Compact course notes

Pure Math 665, Fall 2013

Riemannian Geometry

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Contents

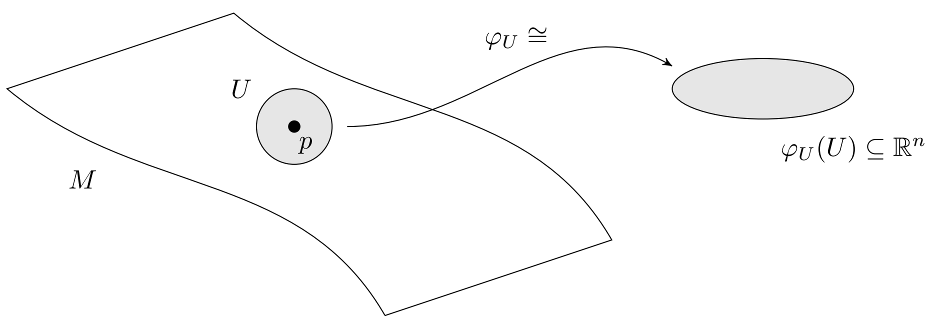
In this course, we will study manifolds equipped with Riemannian metrics, which will allow us to measure different things. These are some of the topics we hope to cover:

- Special curves called geodesics
- Curvature of Riemannian manifolds
- Hopf-Rinow theorem
- Submanifolds and the associated Gauss-Codazzi equations
- Gauss-Bonnet theorem
- Hodge theorem
- Bochner-Weitzenbock formula

1 Review

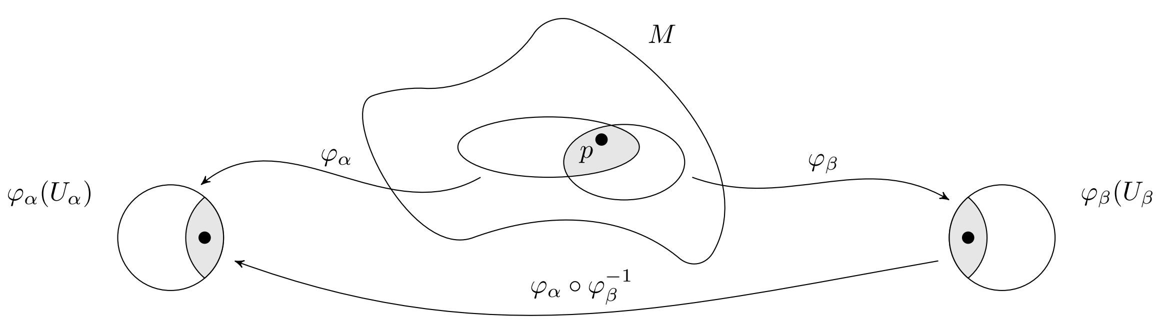
1.1 Smooth manifolds

Definition 1.1.1. A *topological n-manifold M* is a Hausdorff, second-countable, topological space that is locally Euclidean of dimension n. This means that for all $p \in M$, there exists an open set $U \ni p$ and a map $\varphi_U : U \to \varphi_U(U) \subseteq \mathbb{R}^n$ whose image is also open.



The pair (U, φ_U) is called a *coordinate chart*.

Definition 1.1.2. Let M be a topological n-manifold. A *smooth structure* on M is a collection of charts $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$ and $U_\alpha \cap U_\beta \neq \emptyset$ implies that $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.



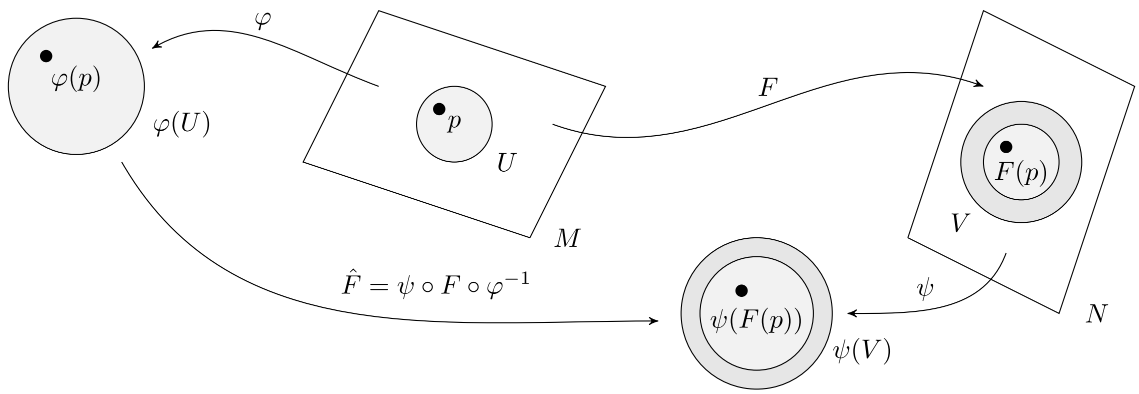
A *smooth n-manifold* is a topological n-matifold together with a choice of smooth structure. In the context of this course, all manifolds wirr be smooth manifolds with fixed structure.

Example 1.1.3. Some examples of smooth n-manifolds are $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{R}P^n$. If M is a smooth m-manifold and N is a smooth n-manifold, then $M \times N$ is a smooth $(m+n)$-manifold.

Note that for manifolds, connectedness is equivalent to path-connectedness.

Definition 1.1.4. Let M, N be manifolds. A map $F : M \to N$ is termed *smooth* if all of its coordinate representations are smooth.

Note that smoothness is a local property. A map is smooth at $p \in M$ if and only if it is smooth when related to an open manifold of p.



Above, (U, φ) is a chart for M and (V, ψ) is a chart for N, with $F(U) \subseteq V$.

In this course, all maps will be smooth.

Definition 1.1.5. A map $F : M \to N$ of manifolds is a *diffeomorphism* iff it is a smooth bijection with a smooth inverse.

Definition 1.1.6. A *Lie group* G is a group that is also a smooth manifold. It must be that the group operation $(x, y) \mapsto xy^{-1}$ is a differentiable map of $G \times G$ to G. Fundamental Lie groups are:

- $GL(n, \mathbb{R})$, the invertible $n \times n$ matrices over $\mathbb{R}$, with size n^2
- $GL(n, \mathbb{C})$ as above, but over $\mathbb{C}$ and with size $4n^2$
- $SO(n),\ O(n),\ U(n),\ SU(n)$

Definition 1.1.7. Let M be a manifold and $U = \{U_\alpha : \alpha \in A\}$ be an open cover of M. Then a *partition of unity* subordinate to U is a collection of maps $\psi_\alpha : M \to \mathbb{R}$ such that for all $\alpha \in A$:

1. $0 \leqslant \psi_\alpha(p) \leqslant 1$ for all $p \in M$
2. $\text{supp}(\psi_\alpha) = \{p \in M : \psi_\alpha(p) \neq 0\} \subseteq U_\alpha$
3. for all $p \in M$, there exists an open set $W_p \ni p$ such that $W_p \cap \text{supp}(\psi_\alpha) \neq \emptyset$ for only finitely many α
4. $\sum_\alpha \psi_\alpha(p) = 1$ for all $p \in M$

Note that there always exist partitions of unity. They are used to "patch together" local constructions to get global constructions. Besides that, they are also used to:

- Define integration of n-forms on M
- Extend local objects to global objects (functions, vector fields)
- Prove existence of Riemannian metrics
- Prove existence of connections on vector bundles

1.2 Tangent vectors and derivations

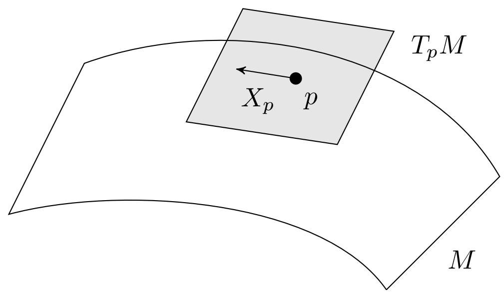
Recall that $C^\infty(M)$ is the set of smooth functions from $M \to \mathbb{R}$, an infinite-dimensional vector space, as well as a commutative algebra with identity.

Definition 1.2.1. Let $p \in M$. A *derivation* at p is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ such that the Leibniz rule is satisfied, namely

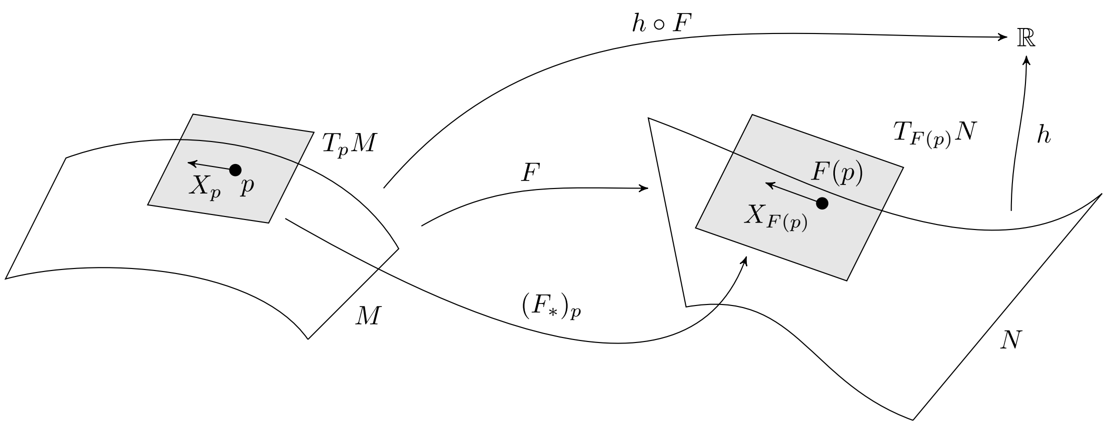
$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$$

Definition 1.2.2. The *tangent space* of M at p, denoted T_pM, is the space of derivations at p. Note that T_pM is an n-dimensional real vector space, where $n = \dim(M)$.

The action of a tangent vector $X_p \in T_pM$ on $C^\infty(M)$ as a derivation "is" the directional derivative of f at p in the direction of X_p.



Definition 1.2.3. Let $F : M \to N$ be a map of manifolds, with $p \in M$. Then there is an induced linear map $(F_*)_p : T_pM \to T_{F(p)}N$, termed the *pushforward* of F at p, or *differential* of F at p.

