Stability of Universal Constructions for Persistent Homology

by

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THESIS

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To everyone who may find it useful in learning and discovering new and interesting ideas. Including myself, as I tend to forget things.

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Non-mathematically, many more people contributed to this thesis, by positively impacting my state of mind. These contributions are more tangible in the sense that without them I would not be happy about writing this thesis, and less tangible in the sense that there is no one specific statement or remark that came about because of them.

ACKNOWLEDGMENT (Continued)

This thesis is typeset in LaTeX, and the BibTeXentries are generated by AMS MathSciNet for published material and by the SAO / NASA ADS for unpublished material on the arXiv. All the figures in this thesis, except Figure 15 and Figure 14, were drawn using Till Tantau's package `tikz`[1] and Jeffrey Hein's `tikz-3Dplot`[2] extension of it.

Bieži dalu savu identitāti starp matemātiķu un latviešu sabiedrībām. Esmu pateicīgs tiem, kas atbalstīja šīs dalīšanās vienošanos, sākot ar Krišjāņa Barona Latviešu Skolu doktorantūras sākumā un beidzot ar 2x2 kustību doktorantūras beigās. Īpaši vēlos pateikties Ainai Galējai par viņas akadēmisko un sabiedrisko piemēru, kam cenšos sekot lielu un mazu plānu veidošanā. Doktorantūras plānam jau izveidotam satiku Madaru Mazjāni, kuras atbalsts, ticība, atspoguļojums un iedrošinājums pilnveidoja sen noliktos mērķus. Paldies ģimenei par atbalstu, sapratni un pacietību, it īpaši līdzgaitniekiem brālim Pēterim un māsām Annai un Helēnai.

JL

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[1] Available online at `https://ctan.org/pkg/pgf`

[2] Available online at `https://ctan.org/pkg/tikz-3dplot`

CONTRIBUTIONS OF AUTHORS

Chapter 1 contains an introduction to specialists and non-specialists that sets up the main ideas we consider. It continues with a detailed setup of existing definitions and constructions necessary for the main results of this thesis. Lemmas 1.5.7 and 1.5.8 are the only original contributions in this chapter. In Chapter 2 we introduce a central object of interest, a particular stratification of $\mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$ and prove the key original results Theorem 2.1.4 and Theorem 2.2.2. These are my original work and have been submitted for publication (1). The results are then applied in Chapter 3 to construct an original cosheaf $\mathcal{F}$ and a sheaf $\mathcal{G}$ that capture the essence of this stratification, which appear in the same unpublished manuscript (1). Chapter 4 places these constructions in topological data analysis settings, and contains a final key original result in Corollary 4.1.2 that ties this work in with persistent homology. In Chapter 5 we briefly discuss further directions and propose an alternative stratification with more structure, leaving it as a starting point for future work.

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TDA	topological data analysis

SUMMARY

We introduce several new constructions for topological data analysis, specifically persistent homology, and use their stability to assemble collections of persistence modules. Underlying this is a partial order on simplicial complexes, the definition of which is motivated by the need to track changes in persistence by changes in the underlying metric space.

The main construction is a stratification of $\mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$, where $\mathrm{Ran}(\mathsf{M})$ is the space of finite subsets of a manifold M. Changes across the strata are captured by a cosheaf $\mathcal{F}$ valued in diagrams of simplicial complexes, for which the inverse image cosheaf on $\{\mathsf{P}\} \times \mathbf{R}_{\geqslant 0}$ recovers the persistent homology of the finite set P. Recognizing higher structure in the space of entrance paths produces a sheaf and a description of entrance paths over $\mathrm{Ran}(\mathsf{M})$ in terms of entrance paths over M. The key realized idea for both the cosheaf and the sheaf was interpreting entrance paths uniquely as simplicial maps, though we recognize the loss of some, not all, monodromy data.

Simplifying to 2-dimensional stratified subspaces $\mathrm{im}(\gamma) \times \mathbf{R}_{\geqslant 0} \subseteq \mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$, for some path γ in $\mathrm{Ran}(\mathsf{M})$, we find common threads with existing research of describing changes in persistence modules, simplicial modules (their generators), and barcodes (their summaries). The natural zigzag structure of simplicial modules benefits from the developing field of zigzag persistence, similarly encoding differences that are lost when passing to the barcode.

CHAPTER 1

INTRODUCTION

This thesis is about describing the stability of complicated functions on complicated spaces.

1.1 To every reader

My personal goal is to give discrete information a more unified and holistic structure. By "discrete information" I mean something like points in space. This goal can be expanded and made more precise by asking:

- How are topological spaces associated to a finite collection of points?
- How does the associated space change as the point collection changes?
- What is a good language that captures the points, the space, and some notion of change, so that:
  - a small change in the point collection is a small change in the associated space,
  - the space reflects heuristic notions of what the associated space should look like, and
  - the language is consistent with existing notions in topological data analysis (TDA)?

We develop this language here, combining geometry, topology, homotopy theory, and sheaf theory. The unifying theme of TDA takes in "data" in some form, such as a finite collection of points mentioned previously, and outputs "topological information" or a "topological summary" of the data. A key tool of TDA is persistent homology.

Figure 1 gives a somewhat technical impression of our approach, using the common concept of the persistent homology "pipeline," adapted from Ulrich Bauer's slides (2), and our relation to it. For more such intuition-based descriptions, I invite you to flip or scroll through this thesis via the List of Figures on page ix.

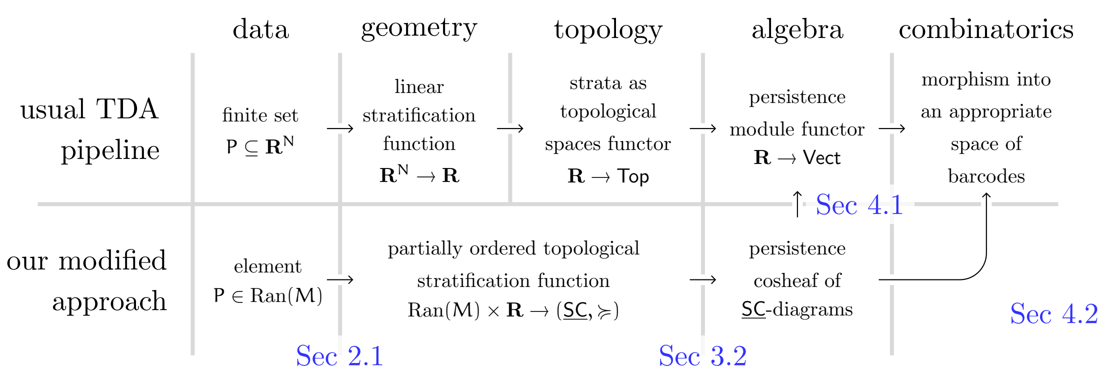


Figure 1: The persistent homology pipeline

If you are still reading now, you probably are interested in some more technical observations from Figure 1. Our approach here generalizes Euclidean space $\mathbf{R}^N$ to a more general manifold M, and combines the geometry and topology steps by uncurrying the stratification and restricting the category of topological spaces Top to a more structured subcategory $\underline{\mathsf{SC}}$ of simplicial complexes. We also generalize the functor $\mathbf{R} \to \mathsf{Top}$, interpreted as a linear diagram valued in Top, to a more general diagram valued in the category $\underline{\mathsf{SC}}$ of simplicial complexes.

If this is still not technical enough for you, then read on for a homotopy theoretic approach to TDA through a stratification of $\text{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ for a universal description of persistent homology. The Ran space $\text{Ran}(M)$ is the space of all finite subsets of a manifold M, and so every subset $\{P\} \times \mathbf{R}_{\geqslant 0} \subseteq \text{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ is well-suited for the persistent homology pipeline of Figure 1. The universal approach is used to give more insight into morphisms between persistence modules, by considering paths between finite subsets in $\text{Ran}(M)$ and the induced geometric, topological, and algebraic morphisms.

We continue with a historical review.

1.2 History

Manifold **stratifications** were introduced by Thom in (3, Section C). This idea was extended to topological spaces by Goresky–MacPherson in (4, Section 1.1) and Siebenmann (5, Definition 1.1), both mentioning the special cases of cone-like stratifications. A simpler view was taken by Lurie in (6, Definition A.5.1), by using the language of partially ordered sets. We take the latter approach.

For any topological space X, its **Ran space**, the space of finite subsets of X endowed with the topology from Hausdorff distance, has a natural stratification coming from the size of the subsets. This space is named after Ziv Ran and was first introduced by Borsuk and Ulam in (7). Some basic properties of the Ran space are discussed in (8, Section 3.4) and in (6, Section 5.5.1). Lurie describes the point-counting stratification of it, as do Ayala–Francis–Tanaka in (9, Definition 3.7.1), in the context of factorization homology. We extend this stratification to the product of the Ran space and $\mathbf{R}_{\geqslant 0}$, combining it with a partial order on isomorphism classes of simplicial complexes.

Stratified spaces admit **constructible sheaves**, which generalize locally constant sheaves on (not necessarily stratified) spaces. Constructible sheaves were introduced by Deligne in (10, Definition IV.3.1) and are given a modern treatment in (11, Section 4.1) and (6, Section A.5), the latter of which requires the base space to be stratified. A more restricted interpretation in TDA with categorical equivalences of these sheaves is presented in (12, Definition 3.6), which also builds them on stratified spaces.

Given a stratification, constructible sheaves on the stratified space are completely described by **exit paths** of the stratified space. This was first described by Treumann in (13) after MacPherson[1], extended in part by Ayala–Francis–Rozenblyum in (14, Corollary 3.3.11), and given in full generality by Lurie in (6, Theorem A.9.3). A good overview of exit paths in modern mathematics is given in (15, Introduction).

Current trends in **persistent homology** developed during the late 1990s and early 2000s by several groups independently[2]. Persistence diagrams, easily-understandable summaries of persistent homology, were shown to be stable by Cohen-Steiner et al. in (17). This stability was described in terms of persistence modules by Chazal et al. in (18, Theorem 4.4). The algebraic and categorical interpretation of persistent homology was successfully continued Edelsbrunner

[1]Treumann mentions that MacPherson's unpublished work was the first to recognize this exit path description.

[2]A proper overview of its history is given in (16).

et al. in (19), and Bauer–Lesnick in (20) and (21), providing some of the key motivation for this thesis.

1.3 Categories and simplicial sets

The word "category" will be used to mean "not necessarily a 1-category." Let Set be the category of all sets, Set_* the category of all pointed sets, Cat_1 the category of all 1-categories, and Cat_∞ the category of all ∞-categories. We use the quasi-category model of ∞-categories. Key to this thesis is constructions in opposite categories and with dual objects, and we endeavor to give complete descriptions from both directions.

We now follow (22, Section 1.2.14, Appendix A.2.7). Let $\mathcal{C}$ be a category.

Definition 1.3.1. Let A be an object of $\mathcal{C}$. The *over category* $\mathcal{C}_{/A}$ of $\mathcal{C}$ over A is the category whose objects are morphisms of $\mathcal{C}$ with target A, and whose morphisms are the natural commutative triangles. The *under category* ${}_{A\backslash}\mathcal{C}$ of $\mathcal{C}$ under A is the category whose objects are morphisms of $\mathcal{C}$ with source A, and whose morphisms are the natural commutative triangles.

The commutative triangles that were described as natural in Definition 1.3.1 are

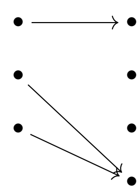
$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & A & \end{array} \quad , \qquad \begin{array}{ccc} & A & \\ & {\scriptstyle g_1}\swarrow \quad \searrow{\scriptstyle g_2} & \\ U_1 & \xrightarrow[g]{} & U_2 \end{array} \quad ,$$

for V_1, V_2, U_1, U_2 all objects of $\mathcal{C}$ and f, f_1, f_2, g, g_1, g_2 all morphisms in $\mathcal{C}$. There are natural forgetful functors $\mathcal{C}_{/A} \to \mathcal{C}$ and ${}_{A\backslash}\mathcal{C} \to \mathcal{C}$. Often the over category is called the *slice category.*

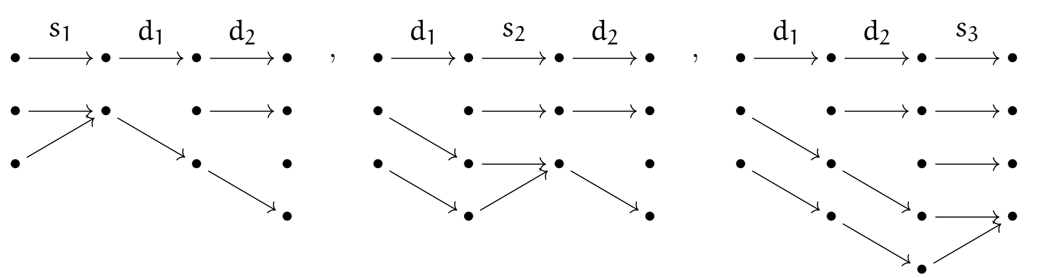
Definition 1.3.2. Let Δ be the 1-category that has finite ordered sets $[n] = (0, 1, \ldots, n)$ as objects, and monotonic (that is, order-preserving) maps $[n] \to [m]$ as morphisms, for all $m, n \in \mathbf{Z}_{\geqslant 0}$.

Note that every morphism in Δ is a composition of *coface maps* $s_i : [n] \to [n-1]$, which hit i twice, and *codegeneracy maps* $d_i : [n] \to [n+1]$, which skip i. This composition is not necessarily unique.

Example 1.3.3. Observe that the morphism $[3] \to [4]$ given by



can be decomposed into any one of the compositions



Definition 1.3.4. A *simplicial set* is a functor $S \colon \Delta^{\mathsf{op}} \to \mathsf{Set}$. For every $n \in \mathbf{Z}_{\geqslant 0}$, an n-*simplex* of S is an element of the set $S([n])$.

We often write S_n for $S([n])$. The maps described after Definition 1.3.2 through the functor S now become *face maps* $s_i \colon S_{n-1} \to S_n$ and *degeneracy maps* $d_i \colon S_{n+1} \to S_n$, where we use the same symbols for convenience. Let sSet be the category of simplicial sets, whose morphisms

$S \to T$ are natural transformations, equivalently described as a collection of level-wise set maps $S_n \to T_n$ satisfying appropriate diagrams with the s_i and d_i.

Every small category[1] $\mathcal{C}$ has a natural simplicial set associated to it.

Definition 1.3.5. The *nerve* $N(\mathcal{C})$ of a small category $\mathcal{C}$ is the simplicial set with $N(\mathcal{C})_0$ the set of objects of $\mathcal{C}$, and $N(\mathcal{C})_{n\geqslant 1}$ the set of n composable morphisms of $\mathcal{C}$.

The ith face map s_i of the nerve inserts the identity morphism at the ith step, and the ith degeneracy map d_i of the nerve composes the map whose target is the ith step with the map whose source is the ith step. A common use of the nerve is for $\Delta^n := N([n])$, the *standard* n*-simplex*[2]. Removing the ith face of Δ^n gives another common simplicial set Λ^n_i, the i*th* n*-horn*.

Definition 1.3.6. Let $\varphi\colon S \to T$ be a morphism of simplicial sets and $\iota\colon \Lambda^n_i \hookrightarrow \Delta^n$ the natural inclusion map. The morphism φ is a *fibration* if whenever we have a commutative diagram of solid arrows

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ {\scriptstyle\iota}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle\varphi} \\ \Delta^n & \longrightarrow & T \end{array} \tag{1.1}$$

in sSet, there exists a morphism, represented by the dashed arrow, the addition of which retains the commutativity of the diagram.

[1] A small category is a category whose collection of objects and collection of morphisms are both sets.

[2] To take the nerve of $[n]$, we view $[n]$ as the poset $(\{0, \dots, n\}, \leqslant)$ interpreted as a category.

Compare this with a special case in Equation 3.2. If the above condition holds for all $0 \leqslant i \leqslant n$, then φ is called a *Kan fibration*; if it holds for $0 < i < n$, it is called an *inner Kan fibration.*

This notion for a morphism to be a "fibration" can be extended to other categories.

Definition 1.3.7. A *model structure* on $\mathcal{C}$ is a choice of three distinguished classes of morphisms of $\mathcal{C}$, called *weak equivalences*, *fibrations*, and *cofibrations*, such that

- the weak equivalences satisfy the two-out-of-three condition of composition,
- every morphism can be factored into a trivial cofibration followed by a weak equivalence,
- every morphism can be factored into a weak equivalence followed by a trivial fibration.

