A gentle introduction to knots and knot invariants

Joint PM C&O colloquium

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Slides available online at

www.math.uwaterloo.ca/~jlazovsk/knottalk

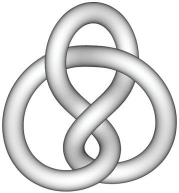
Basics - definitions

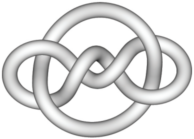
- *knot*: An embedding of S^1 into $\mathbf{R}^3$ without self-intersection

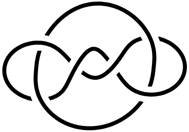
- *link*: An embedding of $n \in \mathbb{N}^*$ copies of S^1 into $\mathbf{R}^3$ without self-intersection

- *directed link*: A link with a choice of direction along each component

- *link diagram*: A projection of a link onto $\mathbf{R}^2$ with an indication of which part is over / under at every intersection (with only transversal intersections). Note $\mathcal{K} \subsetneq \mathcal{L}$.

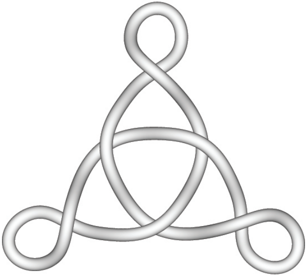






Basics - representing a knot

- Q: How can we identify which knots are the same?



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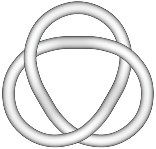
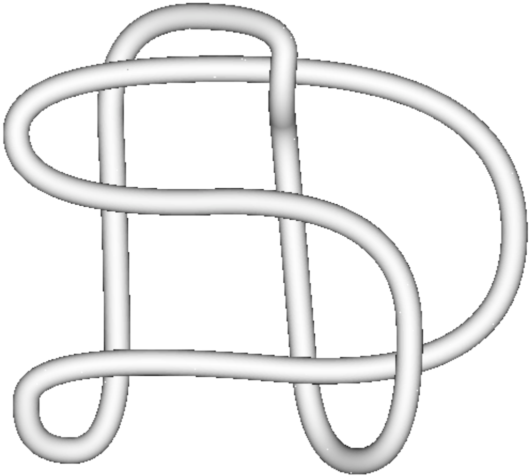
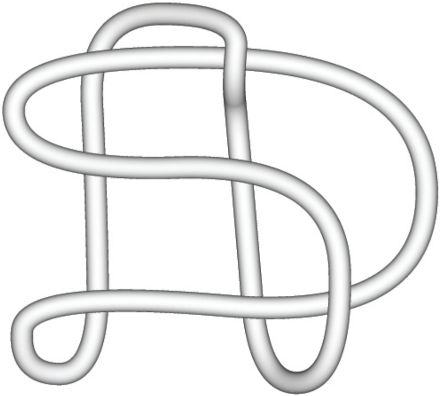


- A: Use the equivalence relation of ambient isotopy.