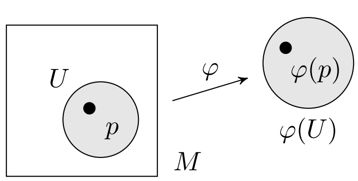


With respect to the diagram above, it is straightforward to check that the following identity is satisfied:

$$(\underbrace{(F_*)_p(X_p)}_{\in T_{F(p)}N})(\underbrace{h}_{\in C^\infty(M)}) = X_p(\underbrace{h \circ F}_{\in C^\infty(M)})$$

Remark 1.2.4. For maps $F, G : M \to N$, we have that $(G \circ F)_* = G_* \circ F_*$ and $(\mathrm{id}_M)_* = \mathrm{id}_{T_pM}$. Hence if F is a diffeomorphism, then $(F_*)_p : T_pM \xrightarrow{\cong} T_{F(p)}N$ is a linear isomorphism.

Also note that the tangent space is local, so if $U \subseteq M$ is open with $p \in U$, then $T_pU = T_pM$. Moreover, we have the following identification, given the situation on the left:



$$(\varphi_*)_p : T_pM \xrightarrow[\text{isom.}]{\cong} T_{\varphi(p)}\mathbb{R}^n \xrightarrow[\text{can. isom.}]{\cong} \mathbb{R}^n$$

Definition 1.2.5. Let $\{(e_1)_p, \ldots, (e_n)_p\}$ be the standard ordered basis of $T_{\varphi(p)}\mathbb{R}^n$. Define

$$\left.\frac{\partial}{\partial x^k}\right|_p = (\varphi_*)_p^{-1}((e_k)_p) \in T_pM$$

Then the coordinate basis of T_pM associated to the chart (U, φ) is given by

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

Let $f \in C^\infty(M)$. The expression $\left.\frac{\partial}{\partial x^k}\right|_p f$ denotes the *partial derivative* in the e_k-direction of the coordinate representation of f at p. With this, we have that

$$\begin{aligned}\left.\frac{\partial}{\partial x^k}\right|_p f &= ((\varphi^{-1})_*(e_k))f \\ &= e_k(f \circ \varphi^{-1}) \\ &= e_k \hat{f} \\ &= \frac{\partial \hat{f}}{\partial x^k}(\varphi(p))\end{aligned}$$

Remark 1.2.6. Let $F : M^n \to N^k$ be a smooth map of manifolds, with $(F_*)_p : T_pM \to T_{F(p)}N$ a linear map. Define the following values:

(U, φ) is a chart containing p	$\left\{\left.\frac{\partial}{\partial x^1}\right\|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right\|_p\right\}$ is a basis of T_pM	$\varphi(q) = (x^1(q), \ldots, x^n(q)),\ q \in U$
(V, ψ) is a chart containing $F(p)$	$\left\{\left.\frac{\partial}{\partial y^1}\right\|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^n}\right\|_{F(p)}\right\}$ is a basis of $T_{F(p)}N$	$\psi(s) = (y^1(s), \ldots, y^k(s)),\ s \in V$

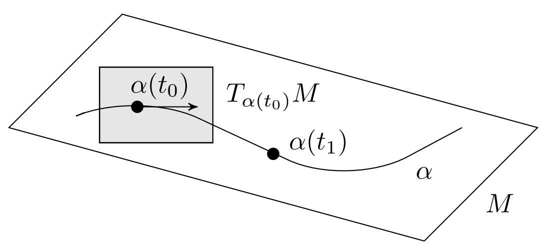
Then the $k \times n$ matrix for $(F_*)_p$ with respect to these bases is

$$((F_*)_p)_{i,j} = \frac{\partial \hat{F}^i}{\partial x^j}(\varphi(p))$$

This is the Jacobian matrix at $\varphi(p)$ of the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$. To make a change of coordinates between two charts $(x^1, \ldots, x^n)$ and $(\tilde{x}^1, \ldots, \tilde{x}^n)$ around p, for $X_p \in T_pM$, we have

$$X_p = \sum_{i=1}^n a^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{i=1}^n \tilde{a}^i \left.\frac{\partial}{\partial \tilde{x}^i}\right|_p \qquad \tilde{a}^i = \sum_{j=1}^n a^j \frac{\partial \tilde{x}^i}{\partial x^j}(p)$$

Definition 1.2.7. A *smooth curve* on a manifold M is a smooth map $\alpha : I \to M$, where I is the open interval in $\mathbb{R}$.



Let $t_0 \in I$ and $p = \alpha(t_0)$, as above. Then we have a map $(\alpha_*)_{t_0} : T_{t_0}\mathbb{R} \to T_{\alpha(t_0)}M$, noting that $T_{t_0}\mathbb{R} \cong \mathbb{R}$. We then define the *velocity vector* of α at $\alpha(t_0)$ to be

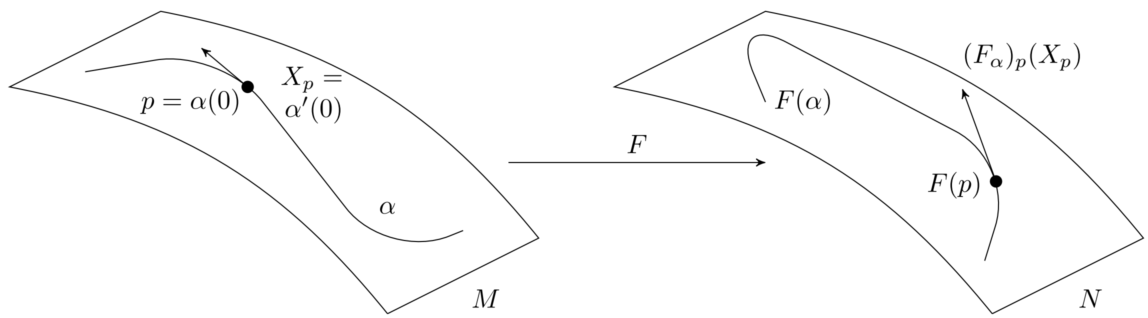
$$\alpha'(t_0) = (\alpha_*)_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{\alpha(t_0)}M$$

In local coordinates, this may be expressed as

$$\alpha(t) = (x^1(t), \dots, x^n(t))$$
$$\alpha'(t_0) = \sum_{i=1}^{n} \frac{dx^i}{dt}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\alpha(t_0)}$$

Note that every tangent vector in T_pM may be expressed as a velocity vector at p of a smooth curve through p, in many ways. Such an interpretation of tangent vectors gives us an easy way to actually compute the pushforward explicitly in practice.

For instance, let $F : M \to N$, $p \in M$, and $X_p \in T_pM$. Choose any smooth curve $\alpha : I \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = X_p$.



Then $(F_*)_p(X_p) = (F \circ \alpha)'(0)$. Observe that $F \circ \alpha$ is a smooth curve on N, and $(F \circ \alpha)(0) = F(p)$.

1.3 Tangent bundles and vector fields

Definition 1.3.1. Let M be a manifold. Then the *tangent bundle* of M is defined to be the smooth $2n$-manifold

$$TM = \bigsqcup_{p \in M} T_pM = \{(p, v_p) \ : \ p \in M, v_p \in T_pM\}$$

Consider a map $\pi : TM \to M$, given by $\pi(p, v_p) = p$. This is termed the *projection map.* Then if (U, φ) is a chart for M, it induces a chart $(\tilde{U}, \tilde{\varphi})$ for TM as follows:

$$\tilde{U} = \pi^{-1}(U) = \{(p, v_p) \ : \ \pi(p) \in U\}$$
$$\tilde{\varphi} : \tilde{U} \to \tilde{\varphi}(\tilde{U}) \subseteq \mathbb{R}^n \oplus \mathbb{R}^n$$
$$\tilde{\varphi}\left(p, \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (\varphi(p), a^1, \dots, a^n)$$
$$= (x^1(p), \dots, x^n(p), a^1(p, v_p), \dots, a^n(p, v_p))$$

Definition 1.3.2. A *smooth vector field* on M is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$, that is, $\pi(X_p) = p$.

Note that here we have expressed the tangent vector as a function, i.e. $X(p) = X_p$. This function in local coordinates may be given by

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \qquad X_p = \sum_{i=1}^{n} a^i(\varphi(p)) \left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$$

Here the a_i are smooth functions defined on the domain $\varphi(U)$ of the chart. We also define $\Gamma(TM)$ to be the *space of smooth vector fields* on M, which is an infinite-dimensional real-valued vector space (or a

$C^\infty(M)$-module). This gives the following relations, for $p \in M$:

$$\begin{array}{rcl} X & \in & \Gamma(TM) \\ f & \in & C^\infty(M) \end{array} \qquad \begin{array}{rcl} fX & \in & \Gamma(TM) \\ (fX)(p) = f(p)X_p & \in & T_pM \end{array}$$

Remark 1.3.3. Given $X \in \Gamma(TM)$, there is an induced map $X : C^\infty(M) \to C^\infty(M)$, such that for $h \in C^\infty(M)$, $(Xh)(p) = X_p h$. In local coordinates, $X = \sum_i a^i \frac{\partial}{\partial x^i}$.

Also note that X is linear over $\mathbb{R}$, and satisfies the Leibniz rule, namely

$$X(fg) = f(Xg) + (Xf)g$$

Definition 1.3.4. Let $F : M \to N$ be a map of manifolds, and $X \in \Gamma(TM)$, $Y \in \Gamma(TN)$. Then we say that X and Y are *F-related* iff $T_{F(p)}N \ni (F_*)_p X_p = Y_{F(p)}$, for all $p \in M$. This follows, by the definition of a pushforward, iff $X(h \circ F) = (Yh) \circ F$ for all $h \in C^\infty(N)$.

2 Operations on vector fields

2.1 The Lie bracket of vector fields

Definition 2.1.1. Let $X, Y \in \Gamma(TM)$. Then the *Lie bracket* of X, Y is denoted by $[X, Y]$. It is a vector field on M, defined by

$$[X, Y]f = X(Yf) - Y(Xf) \qquad f \in C^\infty(M)$$

Locally, the explicity function is given by

$$X = a^i \frac{\partial}{\partial x^i} \quad Y = b^j \frac{\partial}{\partial x^j} \qquad [X, Y] = \left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

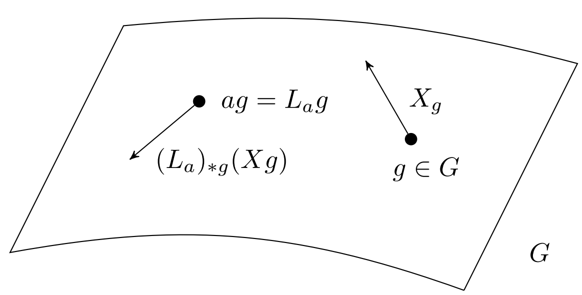
The Lie bracket satisfies the following properties, for all $X, Y, Z \in \Gamma(TM)$ and $f, g \in C^\infty(M)$:

1. $[X, Y]$ is $\mathbb{R}$-linear in X and Y
2. $[X, Y] = -[Y, X]$
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$
4. $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \in \Gamma(TM)$ The first three conditions indicate that $\Gamma(TM)$ is a Lie algebra. The third condition is also termed the *Jacobi identity*.