A *trivial (co)fibration* is a (co)fibration that is also a weak equivalence.

Defining any two of the distinguished classes of morphisms immediately defines the third. A category with a model structure that contains all small limits and colimits is a *model category.* The main model categories we will be interested in are sSet and over categories of sSet.

Definition 1.3.8. Let $\mathcal{C}$ be a model category and A an object of $\mathcal{C}$. Its *cofibrant replacement* is the object QA of $\mathcal{C}$ that fits into the factored sequence $\emptyset \to QA \to A$ guaranteed by Definition 1.3.7 of the initial morphism $\emptyset \to A$. The *fibrant replacement* of A is the object RA of $\mathcal{C}$ that fits into the factored sequence $A \to RA \to *$ guaranteed by Definition 1.3.7 of the final morphism $A \to *$.

We will deal with categories where these objects are unique. This construction will be necessary in Section 3.3, where we use a fibrant-cofibrant replacement, meaning the fibrant replacement of the cofibrant replacement of an object (or equivalently the cofibrant replacement of the fibrant replacement of an object). The following definition is described in (22, Section 2.2.5).

Definition 1.3.9. The *Joyal model structure* on sSet has inner Kan fibrations as fibrations, and level-wise injections as cofibrations.

Given a simplicial set $\mathsf{S} \in \mathsf{sSet}$, from Definition 1.3.1 we have the over category $\mathsf{sSet}_{/\mathsf{S}}$, which inherits the model structure from sSet via the forgetful functor $\mathsf{sSet}_{/\mathsf{S}} \to \mathsf{sSet}$. That is, fibrations (cofibrations, weak equivalences) in $\mathsf{sSet}_{/\mathsf{S}}$ are precisely the morphisms that become fibrations (cofirbations, weak equivalences) in sSet through the functor. In fact, such an inherited model structure exists for any over and under category of a model category.

1.4 Topological spaces

In this section we introduce the topological spaces and associated objects that we will study. All topological spaces are assumed to be paracompact, unless otherwise noted.

Definition 1.4.1. A (*abstract*) *simplicial complex* C is a pair of sets $(\mathsf{V}(\mathsf{C}), \mathsf{S}(\mathsf{C}))$, with $\mathsf{S}(\mathsf{C}) \subseteq \mathsf{P}(\mathsf{V}(\mathsf{C}))$ closed under taking faces. Elements of $\mathsf{V}(\mathsf{C})$ are called *vertices* and elements of $\mathsf{S}(\mathsf{C})$ are called *simplices*.

Let SC be the set of simplicial complexes. To evade problems of at infinitely many vertices, we restrict SC to finite simplicial complexes. Let $\underline{\mathsf{SC}}$ be the category of finite simplicial complexes and simplicial maps, so SC is the set of objects of $\underline{\mathsf{SC}}$. Let $|\mathsf{C}|$ be the geometric realization of a simplicial complex.

Let P be a finite space with metric d, and $r \in \mathbf{R}_{\geqslant 0}$.

Definition 1.4.2. The *Čech complex* of P with radius r is a simplicial complex C with $\mathsf{V}(\mathsf{C}) = \mathsf{P}$ and $\mathsf{P}' \in \mathsf{S}(\mathsf{C})$ whenever the intersection of the all the balls around points in P' is non-empty, for every $\mathsf{P}' \subseteq \mathsf{P}$.

We will usually have $\mathsf{P} \subseteq M$ for some Riemannian manifold M, so distance d is implicit. Instead of checking n-fold intersections, we may choose to only check pairwise intersections. This gives the Vietoris–Rips complex.

Definition 1.4.3. The *Vietoris–Rips complex* of with radius r is a simplicial complex C with $\mathsf{V}(\mathsf{C}) = \mathsf{P}$ and $\mathsf{P}' \in \mathsf{S}(\mathsf{C})$ whenever $\mathsf{B}(p, r) \cap \mathsf{B}(p', r) \neq \emptyset$ for all $p, p' \in \mathsf{P}'$, for every $\mathsf{P}' \subseteq \mathsf{P}$.

There are other constructions of simplicial complexes from finite sets, but we are interested in the Čech and Vietoris–Rips constructions[1]. There are some topological spaces that have properties reminiscent of simplicial complexes.

Definition 1.4.4. A set in $\mathbf{R}^n$ is *semialgebraic* if it can be expressed as

$$\bigcup_{\text{finite}} \{x \in \mathbf{R}^N \ : \ f_1(x) = 0, f_2(x) > 0, \ldots, f_m(x) > 0\},$$

for polynomial functions $f_1, \ldots, f_m$ on $\mathbf{R}^N$. Distance on a semialgebraic set is the restriction of Euclidean distance on $\mathbf{R}^N$ to the set.

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[1]We prefer the Čech approach, because it has a shorter description and is more general. That is, a change in the input that changes the Vietoris–Rips complex must also change the Čech complex, so both constructions are covered by only considering the Čech approach.

Finally, we introduce the Ran space of a given space. Let M be a metric space with a distance function d, and let $n \in \mathbf{Z}_{>0}$. We consider M a topological space with the metric topology induced by d.

Definition 1.4.5. The *Ran space* of M is $\mathrm{Ran}(M) := \{P \subseteq M \ : \ 0 < |P| < \infty\}$, with topology induced by Hausdorff distance d_H of subsets of M.

For a positive integer n, write $\mathrm{Ran}_n(M)$ and $\mathrm{Ran}_{\leqslant n}(M)$ for the subspaces of $\mathrm{Ran}(M)$ with elements exactly of size n and at most size n, respectively. In the former case, $\mathrm{Ran}_n(M) = \mathrm{Conf}_n(M)$ is also called the *configuration space* of n points. Recall the Hausdorff distance between $P, Q \in \mathrm{Ran}(M)$ is defined as

$$\begin{aligned} d_H(P,Q) &:= \max\left\{\max_{p\in P}\min_{q\in Q} d(p,q), \max_{q\in Q}\min_{p\in P} d(p,q)\right\} \qquad (1.2)\\ &= \min\left\{r : Q \subseteq \bigcup_{p\in P} B(p,r),\ P \subseteq \bigcup_{q\in Q} B(q,r)\right\}. \end{aligned}$$

We write B for the closed ball in M and B° for the open ball in M. The Hausdorff distance d_H can be compared with a distance d_M for subsets, for which

$$d_M(X,Y) := \inf_{x\in X, y\in Y}\{d(x,y)\} \leqslant d_H(X,Y), \qquad (1.3)$$

for any $X, Y \subseteq M$. On the product space $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ we use the sup-norm

$$d_\infty((P,r),(Q,s)) := \max\{d_H(P,Q), |r-s|\}.$$

Remark 1.4.6. By scaling and (6, Section 5.5.1), the topology on Ran(M) induced by d_H is equivalent to the coarsest topology that has $\{P \in \text{Ran}(M) : P \subseteq \bigcup_i U_i, P \cap U_i \neq \emptyset\ \forall\ i\}$ as open sets, for all nonempty disjoint collections of open sets $\{U_i \subseteq M\}$.

1.5 Stratifications

Broadly speaking, a *stratification* of a space is a decomposition of the space into disjoint pieces called *strata*.

Let $(A, \leqslant)$ be a poset. If the partial order on A is clear from context, we simply write A. Posets have the upwards-directed, or upset, or Alexandrov topology. This topology has as basis the sets $U_a := \{b \in A : a \leqslant_A b\}$ for all $a \in A$, an example of which is given in Figure 2. We write $A_{>a} := \{a' \in A : a' > a\}$, and analogously for $\geqslant a$, $< a$, $\leqslant a$, which are all posets with the induced partial order from A.

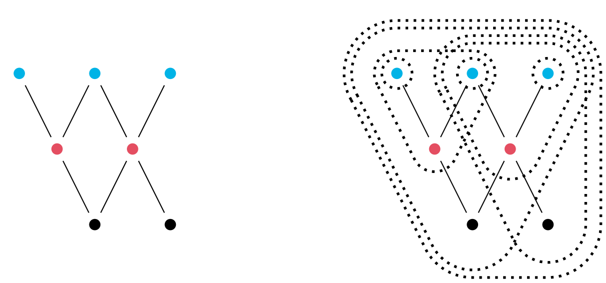

Figure 2: A poset and its basis of open sets.

Let X be a topological space. We follow (6, Section A.5) in the following definitions.

Definition 1.5.1. An A-*stratification* of X, or *poset stratification* of X when A is clear from context, is a continuous map $f\colon X \to A$. A *stratum* of X is $X_a := \{x \in X : f(x) = a\}$, for some $a \in A$.

When f is clear from context, we say X is A-*stratified* or X *is stratified by* A. Some examples of poset-stratifications of the sphere are given in Figure 3.

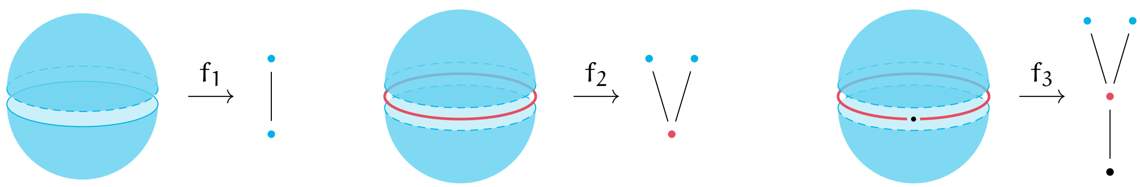

Figure 3: Three different poset stratifications of the sphere.

Remark 1.5.2. Another way to decompose a space is by *filtrations*, which define a larger class of objects than stratifications. An A-filtration of a topological space X, for A a totally ordered set, is a function $\varphi\colon A \to \mathsf{Top}$ so that $a \leqslant b$ implies $\varphi(a) \subseteq \varphi(b)$, and $X = \bigcup_{a \in A} \varphi(A)$. We note that:

- A filtration can be viewed as a functor from the poset category $(A, \leqslant)$ to the category $(\mathsf{Top}, \subseteq)$ of topological spaces and inclusions.

- Given a stratification $f\colon X \to A$, there is a natural associated filtration $\varphi\colon A \to \mathsf{Top}$ given by $\varphi(a) = \bigcup_{b \leqslant a} X_a$, called the *sublevel set* filtration.
- Given a filtration $\varphi\colon A \to \mathsf{Top}$, there is no natural stratification of X, as the subset requirement of a filtration says nothing about the openness of sets.

Definition 1.5.3. An A-stratification of X satisfies the *frontier condition* if $(\overline{X_a} \setminus X_a) \cap X_b \neq \emptyset$ implies $X_b \subseteq \overline{X_a}$, for every pair $b \leqslant a$ in A.

The stratifications f_2, f_3 in Figure 3 satisfy the frontier condition, but f_1 does not.

Definition 1.5.4. An A-stratification of X is *compatible with*, or *refines* a B-stratification of X if, equivalently,

- for every $a \in A$ and $b \in B$, either $X_a \subseteq X_b$ or $X_a \cap X_b = \emptyset$, or
- for every $b \in B$ there is a subset $A' \subseteq A$ such that $X_b = \bigcup_{a \in A'} X_a$.

It is immediate that being compatible is a transitive relation. In Figure 3 the stratification f_2 refines f_1, and f_3 refines f_2.

Definition 1.5.5. Given an A-stratification $f\colon X \to A$ and a B-stratification $g\colon Y \to B$, a *stratified map* ϕ from f to g is a pair of continuous maps $\phi_{XY}\colon X \to Y$ and $\phi_{AB}\colon A \to B$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_{XY}} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\phi_{AB}} & B \end{array}$$

commutes. The stratified map ϕ is an *open embedding* (respectively, *homeomorphism*) if both ϕ_{XY} and $\phi_{XY}|_{X_a}\colon X_a \to Y_{\phi_{AB}(a)}$ are open embeddings (respectively, homeomorphisms), for all $a \in A$.

Stratifications define partitions of the base space, so if an A-stratification f is compatible with a B-stratification g, there is a stratified map ϕ between them, with an order-preserving poset map $\phi_{AB}|_{f(X)}\colon f(X) \to g(X)$.

We are interested in a special class of poset stratifications, already mentioned in Section 1.2. Here we follow (6, Definition A.5.5).

Definition 1.5.6. Let $f\colon X \to A$ be an A-stratification of X. Then X is *conically stratified at* $x \in X$ by f if there exist

- a topological space Z,
- an $A_{>f(x)}$-stratified topological space L, and
- a stratified open embedding $Z \times C(L) \hookrightarrow X$ whose image contains x.

The space X is *conically stratified* by f if it is conically stratified at every $x \in X$ by f, in which case we call f a *conical stratification* of X.

Often Z is Euclidean space. The *stratified cone* $C(L)$ of L is defined as follows. Given an $A_{>f(x)}$-stratification $g\colon L \to A_{>f(x)}$, the open cone $C(L)$, understood as the quotient $L \times [0,1)/L \times \{0\}$, has the $A_{\geqslant f(x)}$-stratification $g'\colon C(L) \to A_{\geqslant f(x)}$ given by $g'(\ell, t \neq 0) = g(\ell)$ and $g'(\ell, 0) = f(x)$. The product $Z \times C(L)$ is naturally $A_{\geqslant f(x)}$-stratified through projection to the cone factor.

A visual description of the construction from Definition 1.5.6 is given in Figure 4. The idea to have in mind is that Z is an open neighborhood of x in its stratum $X_{f(x)}$, and L is a collection of neighborhoods in strata directly above $X_{f(x)}$, called the *link* of x.

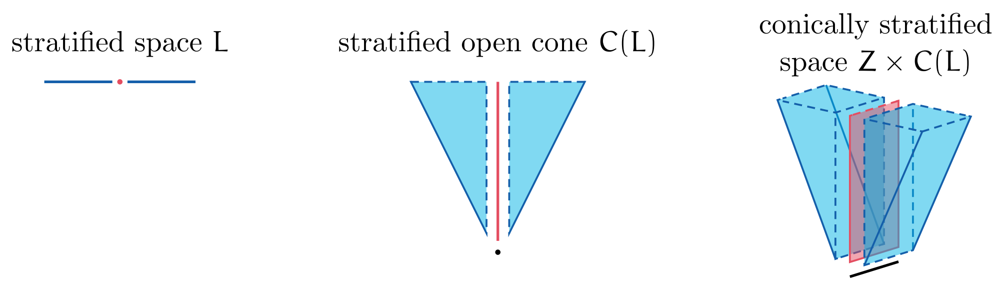

Figure 4: Constructing a neighborhood in a conical stratification.

Lemma 1.5.7. *Let f be an A-stratification of a topological space X whose strata are path-connected. If f is a conical stratification, then f satisfies the frontier condition.*

Proof. Take a pair $b \leqslant a$ in A for which $(\overline{X_a} \setminus X_a) \cap X_b \neq \emptyset$, and let $x \in (\overline{X_a} \setminus X_a) \cap X_b$. Since X is conically stratified at x, we have a stratified open embedding $\mathsf{emb}\colon Z \times C(L) \to X$, as in Definition 1.5.6, for some $A_{>b}$-stratified space L, as $f(x) = b$. Given the stratified cone

$g\colon C(L) \to A_{\geqslant b}$, note that $C(L)_b \subseteq \overline{C(L)_a}$, as $C(L)_b$ is the stratum of the cone point of $C(L)$. Hence $Z \times C(L)_b \subseteq \overline{Z \times C(L)_a}$, both viewed as subsets of $Z \times C(L)$. It follows immediately that

$$x \in \mathsf{emb}(Z \times C(L)_b) \subseteq \overline{\mathsf{emb}(Z \times C(L)_a)}. \tag{1.4}$$

That is, x has an open neighborhood $U_x \subseteq X_b$ such that $U_x \subseteq \overline{X_a}$. Since $(\overline{X_a} \setminus X_a) \cap X_b$ is closed in X_b and X_b is path-connected, such a neighborhood exists for every $x \in X_b$. Hence $X_b \subseteq \overline{X_a}$. □

Semialgebraic sets, introduced in Definition 1.4.4, are well-suited for conical stratifications. A stratification is a *semialgebraic* stratification when every stratum is a semialgebraic set.