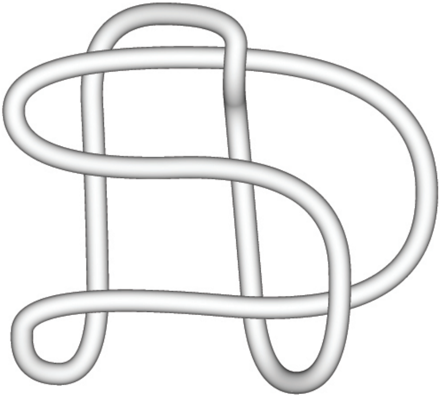
Basics - representing a knot



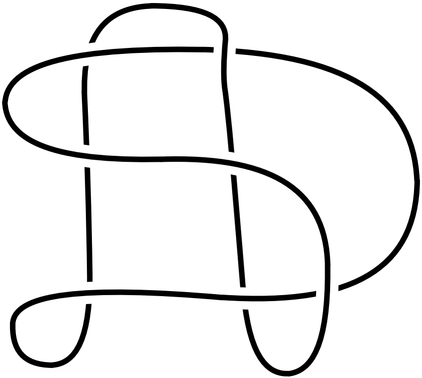
Basics - representing a knot

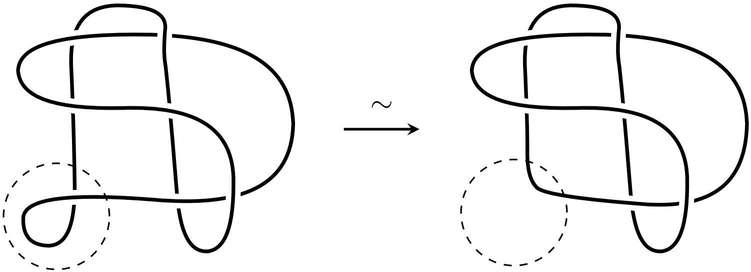


Basics - representing a knot



Basics - representing a knot

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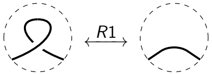
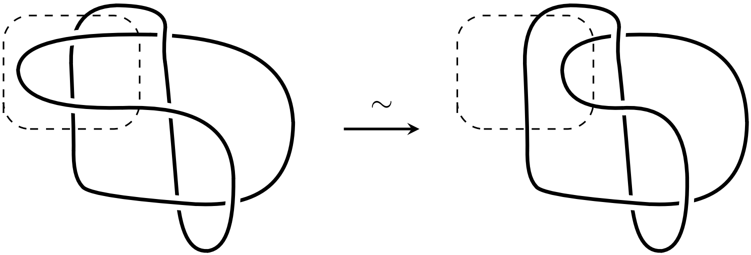
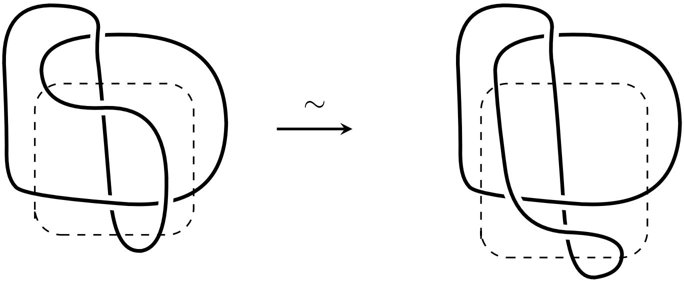

R1

Basics - representing a knot

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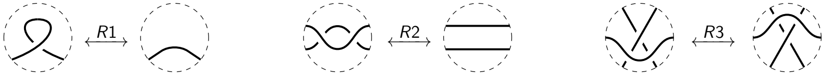
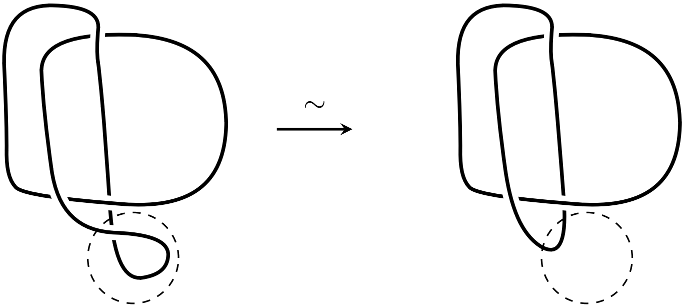
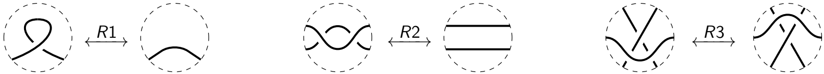
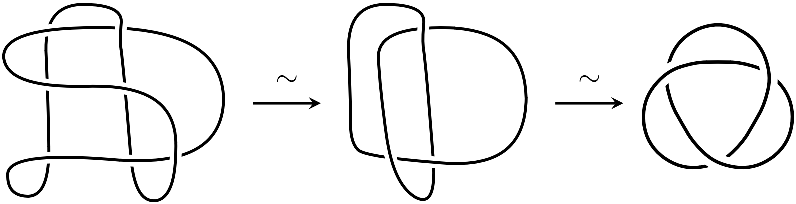