Proposition 2.1.2. Let $F : M \to N$ be a map of manifolds. Suppose $X_1, X_2 \in \Gamma(TM)$ and $Y_1, Y_2 \in \Gamma(TN)$, and that X_j is F-related Y_j for $j = 1, 2$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$. In particular, $F_*[X_1, X_2] = [F_* X_1, F_* X_2]$.

Example 2.1.3. Let $G \ni a$ be a Lie group. Define $L_a : G \to G$ by $L_a(G) = ag$, left-multiplication by a. Then the map is smooth, with $L_{a^{-1}} = (L_a)^{-1}$. This follows as L_a is a diffeomorphism of G for all $g \in G$.

Definition 2.1.4. A vector field $X \in \Gamma(TG)$ is termed *left-invariant* iff $(L_a)_{*g}(X_g) = X_{ag}$ for all $a \in G$.



A left-invariant vector field X on G is L_a-related to itself for all $a \in G$.

Proposition 2.1.5. The set of all left-invariant vector fields on G is a Lie subalgebra (a vector space closed under the Lie bracket) of $\Gamma(TG)$ of dimension $\dim(G)$.

Proof: Since $(L_a)_{*g}$ is linear, $\mathfrak{J}$ is a vector subspace of $\Gamma(TG)$. We need to show that it is closed under the Lie bracket:

$$(L_a)_*[X, Y] = [(L_a)_*X, (L_a)_*Y] = [X, Y]$$

The last equality follows as X, Y are left-invariant. Hence $[X, Y] \in \mathfrak{J}$.

Now we will show that $\mathfrak{J} \cong T_eG$, for e the identity element. Consider the map $\ell : T_eG \to \Gamma(TG)$ given by

$$(\ell(X_e))_g = (L_g)_{*e}(X_e) \in T_gG \qquad \text{with} \qquad (L_g)_{*e} : T_eG \to T_{L_g(e)=g}G$$

We leave it as an exercise to show that $\ell(X_e)$ is a smooth map $G \to TG$. To show this, use local coordinates and the fact that multiplication in G is smooth. Our next claim is that $\ell(X_e) \in \mathcal{J}$ is a left-invariant vector field. This follows as:

$$(L_a)_{*g}((\ell(X_e))_g) = (L_a)_{*g}((L_g)_{*e}(X_e)) = (L_a \circ L_g)_{*e}(X_e) = (L_{ag})_{*e}(X_e) = (\ell(X_e))_{ag}$$

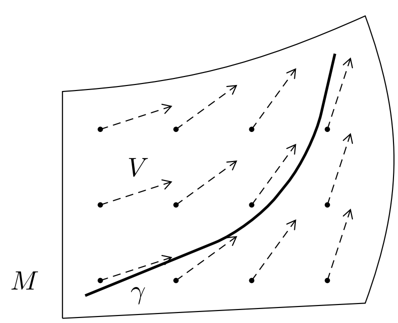
We leave it as an exercise to check that ℓ is injective and surjective onto $\mathcal{J}$, which will give that $\ell : T_eG \to \mathcal{J}$ is an isomorphism. ■

Remark 2.1.6. With respect to the above proposition, $\mathcal{J} \cong T_eG$ is called the Lie algebra of the Lie group G. So if $v, w \in T_eG$, we let $[v, w] = [\ell(v), \ell(w)]_e \in T_eG$. Moreover, if $f : G \to H$ is a Lie group homomorphism (that is, $f(ab) = f(a)f(b)$, and $f(e) = e$), then $(f_*)_e : T_eG \to T_eH$ is a Lie algebra homomorphism. Also note that then $[(f_*)_e v, (f_*)_e w] = (f_*)_e[v, w]$.

2.2 Integral curves of vector fields

In this section, we will see that vector fields are infinitesimal diffeomorphisms.

Definition 2.2.1. Let M be a manifold, and $V \in \Gamma(TM)$. An *integral curve* of V is a smooth curve $\gamma : I \to \mathbb{R}$ such that $\gamma'(t) = V_{\gamma(t)}$.



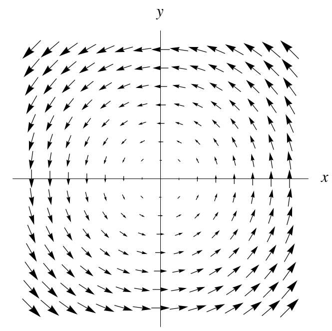
Example 2.2.2. Let $M = \mathbb{R}^2$ with global coordinates (x, y). Any vector field on M is given by $a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ for smooth functions a, b.

Example 2.2.3. Let $M = \mathbb{R}^2$ and consider the following vector field:

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$x' = -y \qquad y' = x$$
$$x'' = -x \quad y'' = -y$$

$$\implies \quad x(t) = x_0\cos(t) + y_0\sin(t)$$
$$\implies \quad y(t) = x_0\sin(t) - y_0\cos(t) \quad \forall\ t \in \mathbb{R}$$



Remark 2.2.4. How is it possible to find integral curves? Let $p \in M$, and let (U, φ) be a chart containing p, with $V = V^i(x^1, \dots, x^n)\frac{\partial}{\partial x^i}$. Then a curve γ in local coordinates looks like $\varphi \circ \gamma = \hat{\gamma}(t) = (x^1(t), \dots, x^n(t))$, giving

$$T_{\gamma(t)}M \ni \gamma'(t) = \frac{dx^i}{dt}(t)\ \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}$$

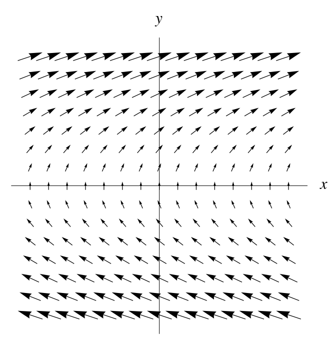
To have $\gamma'(t) = V_{\gamma(t)}$, we need $\frac{dx^i}{dt} = V^i(x^1(t), \dots, x^n(t))$ for all $i = 1, \dots, n$ and $t \in I$, the domain of γ. This is a system of a first order ODE, which leads us to the following theorem:

Theorem 2.2.5. [Picard, Lindelof]
Given initial conditions $x^i(0) = x^i_0 \in \mathbb{R}$ for all $i = 1, \dots, n$, there exists a unique solution $(x^1(t), \dots, x^n(t))$ to the system of first order ODEs above for $t \in (-\epsilon, \epsilon)$, for some ϵ. Furthermore, the solution depends smoothly on the initial conditions.

Example 2.2.6. Let $M = \mathbb{R}^2$, and consider the following vector field:

$$V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$
$$V = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$$

$$y' = 1 \quad \implies \quad y = y_0 + t$$
$$x' = y \quad \implies \quad x' = y_0 + t$$
$$\implies \quad x = x_0 + y_0 t + \tfrac{1}{2}t^2$$

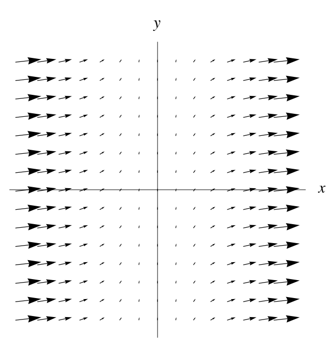


Here we see that solutions do indeed exist for all $t \in \mathbb{R}$.

Example 2.2.7. Let $M = \mathbb{R}^2$ and consider the following vector field:

$$V = x^2\frac{\partial}{\partial y} + \frac{\partial}{\partial x}$$

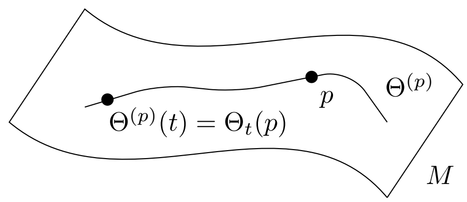
$$x' = x^2 \quad y' = 1$$



Next we will package the integral curves together with a vector field.

2.3 Flows

Definition 2.3.1. Let $V \in \Gamma(TM)$ Suppose for simplicity that there exists an integral curve of V given by $\Theta^{(p)} : \mathbb{R} \to M$ for all $p \in M$, with $\Theta^{(p)}(0) = p$. For all $t \in \mathbb{R}$, define a *flow* on M to be the function $\Theta_t : M \to M$, with $\Theta_t(p) = \Theta^{(p)}(t)$. This is a function that follows the integral curve that starts at p for time t. The space V associated to Θ is termed the *infinitesimal generator* of Θ.



Let $q = \Theta^{(p)}(s)$, and consider the smooth curve $\Theta^{(p)}(t+s) = \gamma(t)$. For this curve,

$$\gamma(0) = \Theta^{(p)}(s) = q$$
$$\gamma'(0) = \frac{d}{dt}\bigg|_{t=0} \Theta^{(p)}(t+s) = \frac{d}{du}\bigg|_{u=s} \Theta^{(p)}(u) = V_{\Theta^{(p)}(s)} = V_q$$

Therefore $\gamma(t) = \Theta^{(q)}(s)$. By uniqueness of integral curves,

$$\Theta_{t+s}(p) = \Theta^{(p)}(t+s) = \Theta_t(q) = \Theta_t(\Theta_s(p))$$

Hence $\Theta_{t+s} = \Theta_t \circ \Theta_s$, meaning that if we follow an integral curve for time s, then time t, it is the same as following the curve for time $s+t$. This law, among others, gives us a complete perspective on flows:

$$\Theta_{t+s} = \Theta_{t+s}$$
$$\Theta_t^{-1} = \Theta_{-t}$$
$$\Theta_0 = \mathrm{id}_M$$
$$\Theta_t : M \to M \text{ is a diffeomorphism of } M \text{ for all } t$$

The last statement holds because the ODE theorem gives smooth dependence on time.