Lemma 1.5.8. *Let* f *be a semialgebraic stratification of a closed semialgebraic set* X*. Then there exists a conical semialgebraic stratification of* X *compatible with* f*.*

Proof. Let $f\colon X \to A$ be as in the statement. By (23, Theorem II.4.2), there exists a simplicial complex K with homeomorphic image $|K| \cong X$ and stratum decomposition $f^{-1}(a) = \bigcup_i \sigma^\circ_{a_i}$, for σ° the interior of a simplex $\sigma \in S(K)$, for every $a \in A$. With the partial order $\sigma^\circ \leqslant \tau^\circ$ whenever σ is a face of τ, there is a natural stratification $g\colon |K| \to \{\sigma^\circ : \sigma \in S(K)\}$, and g is compatible with f by the mentioned result. This stratification of $|K|$ is precisely the S-stratification of $|K|$ given by (6, Definition A.6.7), where $S = S(K)$, which is conical by (6, Proposition A.6.8). The S-stratification is semialgebraic because the interiors of simplices are semialgebraic, and finite unions of semialgebraic sets are semialgebraic. □

Let $\mathsf{Op}(X)$ be the category of open sets of X and inclusions. We refine this category for a conically stratified space. Let $f\colon X \to A$ be a conical stratification.

Definition 1.5.9. A *basic open* of X is an open set $U \in \mathsf{Op}(X)$ that is the image of a stratified open embedding $\mathbf{R}^k \times C(L) \hookrightarrow X$, for $\mathbf{R}^k \times C(L)$ satisfying the conditions of Definition 1.5.6. The stratum $X_{f(\mathbf{R}^k \times *)}$, for $*$ the cone point of $C(L)$, is called the *associated stratum* of U.

Let $\mathsf{Bsc}(X) \subseteq \mathsf{Op}(X)$ be the category of basic opens of X and inclusions.

1.6 Sheaves

Let $\mathcal{C}$ be a category that has all small limits and colimits.

Definition 1.6.1. A *presheaf* on X valued in $\mathcal{C}$ is a functor $\mathsf{Op}(X)^{\mathsf{op}} \to \mathcal{C}$. A *precosheaf* on X valued in $\mathcal{C}$ is a functor $\mathsf{Op}(X) \to \mathcal{C}$.

Every presheaf produces a sheaf through sheafification that retains much of the properties of the presheaf, but a similar analogy does not hold for precosheaves, as discussed in (24, Section 2.5.3). If for every pair $V \subseteq U$ in $\mathsf{Bsc}(X)$ associated to the same stratum, the induced morphism of the pre(co)sheaf functor is an isomorphism, both presheaves and precosheaves uniquely determine sheaves and cosheaves, respectively. See (12) and (25) for more welcome properties of (co)sheaves defined on $\mathsf{Bsc}(X)$.

Definition 1.6.2. A presheaf $\mathcal{F}$ is a *sheaf* if for every $U \in \mathsf{Op}(X)$ and every open cover $\{U_i\}$ of U, the natural map

$$\mathcal{F}(U) \to \lim_i \mathcal{F}(U_i)$$

is an isomorphism. A precosheaf $\mathcal{G}$ is a *cosheaf* if for every $U \in \mathsf{Op}(X)$ and every open cover $\{U_i\}$ of U, the natural map

$$\operatorname*{colim}_i \mathcal{G}(U_i) \to \mathcal{G}(U)$$

is an isomorphism.

The (co)limits are taken over the image in $\mathcal{C}$ of the full subcategory of $\mathsf{Op}(X)$ of all the U_i and their intersections. Let $\mathsf{Shv}(X)$ be the category of sheaves on X. So far we have described sheaves as functors out of $\mathsf{Op}(X)$, but it is common to instead describe them as functors out of its subcategory $\mathsf{Bsc}(X)$, as in (12), (9), and (25). This provides some simplifications, as we see in Section 3.2.

Definition 1.6.3. Let $\mathcal{F}$ be a sheaf or a cosheaf on an A-stratified space X.

- As a functor out of $\mathsf{Op}(X)$, $\mathcal{F}$ is A-*constructible* if $\mathcal{F}|_{X_a}$ is locally constant, for every $a \in A$;
- As a functor out of $\mathsf{Bsc}(X)$, $\mathcal{F}$ is A-*constructible* if $\mathcal{F}(V \subseteq U)$ is an isomorphism for all V, U associated to the same stratum.

For $\mathsf{Op}(X)$, this definition follows (11, Definition 4.1.1), and for $\mathsf{Bsc}(X)$ it follows (25, Definition 3.2). For sets $V \subseteq X$ not necessarily open and $\mathcal{F}$ a sheaf, $\mathcal{F}|_V$ is the inverse image presheaf $V \mapsto \operatorname{colim}_{U \supseteq V} \mathcal{F}(U)$. When $\mathcal{F}$ is a cosheaf, $\mathcal{F}|_V$ is the inverse image precosheaf $V \mapsto \lim_{U \supseteq V} \mathcal{F}(U)$, as in (26, Appendix B).

Let $\mathsf{Shv}^A(X)$ be the category of A-constructible sheaves on X. We return to this category in Section 3.3, and consider the subcategory of factorizable cosheaves is mentioned in Section 5.2.

1.7 Persistent homology

In this section we briefly recall the basic constructions for persistent homology, partly introduced in Section 1.1. As in Section 1.4, let P be a finite metric space. Fix a linear filtration $\mathsf{f}\colon \mathbf{R} \to \mathsf{Top}$ of the complete $|\mathsf{P}|$-simplex on the vertices P. This may be viewed as a functor $(\mathbf{R}, \leqslant) \to \underline{\mathsf{SC}}$, and in view of Definition 1.7.1, we call this functor the *simplicial module* of P, and write $\mathrm{SM}_{\mathsf{P}}\colon \mathbf{R} \to \underline{\mathsf{SC}}$. The precise relationship between the two is explored in Section 4.1.

This is a slightly different approach than the usual $\mathsf{P} \subseteq \mathbf{R}^{\mathsf{N}}$ and $\mathbf{R}^{\mathsf{N}} \to \mathbf{R}$ approach described in Figure 1, but the two are equivalent if P can be embedded in $\mathbf{R}^{\mathsf{N}}$. Otherwise, taking a finite metric space P and a filtration is more general.

Fix a homology degree for the homology functor H, and let Vect be the category of finite-dimensional vector spaces over an algebraically closed field $\mathbf{k}$.

Definition 1.7.1. The *persistence module* of the filtration f is $\mathrm{PM}_{\mathsf{f}} := \mathsf{H} \circ \mathsf{f}\colon \mathbf{R} \to \mathsf{Vect}$. The image of this functor is the collection of *persistent homology groups* of f.

Having homology groups valued in vector spaces over a field allows for unique decomposition, as in (27, Theorem 1.1).

Definition 1.7.2. Let $(\mathsf{Int}, \subseteq)$ be the set of intervals $[a, b) \subseteq \mathbf{R}$, partially ordered by inclusion. A *persistence diagram*, or *barcode*, is a function $\mathsf{Int} \to \mathbf{Z}_{\geqslant 0}$ that is 0 at all but finitely many intervals.

The statement of (27, Theorem 1.1) says that there is a unique persistence diagram associated to the Vect-valued persistence module of the filtration f, which we call PD_{f}. One way to generalize this is to use a different indexing poset, often a *zigzag*, a generalization considered in Section 4.2.

As in Section 1.4, we consider topologies on the spaces introduced here. Given two integers p, q, we have the (p, q)-Wasserstein distance on the set of persistence diagrams, given by

$$d_W(D, E) := \inf_{\varphi\colon D \to E} \left(\sum_{x \in D} \|x - \varphi(x)\|_p^q \right)^{1/q}, \tag{1.5}$$

for two persistence diagrams D, E. Often this approach is too general, and the case $q = 1$ and $p \to \infty$ is taken, without the sum, which is the bottleneck distance

$$d_B(D, E) := \inf_{\varphi\colon D \to E} \left(\sup_{x \in D} \|x - \varphi(x)\|_\infty \right). \tag{1.6}$$

Both of these distances make the set of persistence diagrams, denoted Dgm, into a topological space. We can also consider distances on the set of persistence modules, the most common one being interleaving distance.

Definition 1.7.3. For two functors $\mathcal{M}, \mathcal{N}\colon \mathbf{R} \to \mathsf{Vect}$, we say $\mathcal{M}, \mathcal{N}$ are ϵ-*interleaved* if there exist functors $F\colon \mathcal{M} \to \mathcal{N} \circ T_\epsilon$ and $G\colon \mathcal{N} \to \mathcal{M} \circ T_\epsilon$ such that $G \circ F = \mathcal{M}(T_0 \to T_{2\epsilon})$ and $F \circ G = \mathcal{N}(T_0 \to T_{2\epsilon})$, where $T_\epsilon\colon \mathbf{R} \to \mathbf{R}$ is the functor defined by $t \mapsto t + \epsilon$.

The *interleaving distance* is an extended pseudo metric, and is defined as

$$d_I(\mathcal{M}, \mathcal{N}) := \inf\{\epsilon : \mathcal{M}, \mathcal{N} \text{ are } \epsilon\text{-interleaved}\}. \tag{1.7}$$

In (28) this was shown to be universal among all pseudo metrics of persistence modules. This was generalized in (29) to a homotopy-invariant pseudo metric on persistence modules of filtered spaces (see Remark 1.5.2 for the relation between filtrations and stratifications).

Germane to our work is (19), (20) and (21), the first of which introduces a particular category to ask what a "morphism" of persistence diagrams would look like. A notion of an "induced" morphism between two persistence diagrams, and its limitations, is introduced in (20). We consider these limitations in more detail and identify some of their causes in Section 4.1.

CHAPTER 2

STRATIFYING BY SIMPLICIAL COMPLEXES

This chapter and the next contains material submitted for publication (1).

In this chapter we describe the Čech construction of a simplicial complex as a stratification of $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$, which we call the *Čech stratification.* The stratification will generalize the point-counting map $|\cdot|\colon \mathrm{Ran}(M) \to (\mathbf{Z}, \leqslant)$, which itself is a stratification by the definition of Hausdorff distance. When restricted to $\mathrm{Ran}_{\leqslant n}(M)$, the map $|\cdot|$ is a conical stratification, by (9, Proposition 3.7.5). This section will conclude with an analogous statement of the existence of a conical stratification, in Theorem 2.2.2, for the Čech stratification.

2.1 The Čech stratifications $\check{C}$ and $[\check{C}]$

We interpret every $(P, r) \in \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ as a simplicial complex.

Definition 2.1.1. The *Čech map* is the function $\check{C}\colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to \mathsf{SC}$ given by $V(\check{C}(P, r)) = P$ and $P' \in S(\check{C}(P, r))$ whenever $\bigcap_{p \in P'} B(p, r) \neq \emptyset$, for every $P' \subseteq P$.

Isomorphism of simplicial complexes is an equivalence relation, so let $[\mathsf{SC}] := \mathsf{SC}_{/\cong}$ be the set of isomorphism classes $[C]$ of simplicial complexes.

Definition 2.1.2. Let $\succcurlyeq$ be the relation on $[\mathsf{SC}]$ given by $[C] \succcurlyeq [C']$ whenever there is a simplicial map $C \to C'$ that is surjective on vertices.

This relation is well-defined, irrespective of the choice of class representatives.

Lemma 2.1.3. *The relation* $\succcurlyeq$ *defines a partial order on* $[\mathsf{SC}]$.

Proof. Let $[\mathsf{C}], [\mathsf{C}'], [\mathsf{C}''] \in [\mathsf{SC}]$.

For reflexivity, take any two representatives $\mathsf{C}_1, \mathsf{C}_2$ of $[\mathsf{C}]$. Since $\mathsf{C}_1 \cong \mathsf{C}_2$, the isomorphism is a bijection $\mathsf{C}_1 \to \mathsf{C}_2$ in SC, which is surjective on vertices.

For anti-symmetry, suppose that $[\mathsf{C}] \succcurlyeq [\mathsf{C}']$ and $[\mathsf{C}'] \succcurlyeq [\mathsf{C}]$. If $|\mathsf{V}(\mathsf{C})| > |\mathsf{V}(\mathsf{C}')|$, then we cannot have $[\mathsf{C}'] \succcurlyeq [\mathsf{C}]$, and if $|\mathsf{V}(\mathsf{C}')| > |\mathsf{V}(\mathsf{C})|$, we cannot have $[\mathsf{C}] \succcurlyeq [\mathsf{C}']$. Hence we must have $|\mathsf{V}(\mathsf{C})| = |\mathsf{V}(\mathsf{C}')|$, and so any map $\mathsf{C} \to \mathsf{C}'$ inducing $[\mathsf{C}] \succcurlyeq [\mathsf{C}']$ must be injective on vertices, and so injective on simplices. Similarly, the same properties hold any map $\mathsf{C}' \to \mathsf{C}$ inducing $[\mathsf{C}'] \succcurlyeq [\mathsf{C}]$. Hence we have a map $\mathsf{C} \to \mathsf{C}'$ that is bijective on simplices, so $\mathsf{C} \cong \mathsf{C}'$, and $[\mathsf{C}] = [\mathsf{C}']$.

For transitivity, suppose that $[\mathsf{C}] \succcurlyeq [\mathsf{C}']$ and $[\mathsf{C}'] \succcurlyeq [\mathsf{C}'']$. Then there exists a simplicial map $\mathsf{C} \to \mathsf{C}'$ that is surjective on $\mathsf{V}(\mathsf{C}')$, as well a simplicial map $\mathsf{C}' \to \mathsf{C}''$ that is surjective on $\mathsf{V}(\mathsf{C}'')$. The composition of these two simplicial maps is a simplicial map $\mathsf{C} \to \mathsf{C}''$, and as both were individually surjective on vertices, the composition must also be surjective on vertices. □

The same arguments show that $\succcurlyeq$ defines a preorder on SC.

We now consider the partially ordered set $([\mathsf{SC}], \succcurlyeq)$ as a topological space with the upset topology, as described in Section 1.5. Let $[\check{\mathrm{C}}] \colon \mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SC}]$ be the composition of $\check{\mathrm{C}}$ and the projection to $[\mathsf{SC}]$.

Theorem 2.1.4. *The Čech map* $[\check{\mathrm{C}}]$ *is continuous.*

Proof. A basis for the upset topology on $[\mathsf{SC}]$ consists of the sets $U_{[C]} = \{[C'] \in [\mathsf{SC}] \,:\, [C'] \succcurlyeq [C]\}$ based at $[C] \in [\mathsf{SC}]$, so we show the preimage of all such sets is open in $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$. Take any $(P, r) \in [\check{C}]^{-1}(U_{[C]})$, with $P = \{P_1, \ldots, P_k\}$, which we will show has an open neighborhood contained in $[\check{C}]^{-1}(U_{[C]})$. For every $P' \subseteq P$, let

$$\check{c}s(P') := \bigcap_{p \in P'} B(p, \inf\{r : \bigcap_{p' \in P'} B(p', r) \neq \emptyset\}) \subseteq M, \tag{2.1}$$

$$\check{c}r(P', r) := r - d_M(P', \check{c}s(P')) \in \mathbf{R} \tag{2.2}$$

be the *Čech set*[1] of P' and *Čech radius* of P' at r, respectively[2]. The Čech set is the smallest non-empty intersection of the closed balls on M of increasing radius around P'. The inf is necessary when $|P'| = 1$, otherwise the minimum always exists, as the balls are closed and M is connected. The Čech radius is positive if and only if the intersection $\bigcap_{p \in P'} B(p, r)$ contains an open set of M, negative when the intersection is empty, and 0 otherwise.

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[1] This can be thought of as the circumcenter of some subset of P, whose size is restricted by $\dim(M)$ and whose choice is restricted by its convex hull.

[2] These two constructions are related by the equation $\check{c}r(P', d_M(P', \check{c}s(P'))) = 0$.

Case 1: For every $P' \subseteq P$ with $|P'| > 1$, $\check{c}r(P', r) \neq 0$. Let $B^{\circ}_{\infty}((P, r), \tilde{r}/4)$ be the open ball in the sup-norm on the product $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ around (P, r) of radius $\tilde{r}/4$, where $\tilde{r}$ is the smallest of the two values

$$r_1 := \min_{1 \leqslant i < j \leqslant k} d(P_i, P_j), \tag{2.3}$$

$$r_2 := \min_{P' \subseteq P,\ |P'| > 1} 2|\check{c}r(P', r)|. \tag{2.4}$$

Briefly, having $\tilde{r} \leqslant r_1$ guarantees that points will not merge in the open ball, and having $\tilde{r} \leqslant r_2$ guarantees that simplices among the P_i are neither lost nor gained in the open ball. Figure 5 illustrates these roles of r_1 and r_2.