R1
R2
R3

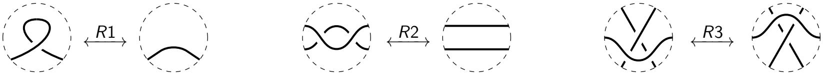
Basics - representing a knot





Basics - representing a knot



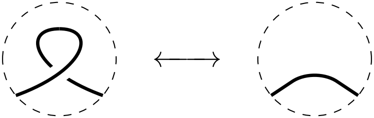


Reidemeister lemma

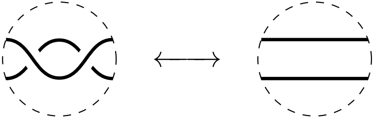
Theorem (Reidemeister lemma, 1927)

$K \sim K'$ if and only if $K \xrightarrow{R1,\ R2,\ R3} K'$ for any $K, K' \in \mathcal{K}$.

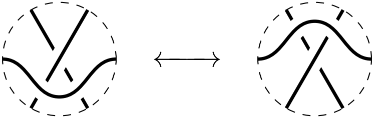
$R1$:



$R2$:



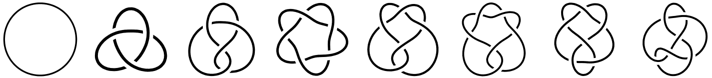
$R3$:



Prime knots

• *connected sum of two knots*: $K \# K'$ is the knot that results in cutting each of K, K' once and gluing ends of different knots together

• *prime knot*: a knot K such that there exist no $K', K'' \in \mathcal{K} \setminus \{0\}$ such that $K = K' \# K''$



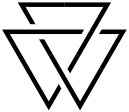
0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 ...

Theorem (Prime decomposition theorem, 1949)
For all $K \in \mathcal{K}$, there exists a unique set of prime knots $\{K_1, \ldots, K_n\} \subset \mathcal{K}$ such that $K = K_1 \# \cdots \# K_n$.

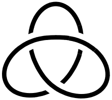
Knot invariant basics

- *knot invariant*: A function $f : \mathcal{K} \to X$ such that $f(K) \neq f(K')$ implies $K \not\sim K'$
- simple knot invariants:
  - crossing number
  - stick number
  - unknotting number

crossing number: 3



stick number: $\leqslant 6$



unknotting number: 1

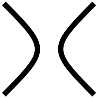
- infeasible knot invariants: $\pi_1(\mathbf{R}^3 \setminus K)$, $H^*(\mathbf{R}^3 \setminus K)$

Kauffman bracket - motivation

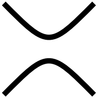
- angles and spin:

a-angle

b-angle



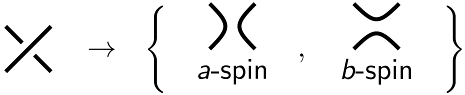
a-spin

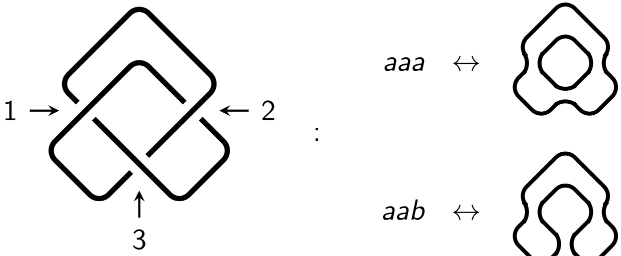


b-spin

Kauffman bracket - definition

• *state*: assignment of spin to each intersection of a link $L \in \mathcal{L}$, i.e. an element $s \in \{a, b\}^{\#\text{ of crossings of } L}$. S_L is the set of all states of L.