Definition 2.3.2. Let M be a manifold, and for $p \in M$, suppose that there exists $\epsilon > 0$ such that $\gamma : (-\epsilon, \epsilon) \to M$ is an integral curve of V with $\gamma(0) = p$. If all integral curves of V are defined for all $t \in \mathbb{R}$, then V is termed a *complete vector field*.

Definition 2.3.3. A *global flow* on a manifold M is a smooth map $\Theta : \mathbb{R} \times M \to M$ such that for all $t, s \in \mathbb{R}$ and $p \in M$,

$$\Theta(t, \Theta(s,p)) = \Theta(t+s, p)$$
$$\Theta(0,p) = p$$

Notationally, we may associate this to the presentation of flows above:
- define $\Theta_t : M \to M$ by $\Theta_t(p) = \Theta(t,p)$
- define $\Theta^{(p)} : \mathbb{R} \to M$ by $\Theta^{(p)}(t) = \Theta(t,p)$

Now these statements are equivalent to the ones above.

Theorem 2.3.4. [Fundamental theorem of global flows]
Let $\Theta : \mathbb{R} \times M \to M$ be a global flow. For all $p \in M$, define $V_p = \Theta^{(p)\prime}(0) \in T_pM$. Then $p \mapsto V_p$ is a smooth vector field on M, and each $\Theta^{(p)}$ is an integral curve of V.

Proof: First, we need to show that V is smooth. It is enough to show that $Vf \in C^\infty(M)$ for all $f \in C^\infty(M)$. This is left as an exercise. So first we observe that

$$V_pf = \Theta^{(p)\prime}(0)f = \frac{d}{dt}\bigg|_{t_0} f(\Theta^{(p)}(t)) = \frac{d}{dt}\bigg|_{t_0} f(\Theta(t,p))$$

is smooth as a function of p. Hence V is smooth. Next, we need to show that $\Theta^{(p)}$ is an integral curve of V. This implies showing that $\Theta^{(p)\prime}(t_0) = V_{\Theta^{(p)}(t_0)}$ for all $p \in M$ and $t_0 \in \mathbb{R}$. Define some values as follows:

$$q = \Theta^{(p)}(t_0) = \Theta_{t_0}(p)$$
$$\Theta^{(q)}(t) = \Theta_t(q) = \Theta_t(\Theta_{t_0}(p)) = \Theta_{t+t_0}(p) = \Theta^{(p)}(t+t_0)$$

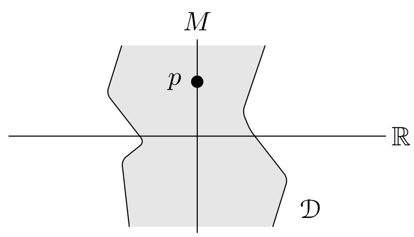
Now observe that

$$\Theta^{(q)\prime}(t) = \Theta^{(p)\prime}(t+t_0) \qquad \text{and} \qquad \Theta^{(q)\prime}(0) = \Theta^{(p)\prime}(t_0) = V_q$$

This completes the proof. ■

However, in general we have the problem that a vector field does not determine a global flow. To resolve this issue, we introduce the idea of a flow domain.

Definition 2.3.5. For a manifold M, a *flow domain* is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that for all $p \in M$, $\mathcal{D}^p = \{t \in \mathbb{R} \ : \ (t,p) \in \mathcal{D}\}$ is an open interval containing 0.



Definition 2.3.6. A *local flow* on a manifold M is a smooth map $\Theta : \mathcal{D} \to M$ such that $\Theta(0,p) = p$ for all p, and $t \in \mathcal{D}^{\Theta(s,p)}$ for all $s \in \mathcal{D}^{(p)}$ such that

$$s + t \in \mathcal{D}^{(p)} \implies \Theta(t, \Theta(s,p)) = \Theta(t+s, p)$$

In other words, the group law holds whenever both sides are defined.

Remark 2.3.7. The fundamental theorem of flows may be equivalently stated by replacing the first sentence with: Let $\Theta : \mathcal{D} \to M$ be a flow. The proof is the same, as the fact that Θ was global was never used.

Theorem 2.3.8. [Fundamental theorem of global flows, part 2]
Let $V \in \Gamma(TM)$. Then there is a unique maximal flow $\Theta : \mathcal{D} \to M$ whose infinitesimal generator is V. This flow satisfies the following:

1. for all $p \in M$, $\Theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p
2. if $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{\Theta(s,p)} = \mathcal{D}^{(p)} - s = \{t - s : t \in \mathcal{D}^{(p)}\}$
3. for all $t \in \mathbb{R}$, $M_t = \{p \in M \ : \ (t,p) \in \mathcal{D}\}$ is open in M
4. for all $(t,p) \in \mathcal{D}$, $(\Theta_t)_* V_p = V_{\Theta_t(p)}$

The last statement asserts that V is Θ_t-related to itself. We say that V is then invariant under the diffeomorphism Θ_t, written $(\Theta_t)_*V = V$, though here, V is restricted to M_t.

Proof: **1.** By the ODE theorem, there exists an integral curve γ starting at each $p \in M$. By uniqueness, any two such curves agree on their common domain. For $p \in M$, let $\mathcal{D}^{(p)} =$ (union of al open intervals $I \subset \mathbb{R}$ containing 0 on which an integral curve starting at p is defined). Define $\Theta^{(p)} : \mathcal{D}^{(p)} \to M$ by $\Theta^{(p)}(t) = \gamma(t)$ for any γ whose domain contains t. Then this is a maximal integral curve by construction that is well-defined and smooth.

2. This is clear, and is left as an exercise. It relies on the (easily provable) fact that if $\mathcal{D}$ is open, then $\Theta : \mathcal{D} \to M$ is smooth.

3. Let $M_t = \{p \in M \ : \ (t,p) \in \mathcal{D}\}$, which is open as $\mathcal{D}$ is open and Θ is continuous. Then $p \in M_t$ implies that $t \in \mathcal{D}^{(p)}$, and $\mathcal{D}^{\Theta_t(p)} - t$ by **2.** Hence $-t \in \mathcal{D}^{\Theta_t(p)}$, so $\Theta_t(p) \in M_{-t}$. Therefore $\Theta_t : M_t \to M_{-t}$. And by the group law as before, $\Theta_t^{-1} = \Theta_{-t}$, so $\Theta_t : M_t \to M_{-t}$ is a diffeomorphism.

4. Let $(t_0, p) \in \mathcal{D}$, and $q \in \Theta_{t_0}(p)$. We want to show that $((\Theta_{t_0})_*)_p V_p = V_q$. To show this, we apply both sides to $f \in C^\infty(V)$ for $U \subset M$ open, where $q \in U$. This gives:

$$\begin{aligned}
(\Theta_{t_0})_* V_p f &= V_p(f \circ \Theta_{t_0}) && \text{(by definition of } (\Theta_{t_0})_*) \\
&= \Theta^{(p)\prime}(0)(f_0 \Theta_{t_0}) && \text{(as } \Theta^{(p)} \text{ is an integral curve of } V \text{ starting at } p) \\
&= \frac{d}{dt}\bigg|_{t=0} (f \circ \Theta_{t_0})(\Theta^{(p)}(t)) \\
&= \frac{d}{dt}\bigg|_{t=0} f(\Theta_{t+t_0}(p)) \\
&= \frac{d}{dt}\bigg|_{t=0} f(\Theta^{(p)}(t+t_0)) \\
&= \Theta^{(p)}(t_0) f \\
&= V_{\Theta^{(p)}(t_0)} f && \text{(as } \Theta^{(p)} \text{ is an integral curve of } V) \\
&= V_q f
\end{aligned}$$

This completes the proof. ■

Our next goal for the following lectures will be to prove the theorem below:

Theorem 2.3.9. [Frobenius theorem]
Let M be a manifold. Let $V_1, \ldots, V_n$ be smooth vector fields on an open subset of M. Suppose that $\{V_1|_p, \ldots, V_n|_p\}$ is a linearly independent set. Then there exists a coordinate chart (w, φ) containing p such that in these coordinates, $V_i = \frac{\partial}{\partial x^i}$.

This is equivalent to saying that $[V_i, V_j] = 0$ for all i, j.

2.4 Regular and singular points

Definition 2.4.1. Let $V \in \Gamma(TM)$. A point $p \in M$ is termed a *singular point* of V if $V_p = 0$. If $V_p \neq 0$, then p is termed a *regular point*.

Lemma 2.4.2. Let $V \in \Gamma(TM)$. Let $\Theta : \mathcal{D} \to M$ be the flow whose infinitesimal generator is V. If p is a singular point of V, then $\mathcal{D}^{(p)} = \mathbb{R}$, and $\Theta^{(p)} : \mathcal{D}^{(p)} \to M$ is an immersion. That is, $(\Theta^{(p)}_*)_t : T_t\mathcal{D}^{(p)} \to T_{\Theta^{(p)}(t)}M$ is injective.

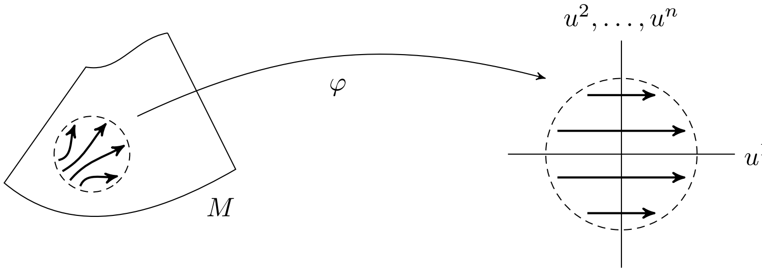
Proof: If $V_i = 0$, then $\gamma : \mathbb{R} \to M$ is defined by $\gamma(t) = p$ for all t, which is a smooth curve on M with $\gamma'(t) = 0 = V_{\gamma(t)} = V_p$. By uniqueness, this is the maximal integral curve starting at p.

For the second part, note that if $V_p \neq 0$, then we let $\gamma = \Theta^{(p)}$. We need to show that $\gamma'(t) \neq 0$ for all $t \in \mathcal{D}^{(p)}$. Let $t_0 \in \mathcal{D}^{(p)}$, and $q = \gamma(t_0)$. Then by the fundamental theorem of global flows, $V_q = (\Theta_{t_0})_*(V_p)$.