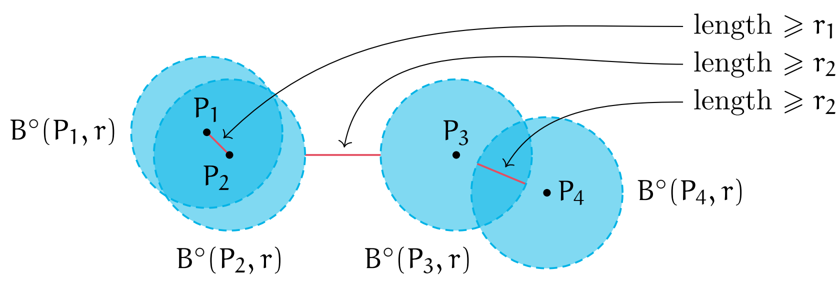

Figure 5: A finite subset of M and open balls in M around its elements.

Let $(Q, s) \in B^\circ_\infty((P, r), \tilde{r}/4)$. Since $\tilde{r} \leqslant r_1$, we have that $d_H(P, Q) < \tilde{r}/4$, which implies that $Q \subseteq \bigcup_{i=1}^{k} B^\circ(P_i, \tilde{r}/4)$. Similarly, the $B^\circ(P_i, \tilde{r}/4)$ are disjoint. Also, for every $1 \leqslant i \leqslant k$, note that $Q \cap B^\circ(P_i, \tilde{r}/4) \neq \emptyset$, as

$$d_M(\{P_i\}, Q) = \min_{q \in Q} d(P_i, q) \leqslant d_H(P, Q) \leqslant d_\infty((P, r), (Q, s)) < \tilde{r}/4. \tag{2.5}$$

In other words, there is a well-defined, natural, and surjective map $\phi\colon Q \to P$ for which $\phi(q) = P_i$ whenever $q \in B^\circ(P_i, \tilde{r}/4)$.

Next, we claim ϕ is a simplicial map. Take $Q' \subseteq Q$ and suppose that $\check{C}(Q', s)$ is a $(|Q'|-1)$-simplex. Let $P' = \{P'_0, \ldots, P'_\ell\} \subseteq P$ be such that $Q' \subseteq \bigcup_{i=1}^{\ell} B^\circ(P'_i, \tilde{r}/4)$ and $Q \cap B^\circ(P'_i, \tilde{r}/4) \neq \emptyset$, for $1 \leqslant i \leqslant \ell$. Suppose, for contradiction, that $\check{C}(P', r)$ is not a $(|P'|-1)$-simplex, or equivalently, that $\check{c}r(P', r) < 0$. Then

$$\begin{aligned}
0 &\geqslant \check{c}r(P', r) + \tilde{r}/2 && \text{(by Equation 2.4 and that } \tilde{r} \leqslant r_2) \\
&= r - d_M(P', \check{c}s(P')) + \tilde{r}/2 && \text{(by definition of Čech radius)} \\
&> r - d_M(Q', \check{c}s(Q')) - \tilde{r}/4 + \tilde{r}/2 && \text{(since } d_H(P, Q) < \tilde{r}/4) \\
&\geqslant s - |s - r| - d_M(Q', \check{c}s(Q')) + \tilde{r}/4 && \\
&> \check{c}r(Q', s) - \tilde{r}/4 + \tilde{r}/4 && \text{(since } |s - r| < \tilde{r}/4) \\
&= \check{c}r(Q', s),
\end{aligned}$$

contradicting the assumption that $\check{c}r(Q', s) \geqslant 0$, as $\check{C}(Q', s)$ was assumed to be a $(|Q'|-1)$-simplex. Hence $\check{C}(P', r)$ is a $(|P'|-1)$-simplex, and so the image of $\check{C}(Q', s)$ under ϕ is the simplex $\check{C}(P', r)$. Since simplices get taken to simplices, the map $\phi\colon Q \to P$ extends to a simplicial map $\check{C}(Q, s) \to \check{C}(P, r)$ that is surjective on vertices. That is, $[\check{C}(Q, s)] \succcurlyeq [\check{C}(P, r)] = [C]$, and so $B^\circ_\infty((P, r), \tilde{r}/4) \subseteq [\check{C}]^{-1}(U_{[C]})$, meaning that $[\check{C}]^{-1}(U_{[C]})$ is open.

Case 2: There is some $P' \subseteq P$ with $|P'| > 1$ and $\check{c}r(P', r) = 0$. Then $r_2 = 0$ from Equation 2.4, so let

$$r_2' := \min_{P' \subseteq P,\ \check{c}r(P', r) \neq 0} 2|\check{c}r(P', r)|, \tag{2.6}$$

and let $\tilde{r}$ be the smallest of the two values r_1 and r_2'. As in Case 1, we claim the open neighborhood $B^\circ_\infty((P, r), \tilde{r}/4)$ of (P, r) is contained within $[\check{C}]^{-1}(U_{[C]})$. The proof of this claim proceeds as in the first case: the only place that r_2 was used was to state that $0 \geqslant \check{c}r(P', r) + \tilde{r}/2$, in showing that $\check{C}(P', r)$ is indeed a $(|P'|-1)$-simplex. If $\check{c}r(P', r) = 0$, then we already have this conclusion, and it is unnecessary to get to the contradiction. That is, ϕ still extends to a simplicial map, and $[\check{C}]^{-1}(U_{[C]})$ is open in this case as well. □

It follows that $[\check{C}]$ is a $[SC]$-stratification of $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$.

Corollary 2.1.5. *Every path* $\gamma\colon I \to \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ *with* $[\check{C}](\gamma(t))$ *constant for* $t \in [0, 1)$ *induces a unique simplicial map* $\check{C}(\gamma(0)) \to \check{C}(\gamma(1))$.

Proof. Let $\gamma(0) = (Q, s)$, with $Q = \{Q_1, \dots, Q_k\}$. For $i = 1, \dots, k$, let $\gamma_i\colon I \to M$ be the induced paths on M. That is, for π_1 the projection to the first factor of $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$, the induced paths are described by $\pi_1(\gamma(t)) = \{\gamma_1(t), \dots, \gamma_k(t)\}$. Such a decomposition exists because $[\check{C}](\gamma([0,1))$ is constant. We immediately get an induced set map $Q \to P$ from these paths.

If $[\check{C}](\gamma(t))$ is constant for all $t \in I$, the set map $Q \to P$ induced by the paths γ_i extends to a simplicial map $\check{C}(\gamma(0)) \to \check{C}(\gamma(1))$, by renaming of vertices.

If $[\check{C}](\gamma(t < 1)) \neq [\check{C}](\gamma(1))$, write γ as the concatenation of the paths γ_a with γ_b, where the image of γ_b is completely within[1] $B^\circ_\infty((P,r),\tilde{r}/4)$, with $\tilde{r}$ defined as in Case 2 of the proof to Theorem 2.1.4. In that proof, the unique simplicial map was only described based on the induced set map $\gamma_b(0) \to \gamma_b(1) = P$. Take that simplicial map and precompose it with the simplicial map from γ_a (for which $[\check{C}]$ is constant), to get a unique simplicial map $\check{C}(\gamma(0)) \to \check{C}(\gamma(1))$ that extends the set map $Q \to P$. □

Given a path γ as in Corollary 2.1.5, let $\check{\gamma}$ be the induced simplicial map. It is important to note that two maps $\gamma, \gamma' \colon I \to \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ that have the same endpoints may not induce the same simplicial maps $\check{\gamma}$, $\check{\gamma}'$. This observation is revisited in Lemma 3.2.1.

Remark 2.1.6. Paths in Ran(M) capture the monodromy of individual points, but simplicial maps do not capture the effect on vertices. We ignore monodromy for most of this work, revisiting it only in Section 4.3.

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[1] It may be that γ_a is the constant path at $\gamma(0)$.

The Čech stratification $[\check{C}]$ extends the point-counting stratification of $\mathrm{Ran}(M)$ mentioned at the beginning of Chapter 2. Indeed, for $|V(\,\cdot\,)| \colon [\mathsf{SC}] \to \mathbf{Z}_{\geqslant 0}$ the map that takes an equivalence class of simplicial complexes to the size of its vertex set,

$$|V(\,\cdot\,)| \circ [\check{C}]|_{\mathrm{Ran}(M)\times\{0\}} \colon \mathrm{Ran}(M) \to \mathbf{Z}_{\geqslant 0}$$

is the point-counting stratification of $\mathrm{Ran}(M)$. Moreover, $[\check{C}]$ is a refinement of the stratification induced by $|\,\cdot\,|$ on $\mathrm{Ran}(M) \times \mathbf{Z}_{\geqslant 0}$, by viewing $\mathbf{Z}_{\geqslant 0}$ as a subposet of $[\mathsf{SC}]$, by the map $n \mapsto (\{1, \dots, n\}, \{\{1\}, \dots, \{n\}\})$.

2.2 A conical refinement

The Čech map $[\check{C}]$ is not a conical stratification of $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$. This follows immediately from Lemma 1.5.7.

Example 2.2.1. Take $(P, r) \in \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ with $|P| = 2$ and $r = d(P_1, P_2)/2$. Since M is a manifold, there exists $(P, r_a) \in \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ with $r_a \in [0, r)$, whose image in $[\check{C}]$ differs from the image of (P, r). Consider the two strata $S_b = [\check{C}]^{-1}(\bullet\!\!-\!\!\bullet)$ and $S_a = [\check{C}]^{-1}(\bullet\;\bullet)$, let $r_b > r$, and observe that

$$\bullet\;\bullet \;=\; [\check{C}](P, r_a) \;\not\succcurlyeq\; [\check{C}](P, r) \;=\; [\check{C}](P, r_b) \;=\; \bullet\!\!-\!\!\bullet\,.$$

Since $(P, r) \in S_b$ and $(P, r) \in \overline{S_a} \setminus S_a$, the set $(\overline{S_a} \setminus S_a) \cap S_b$ is not empty. However, $(P, r_b) \in S_b$ and $(P, r_b) \notin \overline{S_a}$, so $S_a \not\subseteq \overline{S_b}$. Hence $[\check{C}]$ does not satisfy the frontier condition, and so cannot be a conical stratification.

A first solution that presents itself is to make a new stratum for points similar to (P, r) in the example above. That is, for every $[C] \in [SC]$, declare the stratum $S_{[C]} = \{(P, r) \in \text{Ran}(M) \times \mathbf{R}_{\geqslant 0} : [\check{C}](P, r) = [C], \check{c}r(P, r) = 0\}$. It is not immediately clear if this refinement is a conical stratification, but it is explored in more detail in Section 5.2.

We instead specialize to get a more immediate result, by restricting to semialgebraic sets and fixing an upper bound $n \in \mathbf{Z}_{>0}$. The function $[\check{C}]$ will now also refer to the restriction of $[\check{C}]$ to $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$.

Theorem 2.2.2. *If M is a semialgebraic set, there exists a conical semialgebraic stratification of $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ compatible with $[\check{C}]$.*

Proof. As M is a semialgebraic set, (23, I.2.9.1) gives that $M^n \times \mathbf{R}_{\geqslant 0}$ is semialgebraic. Since M^n is closed and bounded and $\text{Ran}_{\leqslant n}(M)$ can be described as a quotient of M^n by a semialgebraic equivalence relation, (30, Corollary 1.5) gives that $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is semialgebraic.

Now we show the strata are semialgebraic sets. Consider the set $[\check{C}]^{-1}([C]) \subseteq \text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, which is defined by functions which use the distance from a point (P, r) to its Čech set $\check{c}s(P)$. The Čech set, from Equation 2.1, is a semialgebraic set, as it is the intersection of balls, and the function that measures distance to a semialgebraic set is also semialgebraic, by (23, I.2.9.11). Finally, a subset of a semialgebraic set defined by semialgebraic functions on the first set is itself semialgebraic in $\mathbf{R}^N$, by (31, Theorem 9.1.6). Hence $[\check{C}]^{-1}([C])$ is semialgebraic, so $[\check{C}]$ is a semialgebraic stratification of $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. Apply Lemma 1.5.8 to get a conical semialgebraic stratification of $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ compatible with $[\check{C}]$. □

The construction of $\mathrm{Ran}_{\leqslant n}(M)$ as a semialgebraic set in the proof of Theorem 2.2.2 is not the common approach: the spaces $(M^k \setminus \Delta)/S_k = \mathrm{Ran}_k(M)$ are usually glued together, for all $k = 1, \ldots, n$, where $\Delta \subseteq M^k$ contains all k-tuples with at least two identical entries. Here we instead quotient out M^n once by a collection of relations with no gluing, as in Example 2.2.3.

Example 2.2.3. To apply (30, Corollary 1.5) in Theorem 2.2.2, we need to find a semialgebraic set $E \subseteq M^n \times M^n$ and interpret it as an equivalence relation, so that $M^n/E = \mathrm{Ran}_{\leqslant n}(M)$. For $M = \mathbf{R}^d$ and $n = 3$, the set E must contain, among others, the set

$$\{(x_1, \ldots, x_6) \in (\mathbf{R}^d)^3 \times (\mathbf{R}^d)^3 : x_1 - x_5 = 0, x_2 - x_4 = 0\}$$

to identify the symmetries (a, b, c) and (b, a, c) in $\mathrm{Ran}_{\leqslant 3}(\mathbf{R}^d)$. It must also contain

$$\{(x_1, \ldots, x_6) \in (\mathbf{R}^d)^3 \times (\mathbf{R}^d)^3 : x_1 - x_4 = x_1 - x_5 = 0, x_2 - x_6 = x_3 - x_6 = 0\}$$

to identify the coincidences (a, b, b) and (a, a, b).

CHAPTER 3

CONSTRUCTIBLE SHEAVES ON THE RAN SPACE

From now on assume M is semialgebraic. Let $[\check{C}C]\colon \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SCC}]$ be a conical semialgebraic stratification given by Theorem 2.2.2, for some appropriate poset refining $[\mathsf{SC}]$, whose restriction to simplicial complexes of at most n vertices is $[\mathsf{SCC}]$.

3.1 The homotopy category of entrance paths

For X, a topological space, recall $\mathrm{Sing}(X)$ is the simplicial set of continuous maps $|\Delta^k| \to X$. Let A be a poset and $f\colon X \to A$ a stratification.

Definition 3.1.1. An *exit path* in X is a continuous map $\sigma\colon |\Delta^k| \to X$ for which there exists a chain $a_0 \leqslant \cdots \leqslant a_k$ in A such that $f(\sigma(t_0, \dots, t_i, 0, \dots, 0)) = a_i$ and $t_i \neq 0$, for all i. An *entrance path*[1] is the same, but with $f(\sigma(0, \dots, 0, t_i, \dots, t_k)) = a_{k-i}$ and $t_i \neq 0$, for all i.

The categories of exit paths and entrance paths are denoted $\mathrm{Sing}^A(X)$ and $\mathrm{Sing}_A(X)$, respectively. These are sub-simplicial sets of $\mathrm{Sing}(X)$.

Remark 3.1.2. It is tempting to think $\mathrm{Sing}^A(X)^{\mathsf{op}} = \mathrm{Sing}_A(X)$, but the difference between exit and entrance paths is only in the indexing of the underlying complexes, not in the morphisms. The functor from $\mathrm{Sing}^A(X)$ to $\mathrm{Sing}_A(X)$ that precomposes every σ with $(t_0, \dots, t_k) \mapsto (t_k, \dots, t_0)$ is covariant and has inverse itself, so $\mathrm{Sing}^A(X) \cong \mathrm{Sing}_A(X)$.

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[1] The choice of "entrance" instead of "entry" comes from interpreting "exit" as a noun rather than a verb.