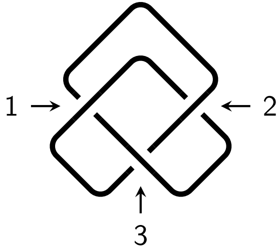




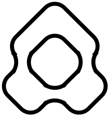
Kauffman bracket - definition

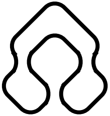
- *Kauffman bracket* (1987): a function $L \mapsto \langle L \rangle = \sum_{s \in S_L} a^{\alpha(s)} b^{\beta(s)} c^{\gamma(s)}$

$$
\begin{aligned}
\alpha(s) &= \ \# \text{ of } a\text{-spins} \\
\beta(s) &= \ \# \text{ of } b\text{-spins} \\
\gamma(s) &= \ \# \text{ of circles } -1
\end{aligned}
$$



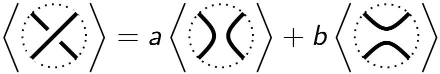
:

$aaa \leftrightarrow$  $\leftrightarrow a^3 b^0 c^1$

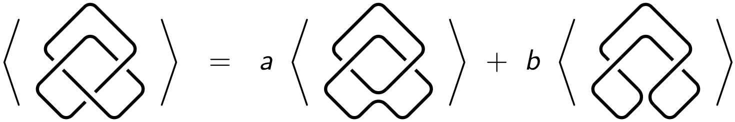
$aab \leftrightarrow$  $\leftrightarrow a^2 b^1 c^0$

Kauffman bracket - properties

- Observe that $\langle L \rangle \in \mathbb{Z}[a, b, c]$ and:

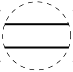
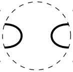
  1. $\langle \bigcirc \rangle = 1$
  2. $\langle L \sqcup \bigcirc \rangle = c \, \langle L \rangle$
  3. 

  For example,



- Is it invariant under the Reidemeister moves?

Kauffman bracket - invariance

- R3: $\langle$  $\rangle = \langle$  $\rangle$
- R2: $\langle$  $\rangle = ab \langle$  $\rangle + (abc + a^2 + b^2) \langle$  $\rangle$

  Resolve: set $b = a^{-1}$ and $c = -a^2 - b^2$, so now $\langle L \rangle \in \mathbb{Z}[a, a^{-1}]$

- R1: $\langle$  $\rangle = -a^3 \langle$  $\rangle$

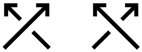
  $\langle$  $\rangle = -a^{-3} \langle$  $\rangle$

  Resolve: orient and count different types of crossings

Jones polynomial

- *writhe* (1971): a function of an oriented link $L \mapsto w(L) = \sum_{\text{crossings } i} e_i$

type of crossing i :



e_i : $+1$ -1

- *Jones polynomial (1984)*: a function of an oriented link

$$L \mapsto V(L) = -a^{-3w(L)} \langle L \rangle \Big|_{q=a^{-4}}$$

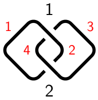
This is invariant under all three Reidemeister moves.

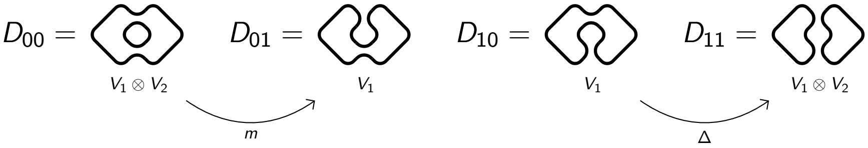
Khovanov invariant - definitions

- Terms: 0-resolution (a-spin), 1-resolution (b-spin), resolution (state)
- Define a vector space $V = \text{span}\{v_+, v_-\}$ with maps

$$\text{multiplication } m: \begin{array}{ccc} V \otimes V & \to & V \\ v_- \otimes v_- & \mapsto & 0 \\ v_- \otimes v_+ & \mapsto & v_- \\ v_+ \otimes v_- & \mapsto & v_- \\ v_+ \otimes v_+ & \mapsto & v_+ \end{array} \qquad \text{co-multiplication } \Delta: \begin{array}{ccc} V & \to & V \otimes V \\ v_- & \mapsto & v_- \otimes v_- \\ v_+ & \mapsto & v_+ \otimes v_- + v_- \otimes v_+ \end{array}$$