But $(\Theta_{t_0})_*$ is an isomorphism $T_pM \to T_{\Theta_{t_0}(p)}M$, since Θ_{t_0} is a diffeomorphism. So $V_p \neq 0$, which implies that $V_q \neq 0$. ■

Theorem 2.4.3. [Canonical form theorem (or Flow box theorem)]
Let $V \in \Gamma(TM)$ with $V_p \neq 0$. Then there exist local coordinates $(u^1, \dots, u^n)$ on a neighborhood of p in which V has the form $V = \frac{\partial}{\partial u^1}$.

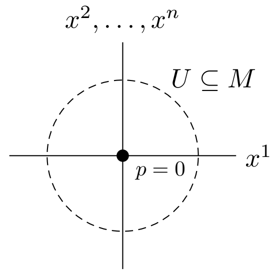
Heuristically, the theorem changes the vector field of a neighborhood on the manifold as follows:



$V \xrightarrow{\varphi} a^i(u^1, \dots, u^n)\frac{\partial}{\partial u^i}$

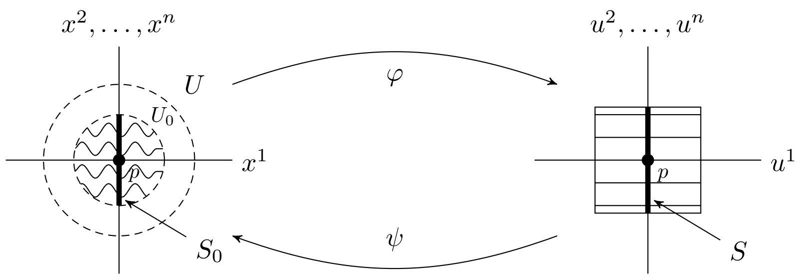
The theorem states that
$a^1 = 1$, $a^i = 0$ for all other i

Proof: We need to find a coordinate chart (U, φ) such that $(\varphi^{-1})_*\left(\frac{\partial}{\partial u^1}\right) = V$. This question is local, so WLOG we may assume that M is an open subset $U \subset \mathbb{R}^n$ with coordinates $(x^1, \dots, x^n)$ centered at p.



By reordering the coordinates, WLOG V_p has a non-zero component in the direction $\frac{\partial}{\partial x^1}\big|_p$. Let $\Theta : \mathcal{D} \to U$ be the flow of V. Then there exists $\epsilon > 0$ and a neighborhood $U_0 \subset U$ of p such that $(-\epsilon, \epsilon) \times U_0 \subset \mathcal{D}$ by the ODE theorem.

Now, let $S_0 \subset U_0$ be defined by $S_0 = U_0 \cap \{x^1 = 0\}$, and $S \subset \mathbb{R}^n$ be defined by $S = \{(u^2, \dots, u^n) : (0, u^2, \dots, u^n) \subset S_0\}$. Define the map $\psi : (-\epsilon, \epsilon) \times S \to U$ by $(u^1, \dots, u^n) \mapsto \Theta_{u^1}(0, u^2, \dots, u^n)$, giving:



Then for each flow $(u^2, \dots, u^n)$, ψ maps $(-\epsilon, \epsilon) \times \{(u^2, \dots, u^n)\}$ to the integral curves starting at $(0, u^2, \dots, u^n) \in U_0$. Next, we claim that ψ pushes $\frac{\partial}{\partial u^1}$ forward to V. To see this, let $(t_0, u_0) \in (-\epsilon, \epsilon) \times S$. Then

$$\psi_*\left(\frac{\partial}{\partial u^1}\bigg|_{(t_0,u_0)}\right) f = \frac{\partial}{\partial u^1}\bigg|_{(t_0,u_0)} (F \circ \psi) = \frac{\partial}{\partial u^1}\bigg|_{(t_0,u_0)} f(\Theta_{u^1}(0, u_0)) = V_{\psi(t_0,u_0)} f$$

The first equality follows from the definition of a pushforward, and the second follows from the definition of ψ. The last equality follows as Θ_t is an integral curve of V. note also, that when restricted to $\{0\} \times S$, we have $\varphi(0, u^2, \ldots, u^n) = (0, u^2, \ldots, u^n)$, because $\Theta_0 = \text{id}$, and

$$(\psi_*) \left. \frac{\partial}{\partial u^1} \right|_{(0,0)} = \left. \frac{\partial}{\partial x^i} \right|_p \qquad \text{for } i = 1, 2, \ldots, n$$

Hence at $(0,0)$, the map ψ_* takes

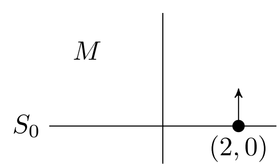
$$\left(\left. \frac{\partial}{\partial u^1} \right|_{(0,0)}, \left. \frac{\partial}{\partial u^2} \right|_{(0,0)}, \ldots, \left. \frac{\partial}{\partial u^n} \right|_{(0,0)} \right) \qquad \text{to} \qquad \left(V_p, \left. \frac{\partial}{\partial x^2} \right|_p, \ldots, \left. \frac{\partial}{\partial x^n} \right|_p \right)$$

So $(\psi_*)_{(0,0)}$ takes a basis to a basis, so it is an isomorphism, meaning that by the inverse function theorem, there exists an open neighborhood $W \ni (0,0)$ and an open neighborhood $\tilde{W} = \psi(W) \ni p$ such than $\psi : W \to \tilde{W}$ is a diffeomorphism. Let $\varphi = \psi^{-1} : \tilde{W} \to W$, for $\tilde{W} \subset U_0$ open, and so

$$(\varphi^{-1})_* \left(\frac{\partial}{\partial u^1} \right) = (\psi_*) \left(\frac{\partial}{\partial u^1} \right) = V$$

as desired, thus completing the proof. ■

Example 2.4.4. Let $M = \mathbb{R}^2$, and $V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The integral curves are $x' = y$ and $\gamma(t) = (x_0 + y_0 t + \frac{1}{2}t^2, y_0 + t)$. We take $p = (2,0)$, so $(x_0, y_0) = (2,0)$, which implies that $\gamma(t) = (2 + \frac{1}{2}t^2, t)$, and $V|_{(2,0)} = \frac{\partial}{\partial y}|_{(2,0)} \neq 0$.



Here, $S_0 = \{(u,0) \ : \ u \in \mathbb{R}\}$, and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\psi(u,t) = \Theta_t(u,0) = (u + \frac{1}{2}t^2, t) = (x,y)$. What is V in (u,t)-coordinates? We know that

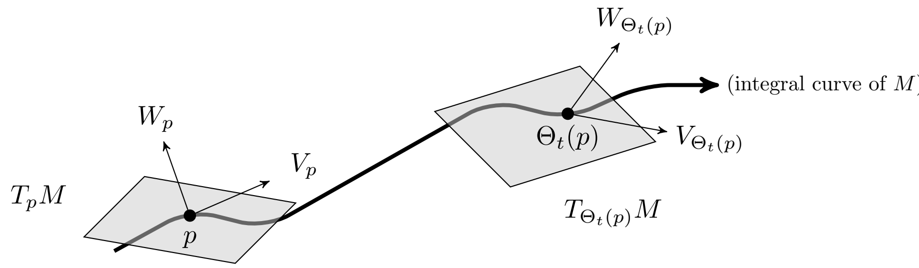
$$\begin{aligned} \frac{\partial}{\partial x} &= \underbrace{\frac{\partial u}{\partial x}}_{1} \cdot \frac{\partial}{\partial u} + \underbrace{\frac{\partial t}{\partial x}}_{0} \cdot \frac{\partial}{\partial t} \\ \frac{\partial}{\partial y} &= \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial u} + \frac{\partial t}{\partial y} \cdot \frac{\partial}{\partial t} \\ &= -y\frac{\partial}{\partial u} + \frac{\partial}{\partial t} \\ &= -t\frac{\partial}{\partial u} + \frac{\partial}{\partial t} \end{aligned}$$

This directly implies that

$$V = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = t\left[\frac{\partial}{\partial u}\right] + \left[-t\frac{\partial}{\partial u} + \frac{\partial}{\partial t}\right] = \frac{\partial}{\partial t}$$

2.5 Lie derivatives

Definition 2.5.1. Let M be a manifold, and $V \in \Gamma(TM)$. Let $\Theta : \mathcal{D} \to M$ be the flow of V. Then $\Theta_t : M_t = \{p \in M \ : \ (t,p) \in \mathcal{D}\} \to M_{-t}$ is a diffeomorphism, and $(\Theta_t)_{*p} : T_pM \xrightarrow{\cong} T_{\Theta_t(p)}M$ is an isomorphism.



Define a section of the tangent bundle $\mathcal{L}_v W : M \to TM$ with $\pi \circ (\mathcal{L}_V W) = \mathrm{id}_M$. The value $(\mathcal{L}_V W)_p \in T_pM$ is termed the *Lie derivative* in the direction of V at p, and defined by

$$\begin{aligned}(\mathcal{L}_V W)_p &= \lim_{t\to 0}\left[\frac{1}{t}\left((\Theta_t)_*^{-1}(W_{\Theta_t(p)}) - W_p\right)\right]\\ &= \lim_{t\to 0}\left[\frac{1}{t}\left(W_p - (\Theta_t)_*^{-1}(W_{\Theta_t(p)})\right)\right]\\ &= \frac{d}{dt}\bigg|_{t=0}(\Theta_t)_*^{-1}(W_{\Theta_t(p)})\end{aligned}$$

Lemma 2.5.2. If $V, W \in \Gamma(TM)$, then $\mathcal{L}_V W \in \Gamma(TM)$.

Proof: Let $p \in M$, and (U, φ) be a coordinate chart containing p. Let $J_0 \ni 0$ be an open interval and $U_0 \subset U$ open such that the flow of V is $\Theta : J_0 \times U_0 \to U$. In these coordinates, $\Theta(t,q) = (\Theta^1(t,q), \dots, \Theta^n(t,q))$ is smooth in t and q. Moreover,

$$(\Theta_t)_*^{-1} = (\Theta_{-t})_* : T_{\Theta_t(q)}M \to T_qM$$

The matrix for this line map (with respect to these coordinate vector fields as bases) is

$$\frac{\partial}{\partial x^i}\Theta^i(-t, \Theta(t,q)) \qquad \Theta_{-t} : \underbrace{(x^1, \dots, x^n)}_{=\,\Theta_t(q)} \mapsto \underbrace{(\Theta^1(-t, x_1, \dots, x_n), \dots, \Theta^n(-t, x^1, \dots, x^n))}_{=\,q}$$

Then this is smooth in t and q, since Θ is smooth in t and q. A vector in the basis is given by

$$W_{\Theta_t(p)} = W^j(\Theta(t,q))\,\frac{\partial}{\partial x^j}\bigg|_{\Theta_t(q)} \implies (\Theta_{-t})_* W_{\Theta_t(q)} = \frac{\partial \Theta^i}{\partial x^j}(-t\Theta(t,q))W^j(\Theta(t,q))\,\frac{\partial}{\partial x^i}\bigg|_q$$

is smooth. Further, $(\mathcal{L}_V W)_p = \frac{d}{dt}|_{t=0}$ of the value above is smooth in q. Hence $\mathcal{L}_V W$ is smooth. ■

Remark 2.5.3. It is intuitively clear that to compute $\mathcal{L}_V W$ at p, we need to know both W and V, not just at p, but in a neighborhood of p. We'll see this explicitly next.