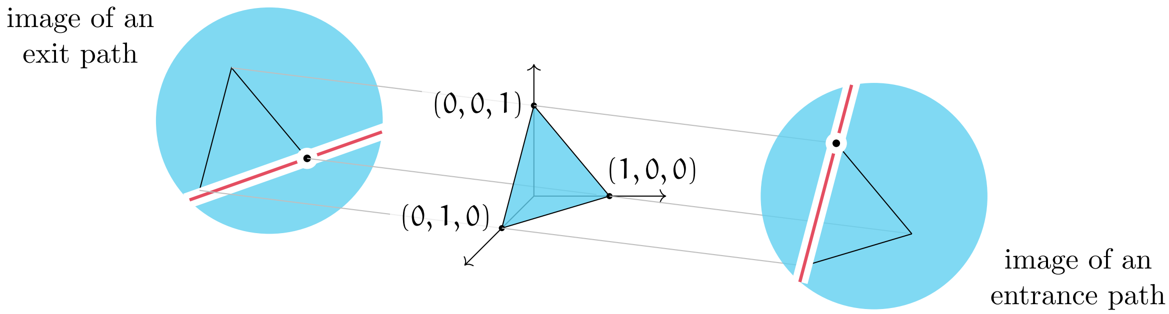


Figure 6: Two continuous maps $|\Delta^2| \to D^2$.

By (6, Theorem A.6.4), $\mathrm{Sing}_{[SCC]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$ is an ∞-category. We follow (22, Section 1.2.3) in constructing the homotopy category of an ∞-category.

Definition 3.1.3. Let $\rho, \sigma \in \mathrm{Sing}_{[SCC]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})_1$ with $\rho(0) = \sigma(0) = (P, r)$ and $\rho(1) = \sigma(1) = (Q, s)$. Then ρ and σ are *homotopic* if there exists a 2-simplex $\tau \in \mathrm{Sing}_{[SCC]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})_2$ for which $d_2\tau = \rho$, $d_1\tau = \sigma$, and $d_0\tau = s_0(Q, s)$.

Recall from Section 1.3 that, for a simplicial set S, the maps $d_* \colon S_i \to S_{i-1}$ are the degeneracy maps and $s_* \colon S_i \to S_{i+1}$ are the face maps. By (22, Proposition 1.2.3.5), homotopy of 1-simplices with common endpoints is an equivalence relation, so let $[\sigma]$ denote the equivalence class of 1-simplices in $\mathrm{Sing}_{[SCC]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$ homotopic to σ.

Definition 3.1.4. The *homotopy category* of $\mathrm{Sing}_{[SCC]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$ has

- pairs $(P, r) \in \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ as objects, and
- homotopy classes $[\sigma]$ as morphisms from $\sigma(0)$ to $\sigma(1)$.

By (22, Proposition 1.2.3.8), this description defines a category. We denote this category by $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$.

3.2 A persistence cosheaf, by construction

Lemma 3.2.1. *Every morphism in* $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ *induces a unique simplicial map.*

Proof. Take a morphism $[\sigma]\colon (P, r) \to (Q, s)$ in $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ and choose a representative $\sigma \in [\sigma]$. Since $\sigma\colon |\Delta^1| = I \to \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is an exit path, $[\check{C}C](\sigma(t))$ is constant for all $t \in [0, 1)$. Since $[\check{C}C]$ is compatible with $[\check{C}]$, we also have that $[\check{C}](\sigma(t))$ is constant for all $t \in [0, 1)$, and so by Corollary 2.1.5, we have a unique simplicial map $\check{\sigma}\colon \check{C}(\sigma(0)) \to \check{C}(\sigma(1))$.

For uniqueness, take some other $\rho \in [\sigma]$, so there exists $\tau \in \mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})_2$ with $d_2\tau = \rho$, $d_1\tau = \sigma$, and $d_0\tau = s_0(Q, s)$. Write $P = \{P_1, \ldots, P_k\}$. As the endpoints of σ and ρ are both fixed, the homotopy between the two extends to k path homotopies from $\sigma_i\colon I \to M$ to $\rho_i\colon I \to M$ on M, with $\sigma_i(0) = \rho_i(0) = P_i$ and $\sigma_i(1) = \rho_i(1)$, constructed explicitly by Construction 3.3.2. Hence the set maps $P \to Q$ induced by both σ and ρ are the same, and as simplicial maps are determined by where vertices are sent, $\check{\sigma} = \check{\rho}$. □

As mentioned in Remark 2.1.6, we ignore monodromy. Key to Lemma 3.2.1 is that a morphism in $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ is induced by a homotopy class of paths $[\sigma]$ for which every representative $\sigma\colon I \to \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ has constant image through $[\check{C}]$ on $[0, 1) \subseteq I$. No such statement can be made for homotopy classes of paths not restricted to a single stratum, as Section 4.3 shows.

Definition 3.2.2. Let $\mathsf{F}\colon \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})) \to \underline{\mathsf{SC}}$ be the functor given by $\mathsf{F}(P, r) = \check{C}(P, r)$ and $\mathsf{F}([\sigma]) = \check{\sigma}$.

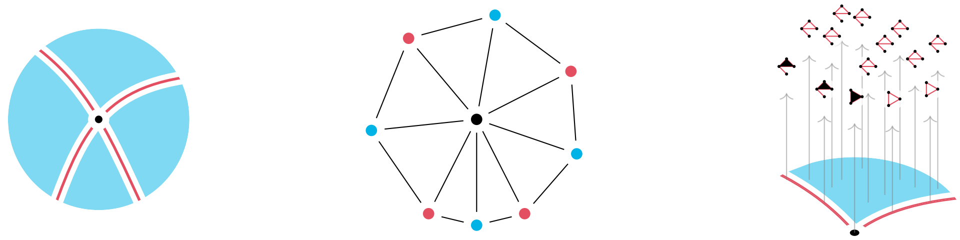
This assignment is well-defined by Lemma 3.2.1 and functorial by (22, Proposition 1.2.3.7). Restriction induces functors $\mathsf{F}_U\colon \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U)) \to \underline{\mathsf{SC}}$ for any subset $U \subseteq \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. This allows us to describe a cosheaf built from F, whose costalks will recover the Čech map $\check{C}$. Let $\mathsf{Cat}_{/\underline{\mathsf{SC}}}$ be the overcategory of functors into $\underline{\mathsf{SC}}$.

Definition 3.2.3. Let $\mathcal{F}\colon \mathsf{Op}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}$ be the functor given by $\mathcal{F}(U) = \mathsf{F}_U$ and $\mathcal{F}(V \subseteq U)$ the natural transformation induced by the inclusion $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(V)) \to \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$.

We call this the *persistence cosheaf*, though verification that it is a cosheaf is still to come.

Since $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(V))$ is a (not necessarily full) subcategory of $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$ whenever $V \subseteq U$, the definition of $\mathcal{F}$ makes sense. For every U, the image of $\mathcal{F}(U)$ is a diagram of simplicial complexes and simplicial maps in SC, as described in Figure 7. Note that in this diagram, simplicial maps within a single stratum may not be the identity, and there are not always unique simplicial maps between strata.

The functor $\mathcal{F}$ with the category $\mathsf{Op}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$ as source is not a cosheaf. Indeed, consider an open set $U = U_1 \cup U_2$ for which $\mathcal{F}(U)$ contains an endomorphism σ with $\check{\sigma}$ not homotopic to the identity simplicial map, and σ in neither $\mathcal{F}(U_1)$ nor $\mathcal{F}(U_2)$. Then the colimit will not contain the full path σ, but the value of $\mathcal{F}$ on the whole set will.



stratified open set U poset image $[\check{\mathsf{C}}\mathsf{C}](\mathsf{U})$ diagram $\mathcal{F}(\mathsf{U})$

Figure 7: A visual description of the [SCC]-stratification and the functor $\mathcal{F}$.

Example 3.2.4. For a concrete example of the just described situation, take $M = S^1$ the unit circle, and consider the open set $U = U_1 \cup U_2$, for the open sets

$U_1 = B^\circ_H\Big($ $, \pi/3\Big) \times \mathbf{R}_{\geqslant 0},$ $\qquad U_2 = B^\circ_H\Big($  $, \pi/3\Big) \times \mathbf{R}_{\geqslant 0}.$

Here $\{U_1, U_2\}$ is a cover of $U \in \mathsf{Op}(\mathrm{Ran}_{\leqslant n}(S^1) \times \mathbf{R}_{\geqslant 0})$ by open sets. It is immediate that the colimit of the nerve of the cover, that is, of the diagram

$$\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U_1)) \longleftarrow \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U_1 \cap U_2)) \longrightarrow \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U_2))$$

only has classes of morphisms $\check{C}(\{\pi/4, 3\pi/4\}, r) \to \check{C}(\{\pi/4, 3\pi/4\}, r)$ that are homotopic to the identity. However, $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$ has one more class that swaps the vertices $\pi/4$ and $3\pi/4$.

Hence the colimit of the image of the nerve of the cover $\{U_1, U_2\}$ can be not isomorphic to the image of U.

We get around this problem by restricting to basic opens. In the context of Example 3.2.4, the union U of U_1 and U_2 is not a basic open.

Theorem 3.2.5. *The functor* $\mathcal{F}\colon \mathsf{Bsc}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}$ *is a cosheaf.*

Proof. Let $V \subseteq U$ bet two basic opens of $\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ with the same associated stratum $[\check{C}C]^{-1}(C)$. For such V, U, by (32, Chapter 4.1) there is a $[\mathsf{SCC}]_{>C}$-stratified space L such that V and U are the homeomorphic images of $\mathbf{R}^k \times C(L)$. In (32) only $\mathbf{Z}_{\geqslant 0}$-stratified spaces are considered, but we get the result by applying the observation to all chains $C < C' < \cdots \subseteq [\mathsf{SCC}]$ and finding a common refinement of the stratifications. This gives a stratified homeomorphism $V \to U$, and as the sets by which V and U are stratified are the same, we get an equivalence $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(V)) \cong \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$. Hence $\mathcal{F}(V) \cong \mathcal{F}(U)$, and by (25, Section 3), this suffices to show that $\mathcal{F}$ is a cosheaf. □

In particular, this cosheaf is $[\mathsf{SCC}]$-constructible. By (25, Section 4), this defines a unique $[\mathsf{SCC}]$-constructible cosheaf on the category of open sets, but we only know existence, not the specific construction, which is of more interest to us.

We now describe some relevant properties of the cosheaf. Since limits and colimits in $\mathsf{Cat}_{/\underline{\mathsf{SC}}}$ are computed in Cat, the proofs are presented as limit and colimit arguments in Cat, which then immediately extend to $\mathsf{Cat}_{/\underline{\mathsf{SC}}}$.

Proposition 3.2.6. *The costalk* $\mathcal{F}^{(P,r)}$ *of* $\mathcal{F}$ *at* (P, r) *is the Čech complex* $\check{C}(P, r)$.

Proof. Let $(P, r) \in \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. Suppose that there is some element $Y \to \underline{\mathsf{SC}}$ of $\mathsf{Cat}_{/\underline{\mathsf{SC}}}$ that has functors $\ell_U \colon Y \to \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$ for every $U \ni (P, r)$, so that $i_{UV} \circ \ell_V = \ell_U$ whenever $V \subseteq U$ both contain (P, r), where i_{UV} is the inclusion map. Note that $\check{C}(P, r)$ is an object of every $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$, whenever $U \ni (P, r)$, so $\check{C}(P, r)$ has preimages $\ell_U^{-1}(\check{C}(P, r))$ as some object(s) of Y. Moreover, $\ell_U^{-1}(\check{C}(P, r))$ is the same as $\ell_V^{-1}(\check{C}(P, r))$, by commutativity of the diagram

$$\begin{tikzcd} & \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(V)) \arrow[d, "i_{UV}"] \\ Y \arrow[ur, "\ell_V"] \arrow[dr, "\ell_U"'] & \\ & \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U)). \end{tikzcd}$$

Notice also every object of Y must be in $\ell_U^{-1}(\check{C}(P, r))$, because every other point (Q, s) is excluded from some $V \ni (P, r)$, as the base space is Hausdorff. Let $\alpha \colon Y \to \mathbf{1}$ be the functor that sends every object to the single object of $\mathbf{1}$, and every morphism to the single morphism of $\mathbf{1}$. Given the natural maps $\alpha_U \colon \mathbf{1} \to \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U))$ that send the single object to $\check{C}(P, r)$, we have the commuting diagram

$$\begin{tikzcd} & & \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(V)) \arrow[d, "i_{UV}"] \\ Y \arrow[r, "\alpha"] \arrow[urr, bend left, "\ell_V"] \arrow[drr, bend right, "\ell_U"'] & \mathbf{1} \arrow[ur, "\alpha_V"] \arrow[dr, "\alpha_U"'] & \\ & & \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U)). \end{tikzcd}$$

Hence Y and the maps ℓ_{U} always factor through $\mathbf{1}$ and the maps α_{U}, and $\mathbf{1}$ is the limit of the diagram $\mathsf{N}(\{\mathsf{U}\}_{\mathsf{U}\ni(\mathsf{P},r)}) \to \mathsf{Cat}$. The same holds in the overcategory $\mathsf{Cat}_{/\underline{\mathsf{SC}}}$, and so $\check{\mathsf{C}}(\mathsf{P}, r)$ is naturally isomorphic to the costalk of $\mathcal{F}$ at (P, r) □

This observation about the costalk holds for $\mathcal{F}$ defined on open sets or on basic opens. We specialize to get a result that will be relevant for applications in Section 4.

Proposition 3.2.7. *For* $\mathsf{P} \in \mathrm{Ran}_{\leqslant n}(\mathsf{M})$*, let* $\mathcal{F}_{\mathsf{P}} \colon \mathsf{Bsc}(\mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}$ *be the functor given by* $\mathsf{U} \mapsto \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\{\mathsf{P}\} \times \mathsf{U}))$*. Then:*

1. $\mathcal{F}_{\mathsf{P}}$ *is a cosheaf.*

2. $\mathcal{F}_{\mathsf{P}}$ *is the inverse image precosheaf of* $\mathcal{F}$ *restricted to* $\{\mathsf{P}\} \times \mathbf{R}_{\geqslant 0}$*.*

Proof. The argument for (1) is the same as in the proof for Theorem 3.2.5. The argument for (2) is a slight generalization of the proof of Proposition 3.2.6, but is essentially the same. That is, here we need to show that $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\{\mathsf{P}\} \times \mathbf{R}_{\geqslant 0}))$ is the limit of the diagram $\{\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathsf{U}))\}_{\mathsf{U}\supseteq\{\mathsf{P}\}\times\mathbf{R}_{\geqslant 0}}$, and instead of the category $\mathbf{1}$, we have the stratified real line $\mathbf{R}_{\geqslant 0}$. □

Implications of the cosheaf $\mathcal{F}$ for persistent homology are described in Section 4.1.

3.3 A persistence sheaf, by existence

In this section we use (6, Theorem A.9.3) to get a $[\mathsf{SCC}]$-constructible sheaf on the open sets of $\mathrm{Ran}_{\leqslant n}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$, by describing a functor $\psi \colon \mathsf{S} \to \mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0})$ of simplicial sets. Part of the mentioned theorem in a simpler setting, and a motivator for our construction of S, is Example 3.3.1.

Example 3.3.1. Vector bundles over manifolds naturally form a 1-category Bun, and the map that gives the manifold base of a vector bundle is a functor $\mathsf{Bun} \to \mathsf{Mfld}$. Given a manifold M, we also have a 1-category Bun_M of vector bundles over M. This gives a sequence of functors

$$\mathsf{Bun} \to \mathsf{Mfld} \to \mathsf{Cat}_1. \tag{3.1}$$

Once M is fixed, choosing a vector bundle $E \to M$ is equivalent to choosing a functor $\Lambda^1_0 \to \mathsf{Bun}$ from the 0th 1-horn into Bun. Whenever we have a commutative diagram[1] of solid arrows

$$\begin{array}{ccccc}
 & 1 & \longmapsto & E \to M & \\
1 & \Lambda^1_0 & \longrightarrow & \mathsf{Bun} & E \to M \\
\downarrow & \downarrow & \nearrow & \downarrow & \downarrow \\
1 & \Delta^1 & \longrightarrow & \mathsf{Mfld}, & M \\
 & (0 \to 1) & \longmapsto & (f\colon N \to M) &
\end{array} \tag{3.2}$$

adding the dashed arrow retains commutativity of the diagram, by defining it to choose the morphism $(f^*E \to N) \to (E \to M)$ with the pullback bundle of E along f as source. For ease of notation we have written Bun and Mfld instead of their nerves in Equation 3.2, which are necessary to have the diagram make sense.