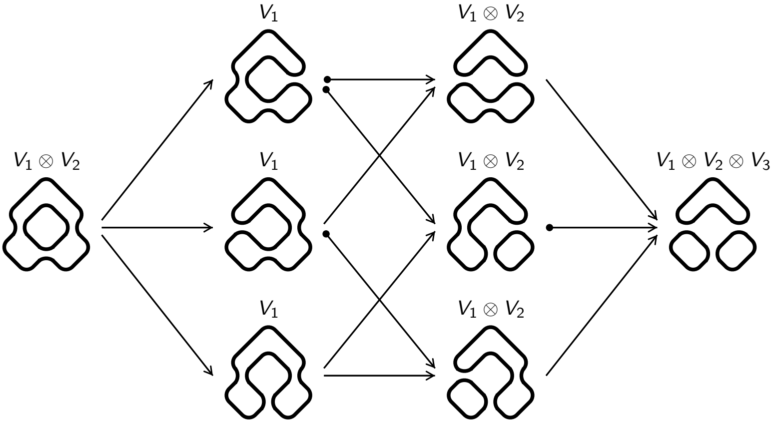
- Give order of $+1$ to v_+, -1 to v_-, to get grading for $V^{\otimes n}$.

$D =$ 



Khovanov invariant - construction

- Construct n-cube. Associate vector spaces to resolutions. Assign connecting maps. Sum and collapse into sequence of vector spaces.



$$C(D): \quad V^{\otimes 2}\{3\} \xrightarrow{d^0} V^{\oplus 3}\{4\} \xrightarrow{d^1} \left(V^{\otimes 2}\right)^{\oplus 3}\{5\} \xrightarrow{d^2} V^{\otimes 3}\{6\}$$

Khovanov invariant

- *Khovanov homology* (2000): the sequence of cohomology groups of $C(D)$. That is, the set of

$$H^i(C(D)) = \ker(d^i)/\mathrm{im}(d^{i-1})$$

- *Khovanov polynomial*: the q-Poincaré polynomial of $H^i(C(D))$;

$$L \mapsto Kh(L) = \sum_r t^r \mathrm{qdim}(\ker(d^i)/\mathrm{im}(d^{i-1}))$$

For example,

$$\ker(d^0) = \langle v_- v_-, v_- v_+ - v_+ v_- \rangle \quad , \quad t^0 \mathrm{qdim}(\ker(d^0)) = q^1 + q^3$$

These are both invariant under the Reidemeister moves.

- *Jones polynomial*: the q-Euler characteristic of $H^i(C(D))$;

$$L \mapsto J(L) = \sum_i (-1)^i \mathrm{qdim}(H^i(C(D)))$$

An overview of invariants

- *complete invariant*: a function $f : \mathcal{K} \to X$ such that $f(K) = f(K')$ if and only if $K \sim K'$

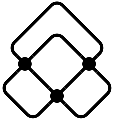
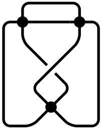
- Khovanov is not complete. Does a complete invariant exist?

- Yes - $H^*(\mathbf{R}^3 \setminus K)$, but it is not practical. No practical invariant exists.

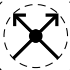
- Consider Vassiliev's invariants: an infinite number of invariants, with increasing strength.

• *singular knot*: a knot with self-intersection. Write Σ_n for the space of knot diagrams with $n > 0$ self-intersections.