Theorem 2.5.4. Let $V, W \in \Gamma(TM)$. Then $\mathcal{L}_V W = [V, W]$.

Proof: Let $\mathcal{R}(V) = \{\text{regular points of } V\} = \{p \in M \ : \ V_p \neq 0\}$. Since V is continuous, $\mathcal{R}(V)$ is open in M. Also, $\overline{\mathcal{R}(V)} = \mathrm{supp}(V)$.

The first step is to show that $\mathcal{L}_V W = [V, W]$ in $\mathcal{R}(V)$. To prove this, we use the canonical form theorem. Let $p \in \mathcal{R}(V)$, so there exist local coordinates near p for which $V = \frac{\partial}{\partial x^1}$. In these coordinates, the flow of

V is $\Theta_t(x^1, \ldots, x^n) = (x^1 + t, x^2, \ldots, x^n) = (y^1, \ldots, y^n)$. Also, $(\Theta_t)_* = \frac{\partial y^i}{\partial x^i}$ is the identity matrix. Next,

$$(\Theta_t)_*^{-1}(W_{\Theta_t(x)}) = (\Theta_t)_*^{-1}\left[W^j(x^1+t, x^2, \ldots, x^n)\ \frac{\partial}{\partial x^i}\bigg|_{\Theta_t(p)}\right] = W^j(x^1+t, x^2, \ldots, x^n)\ \frac{\partial}{\partial x^j}\bigg|_x$$

$$(\mathcal{L}_V W)_* = \frac{d}{dt}\bigg|_{t=0}(\Theta_t)_*^{-1}(W_{\Theta_t(x)}) = \frac{\partial}{\partial x^i}M^j(x^1, \ldots, x^n)\ \frac{\partial}{\partial x^j}\bigg|_x$$

$$[V,W]_x = \left[\frac{\partial}{\partial x^1}, W\right] = \left[\frac{\partial}{\partial x^1}, W^j(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}\right] = \frac{\partial}{\partial x^i}W^j(x^1, \ldots, x^n)\frac{\partial}{\partial x^j} = \mathcal{L}_V W$$

So $\mathcal{L}_V W = [V,W]$ on $\mathcal{R}(V)$.

The next step is to note that by continuity, the same holds for $\overline{\mathcal{R}(V)}$.

The final step is to let $E = W \setminus \overline{\mathcal{R}(V)}$. The set E is open in M, and if $p \in E$, then there exists an open neighborhood W of p on which $V = 0$. Then $(\Theta_t)_* : T_pM \to T_pM$ is the identity map for all $t \in \mathbb{R}$, and $\Theta_t : W \to W$ is also the identity, so $(\Theta_t)_*^{-1}(W_{\Theta_t(p)}) = W_p$ for all t. Hence $(\mathcal{L}_V W)_p = 0_p$, so $[V,W]_p = [0,W]_p = 0_p$, completing the proof. ■

Corollary 2.5.5. Let $X, Y, Z \in \Gamma(TM)$ and $f \in C^\infty(M)$. Then:

1. $\mathcal{L}_X Y = -\mathcal{L}_Y X$
2. $\mathcal{L}_X(fY) = (Xf)Y + f\mathcal{L}_X Y$
3. $\mathcal{L}_X(\mathcal{L}_Y Z) - \mathcal{L}_Y(\mathcal{L}_Y Z) - \mathcal{L}_{[X,Y]}Z = 0$
4. $\mathcal{L}_X[Y,Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$

Proof: The proof of **1.** is trivial, and **2.** and **3.** were proved on the assignment. As for **4.**, we simply consider the following calculation:

$$[X,[Y,Z]] = -[Z,[X,Y]] - [Y,[Z,X]] = [[X,Y],Z] + [Y,[X,Z]]$$

■

Definition 2.5.6. Vector fields $X, Y \in \Gamma(TM)$ are said to *commute* if $[X,Y] = 0$, which holds if and only if $X(Yf) - Y(Xf) = 0$ for all $f \in C^\infty(M)$.

For example, the following vector fields commute:

$$\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] = 0$$

Definition 2.5.7. A vector field Y is termed *invariant* under the flow Θ of $Y \in \Gamma(TM)$ if $((\Theta_t)_*)_p Y_p = Y_{\Theta_t(p)}$. That is, Y is invariant under the flow Θ of Y if the flow pushes Y forward onto itself.

Lemma 2.5.8. Let $F : M \to N$ be smooth, with $X \in \Gamma(TM)$, $Y \in \Gamma(TN)$, and Θ the flow of X and η the flow of Y. Then:

1. If X, Y are F-related, then for all $t \in \mathbb{R}$, $F(M) \subset N_t$, and $\eta_t \circ f = f \circ \Theta_t$ on M_t. That is,

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ \Theta_t \downarrow & & \downarrow \eta_t \\ M_{-t} & \xrightarrow[F]{} & N_{-t} \end{array} \qquad \text{commutes}$$

2. Conversely, if for each $p \in M$ there exists $\epsilon > 0$ such that $(\eta_t \circ F)(p) = (F \circ \Theta_t)(p)$ for all $|t| < \epsilon$, then X and Y are F-related.

Proof: For **1.**, let $\mathcal{D}^{(p)}$ be the domain of $\Theta^{(p)}$. Observe that

$$(\eta_t \circ F)(p) = (F \circ \Theta_t)(p) \iff \eta^{F(p)}(t) = F \circ \Theta^{(p)}(t)$$

Next, let $\gamma : \mathcal{D}^{(p)} \to N$ be given by $\gamma = F \circ \Theta^{(p)}$. We'll show that $\gamma = \eta^{F(p)}$. We first observe that

$$\gamma'(t) = (F \circ \Theta^{(p)})'(t) = (F_*)(\Theta^{(p)\prime}(t)) = (F_*)(X_{\Theta^{(p)}(t)}) = Y_{F\circ\Theta^{(p)}(t)} = Y_{\gamma(t)}$$

Hence γ is an integral curve of Y, with $\gamma(0) = F \circ (\Theta^{(p)}(0)) = F(p)$, so $\gamma = \eta^{F(p)}$ by uniqueness.

For **2.**, suppose that $\eta^{F(p)}(t) = (F \circ \Theta^{(p)})(t)$ for all $|t| < \epsilon$. Then

$$F_*(X_p) = F_*(\Theta^{(p)\prime}(0)) = (F \circ \Theta^{(p)})'(0) = (\eta^{F(p)})'(0) = Y_{F(p)}$$

This implies that X and Y are F-related. ■

Proposition 2.5.9. Let $V, W \in \Gamma(TM)$. Let Θ, ψ be the flows of V, W, respectively. Then equivalently:

1. $[V, W] = 0$
2. $\mathcal{L}_V W = 0$
3. $\mathcal{L}_W V = 0$
4. W is invariant under the flow of V
5. V is invariant under the flow of W
6. For each $p \in M$, if one of $(\Theta_t \circ \psi_s)(p)$ or $(\psi_s \circ \Theta_t)(p)$ are defined, then both are defined and are equal

Proof: The directions **1.** $\iff$ **2.** $\iff$ **3.** are clear. So we first suppose that **4.** holds. Then $W_{\Theta_t(p)} = (\Theta_t)_* W_p$ for $(t, p) \in \mathcal{D} =$ the domain of Θ. Hence $(\Theta_t)_* W_{\Theta_t(p)} = W_p$ and is independent of t. Further,

$$(\mathcal{L}_V W)_p = \frac{d}{dt}\bigg|_{t=0} (\Theta_t)_* W_{\Theta_t(p)} = 0$$

This shows that **4.** $\Longrightarrow$ **2.**. Similarly, **5.** $\Longrightarrow$ **3.**.

Next we suppose that **2.** holds. Define $X(t) = (\Theta_t)_* W_{\Theta_t(p)} \in T_pM$ to be a smooth curve in a fixed vector space, with

$$\begin{aligned}
X'(t_0) &= \frac{d}{dt}\bigg|_{t=t_0} (\Theta_{-t})_* W_{\Theta_t(p)} \\
&= \frac{d}{ds}\bigg|_{s=0} (\Theta_{-t_0-s_0})_* W_{\Theta_{s+t_0}(p)} \\
&= \frac{d}{ds}\bigg|_{s=0} (\Theta_{-t_0})_* (\Theta_{-s})_* W_{\Theta_s(\Theta_{t_0})(p)} \\
&= (\Theta_{-t_0})_* \frac{d}{ds}\bigg|_{s=0} (\Theta_{-s})_* W_{\Theta_s(\Theta_{t_0})(p)} \\
&= (\Theta_{-t_0})_*((\mathcal{L}_V W)_{\Theta_t(p)}) \\
&= 0
\end{aligned}$$

So $X'(t) = 0$ for all t, implying that $X(t) = X(0)$, so $(\Theta_T)_* W_p = W_{\Theta_t(p)}$. Hence **2.** $\Longrightarrow$ **4.**. It remains to show that **3.** $\Longrightarrow$ **5.**.

Let M_s be the domain of ψ_s. The statement **5.** says that $(\varphi_s)_* V_p = V_{\psi_t}(p)$, which also means that $V|_{M_s}$ is ψ_s-related to $V|_{M_{-s}}$. By the previous lemma, this is equivalent to $(\Theta_t \circ \psi_s)(p) = (\psi_s \circ \Theta_t)(p)$, whenever $\Theta_t(p) \in M_s$. ■

Theorem 2.5.10. [Frobenius]
Let M^n be a smooth manifold. Let $V_1, \dots, V_k$ be smooth linearly independent vector fields on an open subset $U \subset M$. Then equivalently:

1. there exists local coordinates $(u^1, \dots, u^n)$ on a neighborhood of each $p \in U$ such that $V_i = \frac{\partial}{\partial u^i}$ for each $i = 1, 2, \dots, k$.