In our case Mfld becomes $\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$, and we endeavor to construct an analogous simplicial set to Bun. To do so, we first observe that, given $\sigma \in \mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M)\times$

[1] Compare with Equation 1.1.

$\mathbf{R}_{\geqslant 0})_\ell$ with $\sigma(1,0,\dots,0) = (P,r)$ and $|P| = k$, there exist $\sigma_1,\dots,\sigma_k \in \mathrm{Sing}(M)_\ell$, unique up to reindexing, and $\sigma_{k+1} \in \mathrm{Sing}(\mathbf{R}_{\geqslant 0})$ such that

$$\sigma(t) = \Big(\{\sigma_1(t),\dots,\sigma_k(t)\},\ \sigma_{k+1}(t)\Big)$$

for all $t \in |\Delta^\ell|$. These ℓ-simplices of $\mathrm{Sing}(M)$ and $\mathrm{Sing}(\mathbf{R}_{\geqslant 0})$ are explicitly defined by Construction 3.3.2.

Construction 3.3.2. For σ as above, fix an ordering $\{P_1,\dots,P_k\}$ of P. This induces an ordering on the vertices in $V(\check{C}(Q,s))$ for all $(Q,s) \in \mathrm{im}(\sigma)$, an ordering that is coherent up to entrance paths (of $\mathrm{im}(\sigma)$, with the inherited stratification). More specifically, for each $j = 0,\dots,\ell-1$ we have a surjective morphism $\varphi_j \in \mathrm{Hom}_\Delta([a_j],[a_{j+1}])$ in the category Δ from Definition 1.3.2, where

$$a_j := |V(\check{C}(\sigma(\underbrace{0,\dots,1,\dots,0}_{1\text{ in position } j})))|. \tag{3.3}$$

Hence there are well-defined continuous maps

$$\mu_i \colon |\Delta^{n-j}| \setminus |d_0\Delta^{n-j-1}| \to M^{a_j},\ \ j = 0,\dots,\ell-1, \tag{3.4}$$

$$\mu_\ell \colon (0,\dots,0,1) \to M^{a_\ell}, \tag{3.5}$$

where every $|\Delta^j|$ is considered as an embedded face of $|\Delta^\ell|$, that satisfy

$$\sigma(0,\dots,0,t_j \neq 0, t_{j+1},\dots,t_\ell) = \mu_j(0,\dots,0,t_j,t_{j+1},\dots,t_\ell) \tag{3.6}$$

for all $j = 0, \dots, \ell$. We can now define the σ_i promised above. Let π_b be the projection onto the bth factor of the (ordered) domain of π_b, and define maps $\sigma_i \colon |\Delta^\ell| \to M$ by

$$\sigma_i(0, \dots, 0, t_j \neq 0, t_{j+1}, \dots, t_\ell) = \begin{cases} \pi_i \circ \mu_0(t_0, \dots, t_\ell), & \text{if } j = 0, \\ \pi_{\varphi_{j-1} \circ \cdots \circ \varphi_0(i)} \circ \mu_\ell(0, \dots, 0, t_j, \dots, t_\ell), & \text{if } j \geqslant 1, \end{cases} \tag{3.7}$$

for every $i = 1, \dots, k$. These functions are continuous by definition of the μ_j and the φ_j.

An example of Construction 3.3.2 is given in Figure 8. We are now ready to construct the simplicial set S, analogous to Bun in Example 3.3.1, that will lie over our category of entrance paths.

Definition 3.3.3. Let S be the simplicial set with

$$S_\ell = \{(\sigma, \tau) : \sigma \in \mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})_\ell, \tau = \sigma_i \text{ for some } i\} \tag{3.8}$$

for every $\ell \in \mathbf{N} \cup \{0\}$, and face and degeneracy maps inherited from $\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$, $\mathrm{Sing}(M)$, and $\mathrm{Sing}(\mathbf{R}_{\geqslant 0})$.

That is, $s_i(\sigma, \tau) = (s_i\sigma, s_i\tau)$ and $d_i(\sigma, \tau) = (d_i\sigma, d_i\tau)$. This completely defines S, as well as a simplicial map $\psi \colon S \to \mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$ by projection to the first factor of S.

Remark 3.3.4. To apply (6, Theorem A.9.3), we need our space to be of locally singular shape. This holds by first observing that every open $U \subseteq \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ may be described as a union $U = \bigcup_i U_i$, where U_i sits in the homeomorphic image of Euclidean space. This in turn

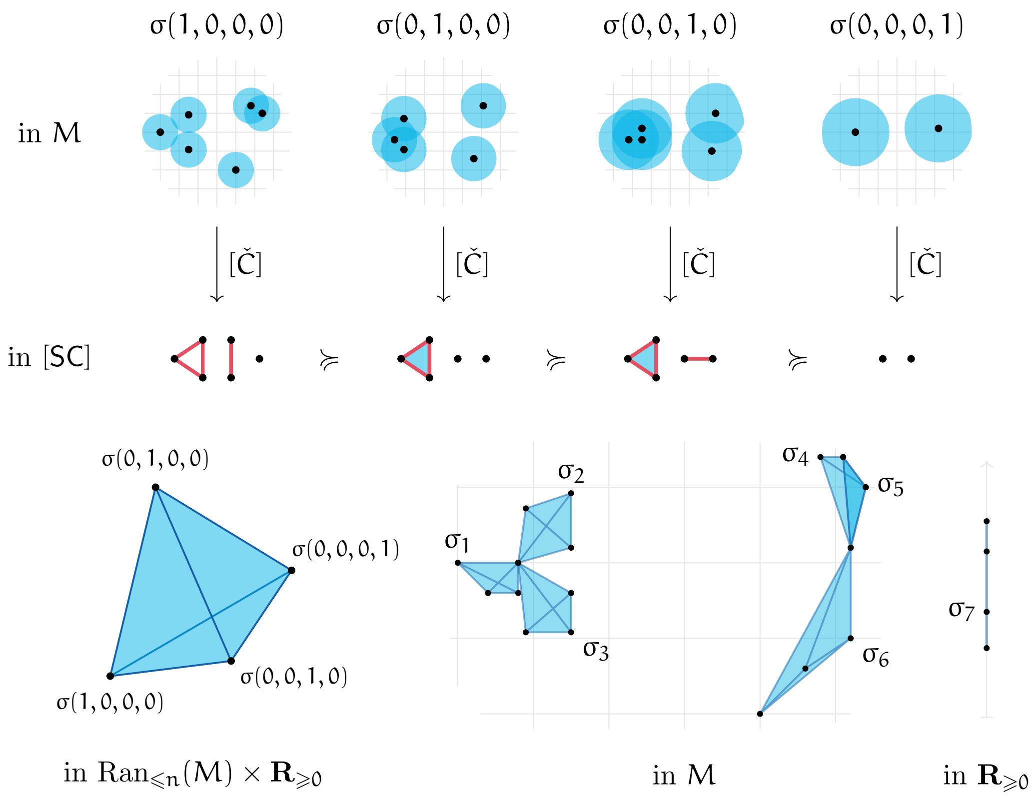


Figure 8: An example 3-simplex σ and its associated σ_i.

follows from (23, Theorem II.4.2) and by using the open star cover on the underlying simplicial complex describing $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. As Euclidean space and its submanifolds are locally of singular shape (by (6, Lemma A.4.14) and the existence of good open covers), (6, Remark A.4.16) gives that $\text{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is locally of singular shape. Our poset $[\mathsf{SCC}]$ satisfies the ascending chain condition because we have bounded the number of vertices by $n \in \mathbf{Z}_{>0}$, and refining $[\check{C}]$ gives a locally finite simplicial complex by (23, Theorem II.4.2).

Finally, recall the simplicial set and model category constructions from Section 1.3. Let ψ° be a fibrant-cofibrant replacement for ψ, in the model structure on $\mathsf{sSet}_{/\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M)\times\mathbf{R}_{\geqslant 0})}$. Viewing an object of $\mathsf{sSet}_{/\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M)\times\mathbf{R}_{\geqslant 0})}$ as a 0-simplex in its nerve, and by Remark 3.1.2 and (6, Construction A.9.2), we get a $[\mathsf{SCC}]$-constructible sheaf

$$\mathcal{G} := \Psi_{\mathrm{Ran}_{\leqslant n}(M)\times\mathbf{R}_{\geqslant 0}}\left(\psi^{\circ}\right) \in \mathsf{Shv}^{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M)\times\mathbf{R}_{\geqslant 0}). \tag{3.9}$$

We call this the *persistence sheaf.* The properties of this sheaf are harder to understand due to its construction, hence we are concerned more with the cosheaf $\mathcal{F}$.

Remark 3.3.5. The simplicial set S does not encode how the σ_i are glued together, that is, when the degeneracy maps from two different simplices $d_j\sigma_i = d_{j'}\sigma_{i'}$ coincide. One way to do this would be to instead decompose σ into the simplices of the simplicial set

$$N(S(\check{C}(\sigma(1,0,\ldots,0)))), \tag{3.10}$$

where the set of simplices $S(\check{C}(\sigma(1,0\ldots,0)))$ of $\check{C}(\sigma(1,0,\ldots,0)) \in \underline{\mathsf{SC}}$ is viewed as a poset under inclusion. In this case there are two natural degeneracy maps, the natural one on σ and the natural one on the nerve from Equation 3.10, the relation among which encodes the gluing of the components of σ.

CHAPTER 4

APPLICATIONS FOR TOPOLOGICAL DATA ANALYSIS

In this section we apply the constructions and results from Chapters 2 and 3 to TDA. We are interested in relating functors $\mathbf{R} \to \underline{\mathsf{SC}}$ to each other by their geometric and topological information, to be able to relate the associated persistence module functors $\mathbf{R} \to \mathsf{Vect}$.

4.1 Universality of the cosheaf $\mathcal{F}$

Here we interpret the results of Section 3.2 in a persistent homology setting.

Proposition 4.1.1. *For every* $\mathsf{P} \in \mathrm{Ran}_{\leqslant n}(\mathsf{M})$*, the image of* $\mathcal{F}|_{\{\mathsf{P}\}\times\mathbf{R}_{\geqslant 0}}$ *is isomorphic to a diagram* $\mathsf{D}_1 \to \cdots \to \mathsf{D}_k$ *in* $\underline{\mathsf{SC}}$*, such that*

- $[\mathsf{D}_i] \succcurlyeq [\mathsf{D}_{i+1}]$ *and* $[\mathsf{D}_i] \neq [\mathsf{D}_{i+1}]$ *for all* $i = 1, \dots, k-1$,
- D_1 *is* $|\mathsf{P}|$ *disconnected 0-simplices, and*
- D_k *is a complete* $|\mathsf{P}|$*-simplex.*

Proof. Since $[\check{\mathsf{C}}\mathsf{C}]$ is a conical $[\mathsf{SCC}]$-stratification of $\mathrm{Ran}_{\leqslant n}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$, every semialgebraic subset of $\mathrm{Ran}_{\leqslant n}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$ inherits a conical stratification by restriction. Since $\{\mathsf{P}\} \times \mathbf{R}_{\geqslant 0}$ is 1-dimensional, the only possible conical stratification of it is

$$\bullet(\!\text{———}\!)\bullet(\!\text{———}\!)\bullet(\!\text{———}\!)\bullet(\!\text{—}\cdots \quad , \tag{4.1}$$

which means the image of $\mathcal{F}|_{\{P\}\times\mathbf{R}_{\geqslant 0}}$ in the poset $[\mathsf{SCC}]$ looks like

$$\color{red}{\bullet} \longleftarrow \color{cyan}{\bullet} \longrightarrow \color{red}{\bullet} \longleftarrow \color{cyan}{\bullet} \longrightarrow \color{red}{\bullet} \longleftarrow \color{cyan}{\bullet} \longrightarrow \color{red}{\bullet} \longleftarrow \cdots \quad . \tag{4.2}$$

By assumption, this stratification is compatible with the restriction of the $[\mathsf{SC}]$-stratification $[\check{C}]$ to $\{P\} \times \mathbf{R}_{\geqslant 0}$, which stratifies $\mathbf{R}_{\geqslant 0}$ as

$$[\!\longrightarrow\!) \, [\!\longrightarrow\!) \, [\!\longrightarrow \cdots \quad , \tag{4.3}$$

where every half-open interval is where the Čech map $[\check{C}]$ is constant. Note that every entrance path in $\mathrm{Sing}_{[\mathsf{SCC}]}(\{P\} \times \mathbf{R}_{\geqslant 0})$ that starts and ends in different strata is either stratified by $\color{red}{\bullet} \longleftarrow \color{cyan}{\bullet}$ or $\color{cyan}{\bullet} \longrightarrow \color{red}{\bullet}$. Given Equation 4.3, the morphism in the former case must be homotopic to the identity morphism, and in the latter case the morphism is either homotopic to the identity or to an inclusion morphism. Hence by collapsing all the identity morphisms, we get the first result. Since M is Hausdorff, we get the second result at $0 \in \mathbf{R}_{\geqslant 0}$. Since P is finite and M is semialgebraic, we get the third result at $r > \mathrm{diam}(P) \in \mathbf{R}_{\geqslant 0}$. □

To complete the connection with persistent homology, we give a morphism for every pair $t \leqslant s$ in $\mathbf{R}_{\geqslant 0}$. Since every basic open must have a unique lowest stratum, the subset $\{P\} \times (t, s)$ is not necessarily an open basic. We can, however, always cover $\{P\} \times [t, s]$ by a collection $\{U_i\}$ of open basics. For every $[t, s] \subseteq \mathbf{R}_{\geqslant 0}$, let $\mathcal{F}_{t\to s} \in \mathrm{Hom}_{\underline{\mathsf{SC}}}(\check{C}(P, t), \check{C}(P, s))$ be the simplicial map given by collapsing, as in the proof of Proposition 4.1.1, the subdiagram of $\mathcal{F}(U_i)$ that starts at $\check{C}(P, t)$ and ends at $\check{C}(P, s)$. Recalling the notion of a simplicial module SM. from Section 1.7, we can summarize this discussion as follows.

Corollary 4.1.2. *For every* $P \in \mathrm{Ran}_{\leqslant n}(M)$*, the image of* $\mathcal{F}|_{\{P\}\times\mathbf{R}_{\geqslant 0}}$ *is isomorphic to* SM_P*.*

That is, the cosheaf $\mathcal{F}$ completely describes the filtered diagram of simplices induced by the Čech construction on a finite subset $P \subseteq M$. As expected, taking homology gives us

$$\mathrm{PM}_{\check{C}(P,-)}(t) = H_d\left(\mathcal{F}^{(P,t)}; \mathbf{k}\right), \qquad \mathrm{PM}_{\check{C}(P,-)}(t \leqslant s) = H_d\left(\mathcal{F}_{t\to s}; \mathbf{k}\right). \tag{4.4}$$

Moreover, as homology preserves colimits of filtered diagrams,

$$H_d \circ \mathcal{F} \colon \mathsf{Bsc}(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\mathsf{Vect}} \tag{4.5}$$

is a cosheaf valued in functors (that is, diagrams) of homology groups. By Proposition 3.2.7 the same holds for the restriction cosheaves $\mathcal{F}|_{\{P\}\times\mathbf{R}_{\geqslant 0}}$.

We continue the discussion by applying the stratified space structure to the problem described in (20, Section 5.2).

4.2 Homology classes along a path

We consider an isomorphism of simplicial modules to be a strictly monotonic automorphism of $\mathbf{R}_{\geqslant 0}$. That is, if there is a strictly monotonic map $\varphi : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$, for which, when interpreted as a functor, we have $\mathrm{SM}_P = \mathrm{SM}_Q \circ \varphi$. In this section we are interested in paths $\gamma \colon I \to \mathrm{Ran}_{\leqslant n}(M)$ for which

1. the subspace $\mathrm{im}(\gamma) \times \mathbf{R}_{\geqslant 0} \subseteq \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is conically stratified,

2. all 1-dimensional strata of $\gamma(I) \times \mathbf{R}_{\geqslant 0}$ are either functions of $t \in I$ or subsets of $\mathbf{R}_{\geqslant 0}$, and

3. the set $T := \{t \in I : U_{(\gamma(t),r)} \cong C(L)$ for some $r \in \mathbf{R}_{\geqslant 0}\}$ is finite and ordered in t, say $T = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$.