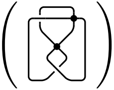
For example,

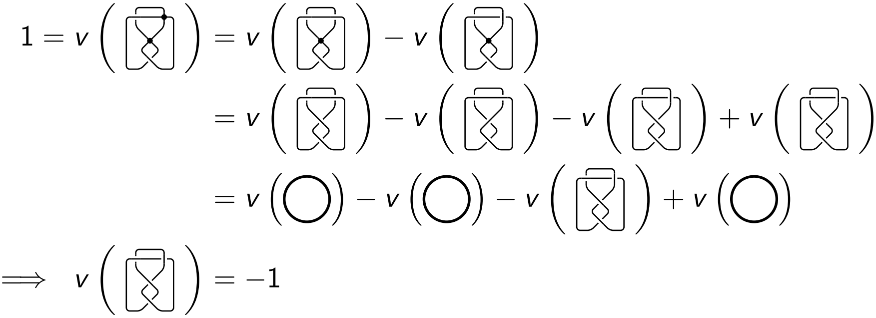
 ,  $\in \quad \Sigma_3 \subset \Sigma_\infty = \mathcal{K} \cup \bigcup_{n=1}^{\infty} \Sigma_n$

• *Vassiliev invariant of order n* (1990): a function $v : \Sigma_\infty \to \mathbb{C}$ such that $v|_{\Sigma_k} = 0$ if $k > n$ and the *Vassiliev relation* is satisfied:

$v\Big($  $\Big) = v\Big($  $\Big) - v\Big($  $\Big)$

This is invariant under the Reidemeister moves.

- To define a Vassiliev invariant v of order n, one must always:
  - fix $v(\bigcirc)$
  - fix $v(K)$ for some $K \in \Sigma_n$ (if $n > 0$)
- For example, set $v(\bigcirc) = 0$ and v  $= 1$. Then

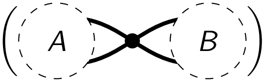


The spaces of Vassiliev invariants

- Define $V_n = \{\text{Vassiliev invariant } v \ : \ \text{ord}(v) \leqslant n\}$.

Lemma (Crossing change lemma)
For $v \in V_n$ and $K, K' \in \Sigma_n$, $v(K) = v(K')$. That is, v does not distinguish between crossing changes.

- *1-term relation*: For any Vassiliev invariant v,

v  $= 0.$

- Order 0: By the crossing change lemma, an invariant $v \in V_0$ does not distinguish knots $K \in \mathcal{K}$. Hence $V_0 \cong \mathbb{C}$.

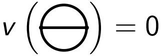
- Order 1: By the 1-term relation, an invariant $v \in V_1$ takes all $K \in \Sigma_1$ to 0. Hence $V_1 \cong \mathbb{C}$.

Gauss diagrams - definitions

• *Gauss (chord) diagram*: A representation of a singular knot.

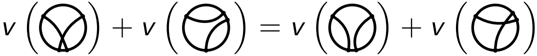
For example,

• Write G_n for the linear space of Gauss diagrams of order n over $\mathbb{C}$.



1-term relation

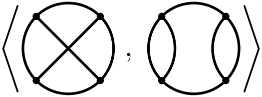
no chords cross
the drawn chord

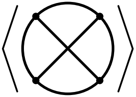


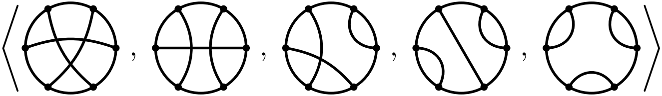
4-term relation

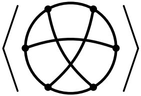
no chords start in the smallest space

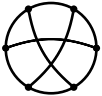
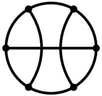
• Write $\mathcal{G}_n = G_n/(\text{all 1-term, 4-term relations})$. This is a Hopf algebra, called the *algebra of Gauss diagrams*.