2. $[V_i, V_k] = 0$ for all i, j

<u>*Proof:*</u> **1.** $\Longrightarrow$ **2.** This follows as coordinates of vector fields must commute.

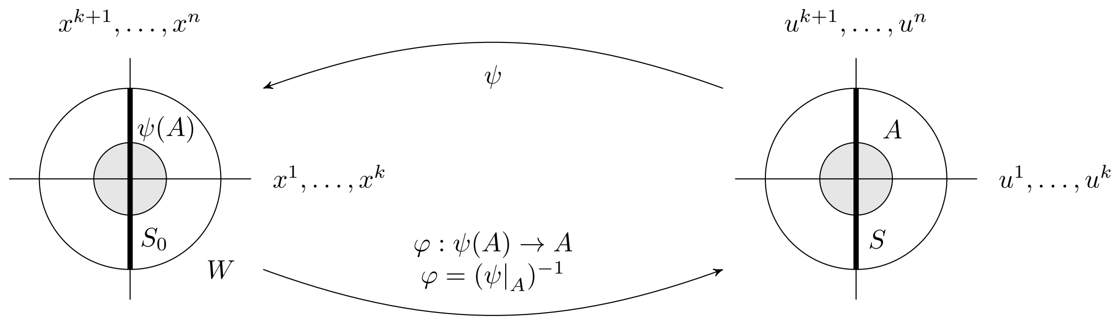
2. $\Longrightarrow$ **1.** Let $p \in U$, and choose local coordinates $x^1, \dots, x^n$ centered at p. The first claim is that by relabelling coordinates,

$$\left(V_1|_p, \dots, V_k|_p, \left.\frac{\partial}{\partial x^{k+1}}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right)$$

is a local basis of T_pM. This follows by letting Θ_i be a flow. Our second claim is that there exists an $\epsilon > 0$ and a neighborhood $W \ni p$ such that $\Theta_k|_{t_k} \circ \Theta_{k-1}|_{t_{k-1}} \circ \cdots \circ \Theta_1|_{t_1}$ is defined on W and maps W into U, whenever $|t_1|, \dots, |t_k| < \epsilon$. This claim holds by choosing $\epsilon_1 > 0$ and $U_1 \subset U$ a neighborhood of p. Then, choose $\epsilon_j > 0$ and $U_j \subset U_{j+1}$ such that $\Theta_j : (-\epsilon, \epsilon) \times U_j = U_{j-1}$. Now proceed as in the proof of the canonical form theorem.

Define the following sets and maps:

$$S_0 = W \cap \{x^{k+1} = \cdots = x^n = 0\}$$
$$S = \{(u^{k+1}, \dots, u^n) \ : \ (0, \dots, 0, u^{k+1}, \dots, u^n) \in W\}$$



$$\begin{array}{rrcl} \psi : & (-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon) \times S & \to & U \\ & (u^1, \dots, u^n) & \mapsto & (\Theta_k)_{u^k} \circ \cdots \circ (\Theta_1)_{u^1}(0, \dots, 0, u^{k+1}, \dots, u^n) \end{array}$$

We will show that ψ is a diffeomorphism on an open subset of its domain, and ψ^{-1} will be the required chart. The first step is to show that for all $i = 1, \dots, k$,

$$(\psi_*)_q \left(\left.\frac{\partial}{\partial u^i}\right|_q \right) = V_i|_{\psi(q)}$$

To show this, we first let $q \in W$, for which

$$\begin{aligned} \psi_* \left(\left.\frac{\partial}{\partial u^i}\right|_q \right) f &= \left.\frac{\partial}{\partial u^i}\right|_q f(\psi(u^1, \dots, u^n)) \\ &= \left.\frac{\partial}{\partial u^i}\right|_q f((\Theta_k)_{u^k} \circ \cdots \circ (\Theta_1)_{u^1}(0, \dots, 0, u^{k+1}, \dots, u^n)) \\ &= \left.\frac{\partial}{\partial u^i}\right|_q f((\Theta_i)_{u^i} \circ (\Theta_k)_{u^k} \circ \cdots \circ \widehat{(\Theta_i)_{u^i}} \circ \cdots \circ (\Theta_1)_{u^1}(0, \dots, 0, u^{k+1}, \dots, u^n)) \\ &= V_i|_{\psi(q)} f \end{aligned}$$

Since for all $x \in M$, the map $t \mapsto (\Theta_t)(x)$ is an integral curve of V_0, we have that $\psi_*(\frac{\partial}{\partial u^i}) = V_i$ for all $i = 1, \ldots, k$, which completes the first step. For the second step, we first note that

$$\psi(0, \ldots, 0, u^{k+1}, \ldots, u^n) = (0, \ldots, 0, u^{k+1}, \ldots, u^n)$$

Hence, in particular, $(\psi_V)(\frac{\partial}{\partial u^i}|_0) = \frac{\partial}{\partial x^i}\Big|_p$ for $i = k+1, \ldots, n$. Therefore ψ_* takes

$$\left\{ \frac{\partial}{\partial u^1}\bigg|_0, \ldots, \frac{\partial}{\partial u^n}\bigg|_0 \right\} \qquad \text{to} \qquad \left\{ V_1|_p, \ldots, V_k|_p, \frac{\partial}{\partial x^{k+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

Note that this is a basis, and so ψ_* is invertible at 0. Hence by the inverse function theorem, there exists an open neighborhood A of 0 such that $\psi(A)$ is open in W, and $\psi|_A : A \to \psi(A)$ is a diffeomorphism. Then ψ is a chart for M near p in which $V_i = \frac{\partial}{\partial u^i}$, completing the proof. ■

2.6 Differential forms and tensors

Definition 2.6.1. Let $M^n \ni p$ be a smooth manifold. Consider the dual space

$$(T_pM)^* = T_p^*M = L(T_pM, \mathbb{R})$$

which is the space of linear maps, and termed the *cotangent space*. Further, the space

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

is termed the *cotangent bundle* of M. We note that T^*M is a smooth $2n$-manifold.

Definition 2.6.2. A smooth map $\alpha : M \to T^*M$ such that $\pi \circ \alpha = \mathrm{id}_M$ is termed a (smooth) *1-form*, or a *covector field* on M, where the projection map is defined by

$$\begin{array}{rccc} \pi : & T^*M & \to & M \\ & (p, \alpha_p) & \mapsto & p \end{array}$$

such that $\alpha(p) = \alpha_p \in T_p^*M$ for all $p \in M$.

Definition 2.6.3. Let (U, φ) be a chart for M. Define $\lambda^i : U \to T^*U$ so that $\pi \circ \lambda^i = \mathrm{id}_U$ by

$$\underset{T_p^*M}{\underset{\Cap}{\lambda_p^i}} \left(\underset{T_pM}{\underset{\Cap}{\frac{\partial}{\partial x^j}\Big|_p}} \right) = \delta_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So λ^i is a 1-form on U, and the set $\{\lambda^1|_p, \ldots, \lambda^n|_p\}$ is termed the *dual basis* of $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$.

The chart (U, φ) induces a dual chart $(\tilde{U}, \tilde{\varphi})$ of T^*M as follows:

$$\begin{aligned} \tilde{U} &= \pi^{-1}(U) \\ \tilde{\varphi}(p, \alpha_p) &= \tilde{\varphi}\left(p, \sum_{k=1}^n \alpha_k \lambda^k|_p \right) = (\varphi(p), \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{2n} \end{aligned}$$

Hence a 1-form (which is a dual object on a vector field) on U can be expressed as $\alpha = \sum_k \alpha_k \lambda^k$, where the α_k are smooth functions on U.

Remark 2.6.4. There exists a pairing between $\Gamma(TM) \times \Gamma(T^*M)$ and $C^\infty(M)$ given by

$$\langle X, \alpha \rangle \mapsto \alpha(X) \in C^\infty(M) \quad \text{such that} \quad \alpha(X)(p) = \alpha_p X_p$$

Note that if $\alpha = \alpha_k \lambda^k$, and $X = x^\ell \frac{\partial}{\partial x^\ell}$, then $\alpha(X) = \alpha_k x^k$.

Definition 2.6.5. Given $f \in C^\infty(M)$, define $df \in \Gamma(T^*M)$ by $(df)(X) = X(f) \in C^\infty(M)$, the *differential* of f. Moreover,

$$\alpha\left(\frac{\partial}{\partial x^j}\right) = \alpha_k \lambda^k \left(\frac{\partial}{\partial x^j}\right) = \alpha_j$$

Observe also that

$$df = (df)_i \lambda^i \quad \text{and} \quad \frac{\partial f}{\partial x^j} = \frac{\partial}{\partial x^j} f = (df)\left(\frac{\partial}{\partial x^j}\right) = (\partial f)_j$$

So $df = \frac{\partial f}{\partial x^i}\lambda^i$ is a special case for $f = x^j \in C^\infty(M)$. Then $dx^j = \frac{\partial x^j}{\partial x^i}\lambda^i = \lambda^j$, so $\lambda^j = dx^j$.

Remark 2.6.6. The differential d has some important properties.