The times $t \in T$ are where the lowest, 0-dimensional, strata occur, which act as endpoints for entrance paths. Both $0, 1 \in T$ because the "corners" at $(\gamma(0), 0)$ and $(\gamma(1), 0)$ must be in their own strata for the space to be conically stratified. We write $U_{(\gamma(t),r)}$ for the neighborhood of $(\gamma(t), r)$ in $\mathrm{im}(\gamma) \times \mathbf{R}_{\geqslant 0}$ that is the image of a stratified open embedding $Z \times C(L) \to \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, guaranteed by Definition 1.5.6. Without loss of generality, assume all strata are connected.

Remark 4.2.1. The above assumptions imply that the 1-dimensional strata of $\mathrm{im}(\gamma|_{[t_{i-1},t_i]}) \times \mathbf{R}_{\geqslant 0}$, stratified by restriction, have a linear order for every $i = 1, \ldots, m$.

An example of such a path γ and the conical stratification of $\mathrm{im}(\gamma) \times [0, N \gg 0]$ is given in Figure 9. There we have used the restriction of the conical [SCF]-stratification of $\mathrm{im}(\gamma) \times [0, N \gg 0]$ from Section 5.1, refined with strata on the boundary of the rectangle.

Lemma 4.2.2. *Let γ be as above.*

1. *For every $t, t' \in (t_i, t_{i+1})$, the simplicial modules $\mathrm{SM}_{\gamma(t)}$, $\mathrm{SM}_{\gamma(t')}$ are isomorphic.*

2. *For every $t \in (t_i - \epsilon, t_i + \epsilon)$, there is a natural morphism $\mathrm{SM}_{\gamma(t)} \to \mathrm{SM}_{\gamma(t_i)}$.*

The ϵ depends on i and is chosen so that $[t_i - \epsilon, t_i + \epsilon] \cap T = \{t_i\}$, and the morphism is natural up to isomorphism of simplicial modules. Lemma 4.2.2 is visually justified by Figure 9.

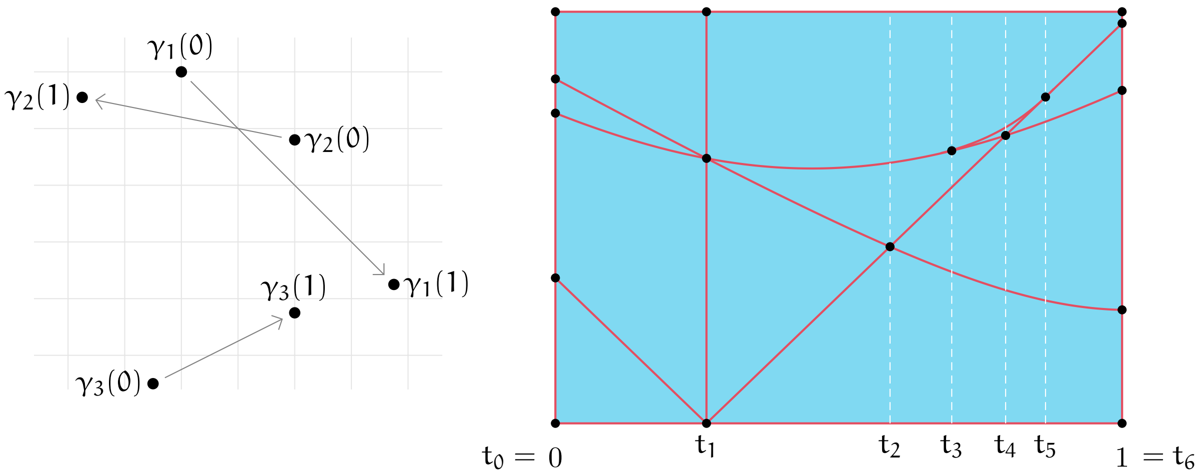


Figure 9: A path in Ran(M) and the induced conically stratified space.

Proof. This follows from Remark 4.2.1. We first consider the stratified subspace $\mathrm{im}(\gamma) \times [0, N]$ for $N \gg \mathrm{diam}(\gamma(t))$ for all $t \in I$, past which there are no changes in strata.

Take $t \in (t_i, t_{i+1})$ and t' in $(t_i, t_{i+1}]$, for which we will describe a morphism $\mathrm{SM}_{\gamma(t)} \to \mathrm{SM}_{\gamma(t')}$. Fix a linear ordering of the restrictions to $[t, t']$ of the 1-dimensional strata that are functions in I. That is, write $e_j \colon [t, t'] \to [0, N]$, for $j = 1, \ldots, w$, so that $e_1 \leqslant e_2 \leqslant \cdots \leqslant e_w$ pointwise on $[t, t']$. Next, note that every $r \in (\gamma(t), -) \cong [0, N]$ can be described as

$e_j(0)a+e_{j+1}(0)(1-a)$ for some $j \in 1,\dots,w-1$ and $a \in [0,1]$. The morphism $\mathrm{SM}_{\gamma(t)} \to \mathrm{SM}_{\gamma(t')}$ is then given by

$$a \cdot e_j(0) + (1-a) \cdot e_{j+1}(0) \quad \mapsto \quad a \cdot e_j(1) + (1-a) \cdot e_{j+1}(1). \tag{4.6}$$

This is well-defined at $a = 0$ and $a = 1$, because $e_j(0)$ always maps to $e_j(1)$. It is never the case that $e_j(0) = e_{j+1}(0)$, as that would imply a 0-dimensional stratum at $\gamma(t)$, and $t \notin T$. In the second case, we may have that $e_j(1) = e_{j+1}(1)$, in which case the interval $[e_j(0), e_{j+1}(0)] \subseteq [0, N]$ gets mapped to the single point $e_j(1) = e_{j+1}(1)$. In the first case this never happens, so the map in Equation 4.6 is simply a monotonic automorphism of $[0, N]$, and so of $\mathbf{R}_{\geqslant 0}$, meaning $\mathrm{SM}_{\gamma(t)} \cong \mathrm{SM}_{\gamma(t')}$.

The proof extends to the unbounded stratified space $\mathrm{im}(\gamma) \times \mathbf{R}_{\geqslant 0}$ by adjusting Equation 4.6 so that every $r \in [e_w(0), \infty)$ gets mapped to $e_w(1) + (r - e_w(0))$. □

Remark 4.2.3. The interleaving distance (from Definition 1.7.3) between the persistence modules $\mathrm{PM}_{\check{C}(\gamma(t),-)}$ and $\mathsf{PM}_{\check{C}(\gamma(t),-)}$ is the largest vertical change among all 1-dimensional strata on $(t, t') \times \mathbf{R}_{\geqslant 0}$. The described isomorphism in the proof of Lemma 4.2.2 witnesses this distance.

To construct morphisms among persistence modules on such a path γ, pick elements $t^i \in [0, 1]$ for $i = 1, \dots, m$ such that $0 = t_0 < t^1 < t_1 < t^2 < \cdots < t^m < t_m = 1$. Let $\nu_i^+ \colon \mathrm{SM}_{\gamma(t^i)} \to$

$\mathrm{SM}_{\gamma(t_i)}$, $\nu_i^- : \mathrm{SM}_{\gamma(t^i)} \to \mathrm{SM}_{\gamma(t_{i-1})}$ be the morphisms given by Lemma 4.2.2, which exist as $t^i \in (t_{i-1}, t_i)$ for all i. Then there is a zigzag of simplicial modules

$$\mathrm{SM}_{\gamma(t_0)} \xleftarrow{\nu_1^-} \mathrm{SM}_{\gamma(t^1)} \xrightarrow{\nu_1^+} \mathrm{SM}_{\gamma(t_1)} \xleftarrow{\nu_2^-} \cdots \xrightarrow{\nu_{m-1}^+} \mathrm{SM}_{\gamma(t_m)} \xleftarrow{\nu_m^-} \mathrm{SM}_{\gamma(t^m)} \xrightarrow{\nu_m^+} \mathrm{SM}_{\gamma(t_m)}. \tag{4.7}$$

We finish this section with an example of two different simplicial module zigzags whose barcodes are the same everywhere. The difference in simplicial modules is also picked up by the *right filtration* (33) of zigzags.

Example 4.2.4. Let $\gamma_0, \gamma_1 : I \to \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ be two paths whose image in $[\check{\mathrm{C}}\mathrm{C}]$ and right filtration after taking homology is given by Figure 11. The stratified space and barcodes induced by the γ_i are described by Figure 10.

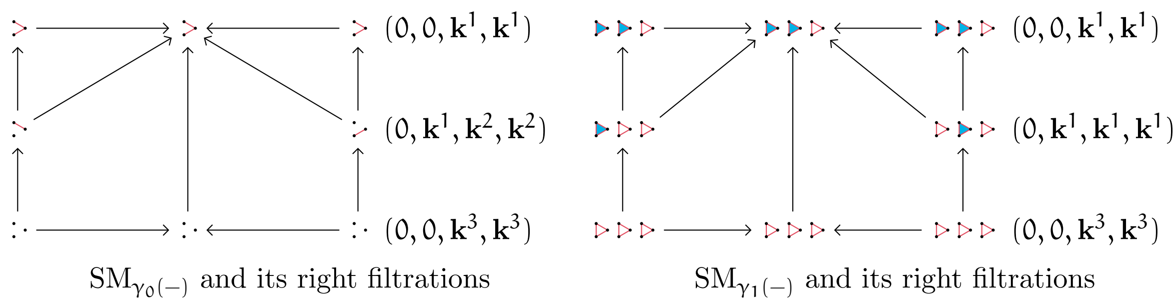
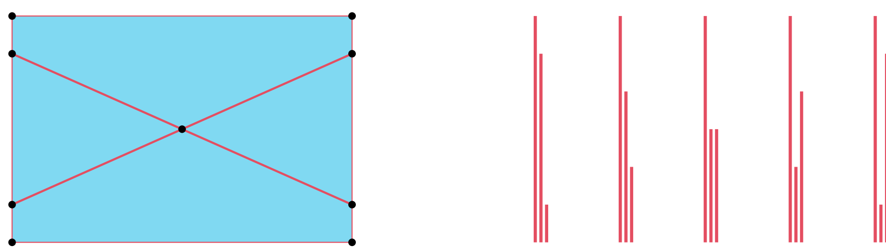


Figure 10: Paths γ_i induce the same data, for $i = 0, 1$.

We choose the paths γ_i them so that the "deaths" of homology classes in the barcodes are different. That is, the simplicial modules for γ_0 will have homology death by merging, and the simplicial modules for γ_1 will have pure homology death. In the figures we see the horizontal simplicial module reflect the different types of homology class death. The difference comes from the difference in preimages of the vector space morphisms $(x, y) \mapsto x + y$ (in the case of γ_0) and $(x, y) \mapsto x$ (in the case of γ_1).



conically stratified space induced by γ_i $\quad$ i-dimensional barcodes for γ_i

Figure 11: Paths γ_i induce different data, for $i = 0, 1$.

This shows that information captured by the simplicial module is similar to information captured by the right filtration of zigzags, and is strictly richer than than the information captured by persistence diagrams.

4.3 Contracting paths in the Ran space

In this section we revisit monodromy by providing explicit constructions for contracting paths in the Ran space. We begin with a direct consequence of Lemma 3.2.1, applying the definitions in Construction 3.3.2.

Corollary 4.3.1. *Let* $\sigma \in \mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})_1$ *be a loop. If there is some associated* $\sigma_i \in \mathrm{Sing}(M)_1$ *that is not a loop, then* $[\sigma] \neq [\mathrm{id}]$.

The converse of the statement does not hold, as a loop $\sigma_i \in \mathrm{Sing}(M)_1$ is not always contractible and can capture the homology of M. The rest of this section explores continuous maps $|\Delta^1| \to \mathrm{Ran}(M)$ that are not necessarily entrance paths. That is, we consider path homotopies not restricted to a single stratum, and use the simpler setting of $\mathrm{Ran}(M)$ to provide motivation for the product space $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$. $\mathrm{Ran}(M)$ has the point-counting $\mathbf{Z}_{\geqslant 0}$-stratification, which is refined by the $[\mathsf{SCC}]$-stratification $[\check{C}C]$ of $\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$.

First we introduce some new terminology.

Definition 4.3.2. Let $\epsilon > 0$. A path $\gamma \colon I \to \mathrm{Ran}(M)$ is ϵ-*conical at* $t \in I$ if $\gamma|_{(t-\epsilon, t+\epsilon)}$ is the image of a stratified embedding $C(*_1 \sqcup *_2 \sqcup \cdots \sqcup *_k) \hookrightarrow \mathrm{Ran}(M)$ with $\gamma(t)$ the image of the cone point.

A visual description of this property is given in Figure 12, where at $t = 1$ γ is not ϵ-conical for any $\epsilon > 0$ as it ends in a Hawaiian earring of circles.

Let $\gamma \colon I \to \mathrm{Ran}(M)$ be a loop with $\gamma(0) = \gamma(1) = \{P_1, \ldots, P_n\}$. We are interested in loops γ for which there exist a contractible $U \subseteq M$ and $\epsilon \in (0, 1)$ such that

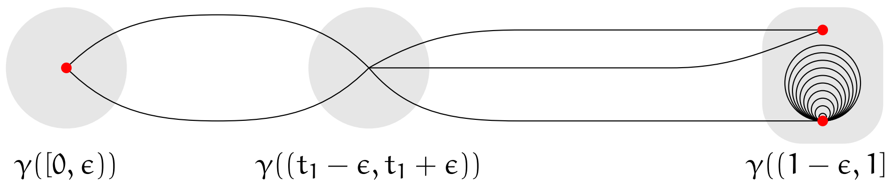


Figure 12: A path $\gamma\colon I \to \mathrm{Ran}(M)$ that is ϵ-conical at 0 and t_1.

1. $\mathrm{im}(\gamma) \subseteq M$ is contained within U,

2. $\gamma|_{[0,\epsilon)}$ is the disjoint union of $\gamma_i^0\colon [0,\epsilon) \to \mathrm{Ran}(M)$, each of which is ϵ-conical at 0,

3. $\gamma|_{(1-\epsilon,1]}$ is the disjoint union of $\gamma_i^1\colon (1-\epsilon,1] \to \mathrm{Ran}(M)$, each of which is ϵ-conical at 1,

for $i = 1, \ldots, n$. Given such a loop, consider a map $H\colon I \times I \to \mathrm{Ran}(M)$ defined as follows. For $s \in [0, 1/2]$, the map H contracts the middle interval $[\epsilon, 1-\epsilon] \subseteq I$ to a point while collapsing γ to a point at $H(1/2, 1/2)$. Then at $s = 1/2$, the path γ has become disjoint loops each based at some P_i, so for $s \in (1/2, 1]$, the map H contracts them to their respective base. A visual description of the interval I contracting under H is given in Figure 13.

For the collapsing of points at $s = 1/2$ and $t = 1/2$, simply scale the elements of $\gamma(t)$ relative to a chosen point in U by an appropriate scalar in s. It follows that $H(0,t) = \gamma(t)$ and $H(1,t)$ is the constant map at $\gamma(0)$. To complement the discussion, we provide an explicit construction of H.

