

Remark: Partial orders on Obj(SC).
- Naturally $C \leqslant D$ whenever $V(C) \subseteq V(D)$ and $S(C) \subseteq S(D)$, up to renaming of vertices.
- Write $C \leqslant_s D$ whenever there is a simplicial map $D \to C$ that is surjective on vertices.

Example: Compare partial orders on 0-simplex, 1-simplex, two 0-simplices.

1.2 Homology

Def: A *chain complex* of abelian groups is a sequence

$$\cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots$$

with $d_i \circ d_{i+1} = 0$ for all i.
Def: Let C_n be the free abelian group on the n-simplices of C and $d_n([a_0, \dots, a_n]) = \sum_{i=0}^n (-1)^i [a_0, \dots, \widehat{a_i}, \dots, a_n]$. Let $H_n = \ker(d_n)/\text{im}(d_{n+1})$ be the nth *singular homology* group of C.
Thm: Every simplicial map $C \to D$ induces a map of groups $H_n(C) \to H_n(D)$, for all n.
Thm: Everything above extends to topological spaces and (triangulations of) manifolds.

Def: Let M be an embedded Riemannian manifold and $f\colon M \to \mathbf{R}$ continuous. The (sub-level set) *persistent homology* of M is the composition of functors

$$\begin{array}{ccccc} (\mathbf{R}, \leqslant) & \longrightarrow & \text{Top} & \xrightarrow{H_n(\,\cdot\,)} & \text{Grp} \\ t & \longmapsto & \{x \in M : f(x) \leqslant t\} =: M_t & \longmapsto & H_n(M_t) \\ (t \leqslant t') & \longmapsto & (M_t \hookrightarrow M_{t'}) & \longmapsto & (H_n(M_t) \to H_n(M_{t'})) \end{array}$$

Example: Sublevel sets of embedded dented sphere.
Def: Let (P, d) be a finite metric space and $f\colon \mathbf{R}_{\geqslant 0} \to \text{Obj}(\mathsf{SC})$ monotonic with $V(f(t)) = P$ for all t. The *persistent homology* of P is the composition of functors

$$\begin{array}{ccccc} (\mathbf{R}_{\geqslant 0}, \leqslant) & \longrightarrow & \mathsf{SC} & \xrightarrow{H_n(\,\cdot\,)} & \text{Grp} \\ t & \longmapsto & f(t) & \longmapsto & H_n(f(t)) \\ (t \leqslant t') & \longmapsto & (f(t) \hookrightarrow f(t')) & \longmapsto & (H_n(f(t)) \to H_n(f(t'))) \end{array}$$

Remark: Often f is the Vietoris–Rips construction or Čech construction.
Example: Points on a circle. Visual demo.

Remark: These perspectives are related by the distance function: Take $P \subseteq \mathbf{R}^d = M$ and $f\colon \mathbf{R}^d \to \mathbf{R}$ given by $f(x) = \min_{p\in P} \|x - p\|$. Then $M_t = \bigcup_{p\in P} B(p,t)$ and $H_n(M_t) = H_n(\check{C}(t))$ for $\check{C}$ the Čech construction of simplicial complexes.

1.3 Universality

Assumption 3: Persistent homology will be on metric spaces that can be embedded in $\mathbf{R}^N$.
Describe "universality" with persistent homology functor factoring through some other category.

Def: Let $\mathrm{Ran}^{\leqslant k}(M) := \{P \subseteq M : 0 < |P| \leqslant k\}$ be the Ran space of M. Topology induced by Hausdorff.
Def: Let $\mathcal{F}\colon \mathrm{Op}(\mathrm{Ran}^{\leqslant k}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathrm{Cat}_{/\mathsf{SC}}$ be the functor that takes U to the appropriate diagram

2 Talk 2: Poset persistent homology

Def: A *persistence module* is a functor $(\mathbf{R}, \leqslant) \to \text{Grp}$. An *interval module* is a persistence module where all the maps are $\mathbf{1}$.
Thm: (Crawley-Boevey, 2015) Every persistence module
- over a totally ordered set
- whose induced sequence of images and sequence of kernels satisfy the descending chain condition,

decomposes uniquely into a direct sum of interval modules.
Example: Three points, persistent homology via Cech

Remark 1: Necessary that we have a filtration, that is, maps always going one way. Is this general?
Remark 2: This can be viewed as a generalization in two ways:
- Had totally ordered set parametrizing sequence, now want partially ordered set.
- Had linear A_n-type quiver representation, now want arbitrary quiver (enriched digraph).

Example: Example of poset, arbitrary quiver.

Def: A*zigzag (persistence) module* is an A_n-type quiver. A sequence of groups (vector spaces) with maps going not necessarily in one direction.

$$A_1 \leftrightarrow A_2 \leftrightarrow A_3 \leftrightarrow \cdots \leftrightarrow A_n$$

Def: The *right filtration* of a zigzag of length 0 is (A_0). Given the right filtration $(V_1, \ldots, V_n)$ of a zigzag Z of length n, the right filtration
- of $Z \xleftarrow{f} V_{n+1}$ is $(f^{-1}(V_1), \ldots, f^{-1}(V_n), V_{n+1})$
- of $Z \xrightarrow{g} V_{n+1}$ is $(g(V_1), \ldots, g(V_n), V_{n+1})$.

Remark: Now have notion of filtration for zigzag. Crawley-Boevey extends to zigzag persistence, with a correspondng condition for the right filtration.

generalized persistence module (bubenik, scott)
right filtration functor (carlsson, de silva, morozov)
rff on γ-ribbon in $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$