$G_2 =$ 

$\mathcal{G}_2 =$ 

$G_3 =$ 

$\mathcal{G}_3 =$ 

since  $- 2$  $= 0$

Theorem (Kontsevich, 1995)

As graded algebras, $V_n \cong \mathcal{G}_n$.

This reduces the problem of finding the dimension of the spaces of Vassiliev invariants.

The Kontsevich integral

- *Kontsevich invariant (1993)*: a function $\mathcal{K} \to \bigoplus_i G_i$ given by

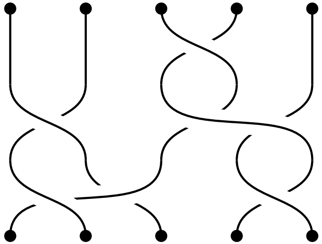
$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\tau < t_1 < \cdots < t_m < T} \sum_{p=\{(z_j, z_j')\}} (-1)^{\downarrow} G_p \bigwedge_{j=1}^{m} \frac{dz_j - dz_i'}{z_j - z_j'}$$

- It is unknown if the Kontsevich integral is a complete knot invariant.
- Related: unknown if for any two knots, there exists a Vassiliev invariant that distinguishes them. Both believed to be true.

Braids - definitions

• *braid*: a finite number of strands in $\mathbf{R}^3$, all beginning on the same line, and all ending on a line parallel to the first one, with no maxima or minima wrt starting lines.

For example, a *braid diagram* of a braid:



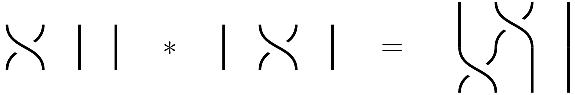
• In contrast to knots, braids form a group.

The braid group - definitions

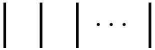
- Write B_n for the group on n strands (Artin, 1925). Generators:

b_1 $\qquad$ b_2 $\qquad$ b_{n-1}

- Group operation:



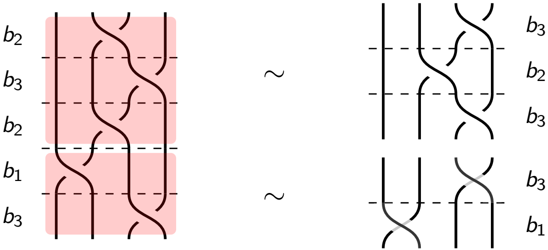
- Identity and inverses:

$e =$  $\qquad$ $b_1^{-1} =$ 

- B_n is not abelian (there exists a homomorphism $B_n \to S_n$).

The braid group - properties

- There exist non-trivial identities among the b_n.



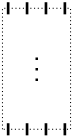
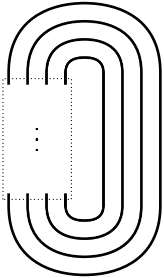
$b_3 b_1 b_2 b_3 b_2$

- far away rule: $b_i b_j = b_j b_i$ for all $1 \leqslant i, j \leqslant n-1$ s.t. $|i-j| \geqslant 2$
- close rule: $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for all $1 \leqslant i \leqslant n-2$

$$B_n = \left\langle b_1, \ldots, b_{n-1} \;:\; \begin{array}{l} b_i b_j - b_j b_i \;\forall\; 1 \leqslant i, j \leqslant n-1,\; |i-j| \leqslant 2, \\ b_i b_{i+1} b_i - b_{i+1} b_i b_{i+i} \;\forall\; 1 \leqslant i \leqslant n-2 \end{array} \right\rangle$$

The braid group - closures

• *closure*: given a braid b, its closure $\mathrm{cl}(b)$ is formed by connecting the ends at the bottom to the ends at the top.

$b =$  , $\mathrm{cl}(b) =$ 

• **Theorem (Alexander)**
For every $L \in \mathcal{L}$, there exists $b \in B_n$ for some n such that $L = \mathrm{cl}(b)$.

FIN