Construction 4.3.3. Assume without loss of generality, for ease of scaling, that $\gamma_i^0(0) = \gamma_i^1(1) = P_i$ for all i, that $U \subseteq \mathbf{R}^N$, and that $0 \in U \setminus \gamma(0)$. Let $\mathbf{H}(2s, t, \epsilon)$ be the map

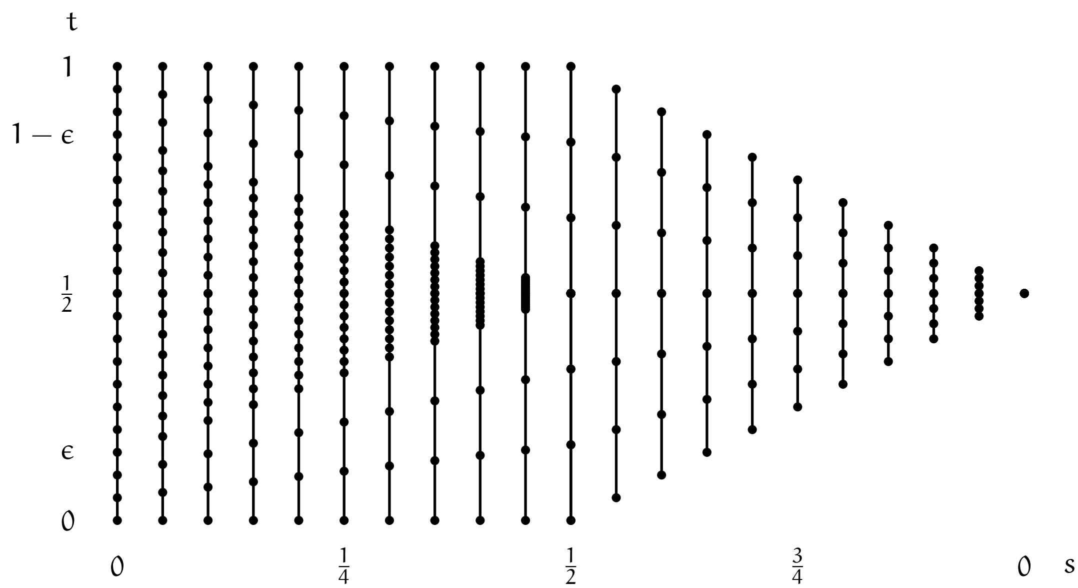


Figure 13: The image of I that H gives γ as input.

$I \times I \times (0, \infty) \to I$ given by Figure 13 for $s \in [0, 1/2]$. Explicitly, one example of such a continuous function is

$$\mathbf{H}(s,t,\epsilon) := \begin{cases} \frac{\overline{\mathbf{H}}(s,2t,2\epsilon)}{2} & t \leqslant 1/2, \\ 1 - \frac{\overline{\mathbf{H}}(s,2-2t,2\epsilon)}{2} & t > 1/2, \end{cases} \tag{4.8}$$

where $\overline{\mathbf{H}}$ is an auxiliary function that continuously reparametrizes a path of unit speed at $\overline{\mathbf{H}}(0,t,\epsilon)$ to the same path $\overline{\mathbf{H}}(1,t,\epsilon)\colon [0,1) \to [0,\epsilon]$, but with speed ϵ. A formula for it is

$$\overline{\mathbf{H}}(s,t,\epsilon) := \begin{cases} ((1-s) + s \cdot \epsilon) \cdot t & t \leqslant \frac{1}{2-s-\epsilon-s\cdot\epsilon}, \\ \frac{s-t}{s-1} & t > \frac{1}{2-s-\epsilon-s\cdot\epsilon}, \end{cases} \tag{4.9}$$

Graphs of both $\mathbf{H}$ and $\overline{\mathbf{H}}$ are given in Figure 14.

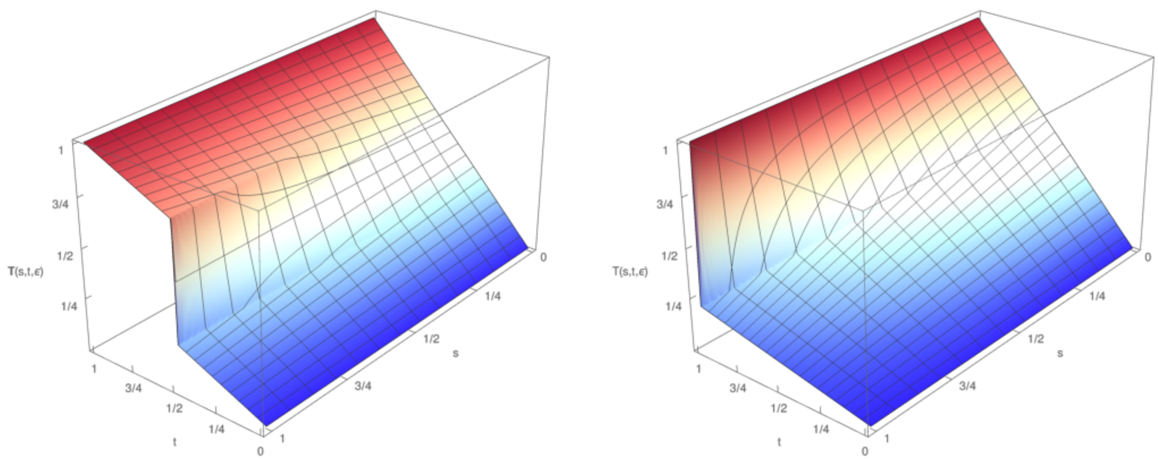


Figure 14: The graphs of $\mathbf{H}(s, t, \epsilon)$ and $\overline{\mathbf{H}}(s, t, \epsilon)$.

The path γ with $\mathbf{H}(2s, t, \epsilon)$ as input will not be continuous at $s = 1/2$. We make it continuous by scaling points on the manifold M through $(\epsilon, 1-\epsilon)$ to their "center", which we assumed to be 0. Hence define

$$H(s,t) := \begin{cases} ((1-2s) + 2s(1-2t))\gamma(\mathbf{H}(2s,t,\epsilon)) & s < 1/2, t < 1/2 \\ ((1-2s) + 2s(2t-1))\gamma(\mathbf{H}(2s,t,\epsilon)) & s < 1/2, t \geqslant 1/2 \\ \bigcup_{i=1}^{n}(2-2s)(1-2t)\gamma_i^0(\mathbf{H}(1,t,\epsilon)) + (2s-1)P_i & s \geqslant 1/2,\ t < 1/2, \\ \bigcup_{i=1}^{n}(2-2s)(2t-1)\gamma_i^1(\mathbf{H}(1,t,\epsilon)) + (2s-1)P_i & s \geqslant 1/2,\ t \geqslant 1/2. \end{cases} \tag{4.10}$$

We check that H is continuous everywhere. Indeed, H is continuous in t as

$$\lim_{t \to 1/2^-} H(s < 1/2, t) = \lim_{t \to 1/2^+} H(s < 1/2, t),$$

$$\lim_{t \to 1/2^-} H(s \geqslant 1/2, t) = \lim_{t \to 1/2^+} H(s \geqslant 1/2, t)$$

by construction. As $\gamma(t)$ is the union of the $\gamma_i^0(t)$ on $t \in [0, \epsilon)$, and the union of $\gamma_i^1(t)$ on $t \in (1-\epsilon, 1]$, we have

$$\lim_{s \to 1/2^-} H(s, t < 1/2) = \lim_{s \to 1/2^+} H(s, t < 1/2),$$

$$\lim_{s \to 1/2^-} H(s, t \geqslant 1/2) = \lim_{s \to 1/2^+} H(s, t \geqslant 1/2),$$

so H is continuous in s as well. Here we used that the γ_i^0 are ϵ-conical at 0 and that the γ_i^1 are ϵ-conical at 1. Hence H is a homotopy equivalence with the desired properties.

We end this section with an example in Figure 15 of the map H, employing the specific maps from Construction 4.3.3. The top progression is $\mathsf{H}(0,t)$ from $t = 0$ to $t = 1$, and the bottom progression is $\mathsf{H}(s,1)$ from $s = 0$ to $s = 1$. This is a loop in $\mathrm{Ran}_{\leqslant 8}(\mathbf{R}^2)$ with $\frac{1}{5}$-conical component paths at $t = 0$ and $t = 1$.

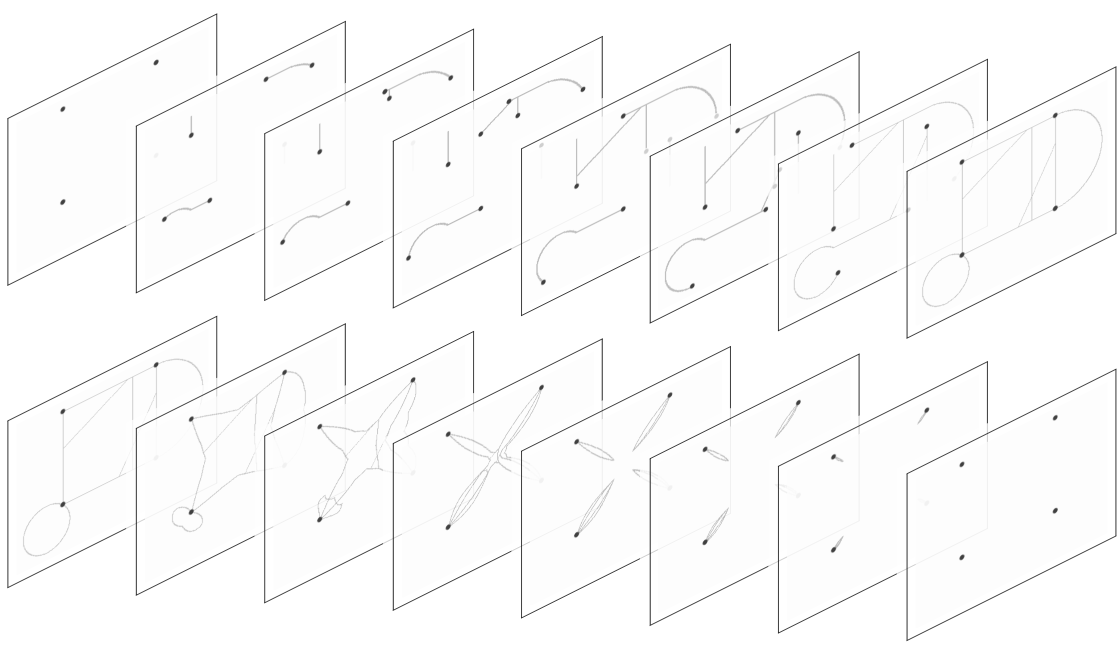


Figure 15: Contracting a loop in $\mathrm{Ran}(M)$.

CHAPTER 5

FURTHER DIRECTIONS

In this final section we consider several extensions of the construction and results shown so far, and investigate (and discount) links to marginally connected topics.

5.1 Building a conical stratification

We first ask if the stratification refinement of $[\check{C}]$ suggested in Section 2.2 after Example 2.2.1 is a conical stratification. Let us be more explicit.

Definition 5.1.1. A *frontier simplicial complex* C is a triple of sets $(\mathsf{V}(\mathsf{C}), \mathsf{S}(\mathsf{C}), \mathsf{F}(\mathsf{C}))$, with $(\mathsf{V}(\mathsf{C}), \mathsf{S}(\mathsf{C}))$ a simplicial complex and $\mathsf{F}(\mathsf{C}) \subseteq \mathsf{S}(\mathsf{C})$ closed under taking cofaces. Elements of $\mathsf{F}(\mathsf{C})$ are called *frontier simplices.*

By "closed under taking cofaces" we mean that $\sigma \in \mathsf{F}(\mathsf{C})$ implies $\tau \in \mathsf{F}(\mathsf{C})$ whenever σ is a subsimplex of $\tau \in \mathsf{S}(\mathsf{C})$. The pair of vertices and simplices of a frontier simplicial complex is called the *underlying* simplicial complex. A morphism $\phi \colon \mathsf{C} \to \mathsf{D}$ of frontier simplicial complexes (also called "simplicial map") is a triple $\phi_\mathsf{V}, \phi_\mathsf{V}, \phi_\mathsf{F}$ of compatible set maps, in the sense that the first two describe a simplicial map $(\mathsf{V}(\mathsf{C}), \mathsf{S}(\mathsf{C})) \to (\mathsf{V}(\mathsf{D}), \mathsf{S}(\mathsf{D}))$ of simplicial complexes, and the third is defined by $\phi_\mathsf{F} = \phi_\mathsf{S}|_\mathsf{F}$.

Let SCF be the set of frontier simplicial complexes, and $[\mathsf{SCF}] := \mathsf{SCF}/_{\cong}$, where an isomorphism of frontier simplicial complexes is an isomorphism on the underlying simplicial complexes

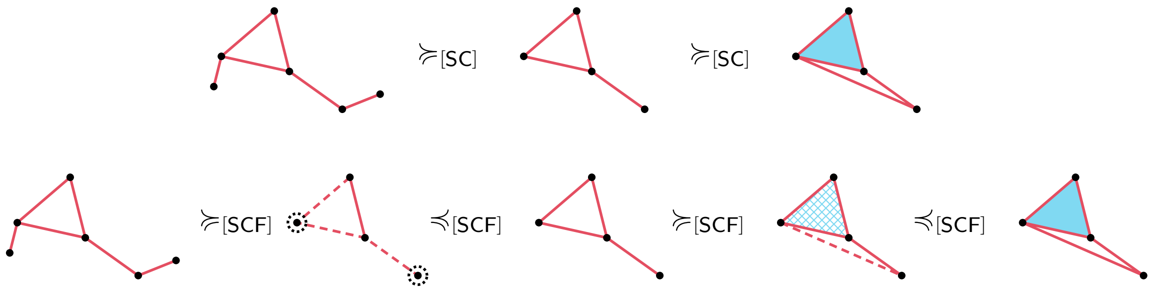


Figure 16: The partial orders on [SC] and [SCF].

that induces an isomorphism on the frontier simplices. The constructions and results for [SC] in Section 2.1 all extend naturally to [SCF]:

- The set of equivalence classes [SCF] has a partial order by letting $[C] \succcurlyeq [C']$ whenever there is a simplicial map $C \to C'$ that is surjective on vertices and injective on frontier simplices. An example of this is given in Figure 16, where frontier simplices are drawn intermittent.

- There is a natural frontier Čech map $\check{C}F\colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to \mathsf{SCF}$ given by

  1. $V(\check{C}F(P, r)) = P$,

  2. $P' \in S(\check{C}F(P, r))$ whenever $\bigcap_{p \in P'} B(p, r) \neq \emptyset$, for every $P' \subseteq P$, and

  3. $P' \in F(\check{C}F(P, r))$ whenever $P' \in S(\check{C}F(P, r))$ and $\check{c}r(P', r) = 0$.

- The frontier Čech map $[\check{C}F]\colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SCF}]$ is continuous.

The Čech radius was defined by Equation 2.2. The key difference between these constructions and the original ones in Section 2.1 is that the $[\mathsf{SCF}]$-stratification of $\mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$ satisfies the necessary frontier condition considered in Lemma 1.5.7, so there is no immediate implication that $[\check{\mathsf{C}}\mathsf{F}]$ is not conical.

Conjecture 5.1.2. *The* $[\mathsf{SCF}]$*-stratification of* $\mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$ *is conical.*

Note the dimension of $\mathrm{Ran}(\mathsf{M})$ is unbounded, and so the dimension of the link L of every point in $\mathrm{Ran}(\mathsf{M}) \times \mathbf{R}_{\geqslant 0}$ should also have unbounded dimension, which may cause some problems.

5.2 Extensions to other fields

First we consider *factorization homology*, the literature (14), (34), (9) on which contains many of the terms discussed here. Central to that approach is a factorization structure on a cosheaf $\mathcal{F}$ over the Ran space for which there is an isomorphism

$$\mathcal{F}_{\mathsf{P}} \cong \bigotimes_{\mathsf{p} \in \mathsf{P}} \mathcal{F}_{\{\mathsf{p}\}} \tag{5.1}$$

on the stalks of the cosheaf. Our setting uses costalks, but more importantly we have not mentioned any symmetric monoidal structure on the category in which the cosheaf is valued, necessary for the above isomorphism. Even if we were to choose some tensor product, it is immediate that, fixing $\mathsf{t} \in \mathbf{R}_{\geqslant 0}$, the value of the cosheaf $\mathcal{F}$ from Definition 3.2.3 on $\mathsf{P} \in \mathrm{Ran}(\mathsf{M})$ is not determined by the value of the cosheaf on each $\mathsf{p} \in \mathsf{P}$.

We end this section with some future directions.

- A natural way to generalize sheaves is to *stacks*, already considered in the case of 2-simplices of exit paths in (13). The approach there has been only with **Z**-stratified spaces, however, so the application of arbitrary posets as developed here may prove fruitful.

- Lemma 4.2.2 has implications for computational complexity. That is, it implies that the barcode along a path can be computed by the barcode at chosen times. Even more, the barcodes at those times are related by precisely defined changes localized to some small region.

- Stratified spaces and sheaves are central to *perverse sheaves*, which are complexes of sheaves. Though we only had one sheaf and one cosheaf here, the theory of perverse sheaves could be applied to, for example, the sequence of homology cosheaves from Equation 4.5.

- Reintroducing order on the finite subsets of M would bring us closer to *FI-modules*, developed in (35). An FI-module is a functor out of the category Δ into the category of commutative rings.

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