- d is $\mathbb{R}$-linear
- $d(fg) = g(df) + f(dg)$, or equivalently, $X_p(fg) = X_p(f)g(p) + f(p)(X_p g)$

Observe also that we may pull back 1-forms. So given $F : M \to N$ with $(F_*)_p : T_pM \to T_{F(p)}N$, the dual map is

$$((F_*)_p)^* = F^* : T^*_{F(p)}N \to T^*_pM \quad \text{with} \quad (F^*)(\alpha_{F(p)})(X_p) = \alpha_{F(p)}((F_*)_p X_p) \quad \text{for} \quad \alpha \in \Gamma(T^*N)$$

We can also define $F^*\alpha$ as a 1-form on M unambiguously: $(F^*_\alpha)_p = F^*(\alpha_{F(p)})$. The pullback has the following properties:

$$\begin{aligned} F^*(dh) &= d(h \circ F) = d(F^*h) && \text{for } h \in C^\infty(M) \\ F^*(h\alpha) &= (h \circ F)(F^*\alpha) = (F^*h)(F^*\alpha) && \text{for } \alpha \in \Gamma(T^*N),\ h \in C^\infty(N) \end{aligned}$$

In local coordinates $x^1, \dots, x^n$ an M and $y^1, \dots, y^k$ an N, with $\alpha = \alpha^i dy^i$, we have that

$$F^*\alpha = (\alpha_i \circ F) d(y^i \circ F) = \alpha_i(F(x^1, \dots, x^n)) \frac{\partial F^i}{\partial x^j} dx^j$$

Definition 2.6.7. Let us formally define what it means to be a tensor. A tensor T of type (k, ℓ) is given by

$$\begin{aligned} T^\ell_k(T_pM) &= \underbrace{T^*_pM \otimes \cdots \otimes T^*_pM}_{k \text{ times}} \otimes \underbrace{T_pM \otimes \cdots \otimes T_pM}_{\ell \text{ times}} \\ &= \text{space of type } (k,\ell)\text{-tensors at } p \\ &= \text{space of multilinear maps} \end{aligned}$$

Example 2.6.8. For some small examples, consider

$$\begin{aligned} T^1_0(T_pM) &= T^*_pM & T^k_\ell(M) &= \bigsqcup_{p \in M} T^k_\ell(T_pM) \\ T^0_1(T_pM) &= T_pM & &= \text{bundle of } (k,\ell)\text{-tensors on } M \\ T^0_0(T_pM) &= \mathbb{R} & &= \text{a smooth manifold of dimension } n + n^{k+\ell} \end{aligned}$$

Remark 2.6.9. Suppose that $\sigma_p \in T^k_\ell(T_pM)$. Then for

$$\begin{array}{rcl} (x_1)_p, \dots, (x_k)_p & \in & T_pM \\ (\alpha_1)_p, \dots, (\alpha_\ell)_p & \in & T^*_pM \end{array} \quad \text{we write} \quad \sigma_p((x_1)_p, \dots, (x_k)_p, (\alpha_1)_p, \dots, (\alpha_\ell)_p)$$

A (k, ℓ)-tensor on M is a smooth map $\sigma : M \to T^{k,\ell}M$ such that $\pi \circ \sigma = \text{id}$ and $\pi(p, \sigma_p) = p$. In local coordinates $(x^1, \dots, x^n)$ for M, we write

$$\sigma = \sigma_{i_1 \cdots i_k}{}^{j_1 \cdots j_\ell} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}}$$

where the σ elements with subscripts and superscripts are smooth functions defined on the domain of the chart. Next, if we let $\beta^i, \alpha^i \in \Gamma(T^*M)$ and $Y_i, X_i \in \Gamma(TM)$, then

$$\bigotimes_{i=1}^{k} \alpha^i \bigotimes_{j=1}^{\ell} X_j(Y_1, \dots, Y_k, \beta^i, \dots, \beta^\ell) = \alpha^1(Y_1) \cdots \alpha^k(Y_k) \beta^1(X_1) \cdots \beta^\ell(X_\ell) \in C^\infty(M)$$

Definition 2.6.10. Now we will introduce smooth differential forms. Let $0 \leqslant k \leqslant n$. Define

$$\begin{aligned} \Lambda^k(T_p^*M) &= \text{the space of } k\text{-forms at } p \\ &= \text{the space of } k\text{-linear, alternating (totally skew-symmetric) maps} \\ &\subset T_0^k(T_p^*M) \end{aligned}$$

An element in $\Lambda^k(T_p^*M)$ is given by $\alpha_p = T_pM \times \cdots \times T_pM \to \mathbb{R}$, where the product is taken k times. As before, we may also define the bundle of k-forms on M by

$$\Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda^k(T_p^*M)$$

This is a smooth manifold of dimension $n + \binom{n}{k}$, with the projection map $\pi : \Lambda^k(T^*M) \to M$. A *$k$-form* α on M is a smooth map $\alpha : M \to \Lambda^k(T^*M) \to M$ such that $\pi \circ \alpha = \text{id}_M$. In local coordinates, this map may be expressed as

$$\alpha = \alpha_{i_1 \cdots, _k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where the coefficients α on the right side are locally defined smooth functions that alternate in i_j.

If $\alpha^1, \dots, \alpha^k \in \Gamma(T^*M)$, then $\alpha^1 \wedge \cdots \wedge \alpha^k$ is a k-form given by $(\alpha^1 \wedge \cdots \wedge \alpha^k)(x_1, \dots, x_k) = \det(\alpha^i(x_j))$, which is a locally defined smooth function.

Definition 2.6.11. Define the infinite-dimensional space of all k-forms on M to be

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M))$$

This is a real vector space, as well as a $C^\infty(M)$-module. For small k we have

$$\Omega^1(M) = \Gamma(T^*M) \qquad\qquad \Omega^0(M) = C^\infty(M)$$

Definition 2.6.12. There exists a product on forms, called the *wedge product*, defined by

$$\wedge : \Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M)$$

The wedge product has the following properties:

- $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$
- $\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha$
- $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$ for $\alpha \in \Omega^k(M)$, $\beta \in \Omega^\ell(M)$

The ring of all the k-form bundles is given by

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

This is an associative algebra with identity, and so is a $C^\infty(M)$-module. In local coordinates,

$$\begin{aligned} \alpha &= \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ \beta &= \beta_{j_1 \cdots j_\ell} dx^{j_1} \wedge \cdots \wedge dx^{j_\ell} \end{aligned} \qquad \Longrightarrow \qquad \alpha \wedge \beta = \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_\ell}$$

Definition 2.6.13. Let $F : M \to N$ be smooth, $\alpha \in \Omega^k(N)$. Define the *pullback* $F^*\alpha \in \Omega^k(M)$ by

$$(F^*\alpha)_p((x_1)_p, \dots, (x_k)_p) = \alpha_{F(p)}((F^*)_p(x_i)_p, \dots, (F_*)_p(x_k)_p)$$

The pullback has some important properties:

· F^* is linear
· $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$
· If $(x^1, \dots, x^k)$ and $(y^1, \dots, y^m)$ are local coordinates on M and N, respectively, then $\alpha = \alpha_{i_1 \cdots i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$

The second property states that the pullback is a homomorphism of algebras. Using these local coordinates,

$$\begin{aligned}(F^*\alpha) &= (\alpha_{i_1 \cdots i_k} \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F) \\ &= (\alpha_{i_1 \cdots i_k} \circ F) \left(\frac{\partial F^{i_1}}{\partial x^{j_1}} dx^{j_1} \right) \wedge \cdots \wedge \left(\frac{\partial F^{i_k}}{\partial x^{j_k}} dx^{j_k} \right)\end{aligned}$$

Definition 2.6.14. Let M be a smooth manifold. There exist unique linear maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all $k = 0, 1, \dots, n$, called *exterior derivatives*, such that

· if $f \in \Omega^0(M) = C^\infty(M)$, then $df \in \Omega^1(M)$, where $(df)(X) = Xf$
· $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge (d\beta)$
· $d^2 = 0$, that is, $d(d\alpha) = 0$ for all $\alpha \in \Omega^k(M)$

In local coordinates, for $\alpha = \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, we have

$$d\alpha = \frac{\partial \alpha_{i_1 \cdots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = (d\alpha_{i_1 \cdots i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

However, there exists a coordinate-free definition of d. It is given by:

$$(d\alpha)(X_1, \dots, X_{k+1}) = \sum_{1 \leqslant i \leqslant k+1} (-1)^{i-1} X_i(\alpha(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})) + \sum_{1 \leqslant i \leqslant j \leqslant k+1} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+n})$$

3 Introduction to Riemannian geometry

3.1 Connections on the tangent bundle

This section may also be titled "affine connections" or "covariant derivatives on the tangent bundle."

Remark 3.1.1. Recall that if $X \in \Gamma(TM)$ and $f \in C^\infty(M)$, then $(Xf) \in C^\infty(M)$, where $(Xf)_p = X_p f$, which is the directional derivative of f at p in the direction $X_p \in T_pM$.

In particular, $X_p f = (Xf)_p$ depends only on the value of f at p. By contrast, if $Y \in \Gamma(TM)$, we defined $\mathcal{L}_X Y \in \Gamma(TM)$, but $(\mathcal{L}_X Y)_p$ does not depend on the value of X at p, but rather on the values of X in a neighborhood of p, as $\mathcal{L}_X Y = [X, Y]$. Hence it does not make sense to call this a directional derivative.

We wil now introduce more structure on M to be able to take directional derivatives of vector fields.

Remark 3.1.2. Consider $M = \mathbb{R}^n$ with its metric space structure. A question one may ask is what are the nicest (and what defines the nicest?) curves in M?

It is intuitive that a constant speed straight line in $\mathbb{R}^n$ is simple (hence nice), with

$$\gamma(t) = p + tv \qquad \gamma'(t) = v \qquad \gamma''(t) = 0$$

These curves are characterized as being with zero acceleration. Now suppose that M is a manifold and $\gamma : I \to M$ is a smooth curve. What should be the "acceleration" of γ at $\gamma(t_0)$, for $t_0 \in I$? As velocity of γ at t_0 is given by $\gamma'(t_0) \in T_{\gamma(t_0)}M$, we could naively say that

$$\gamma''(t_0) = \lim_{h \to 0} \left[\frac{\gamma'(t_0 + h) - \gamma'(t_0)}{h} \right]$$

This approach works in $\mathbb{R}^n$, as $T_p\mathbb{R}^n$ is canonically isomorphic to $\mathbb{R}^n$ for all p, but in general this does not make sense, as $\gamma'(t_0+h)$ and $\gamma'(t_0)$ are vectors in different tangent spaces, so subtracting them is not possible. A canonical isomorphism does not exist in general.

Definition 3.1.3. Let M be a smooth manifold. A *connection* (or *covariant derivative*) ∇ on M is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \quad \text{with} \quad \nabla_X Y = \nabla(X, Y)$$

This satisfies the following conditions:

1. linearity over $C^\infty(M)$ in X, i.e. for all $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2, Y \in \Gamma(TM)$,

$$\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y$$

2. linearity over $\mathbb{R}$ in Y, i.e. for all $a_1, a_2 \in \mathbb{R}$ and $X, Y_1, Y_2 \in \Gamma(TM)$,

$$\nabla_X(a_1Y_1 + a_2Y_2) = a_1\nabla_X Y_1 + a_2\nabla_X Y_2$$

3. the Leibniz rule, i.e. for all $f \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$,

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

Remark 3.1.4. Connections may be more general, as they can be defined on different spaces, such as vector bundles. However, it is not yet obvious that such things exist. They do indeed exist (as proved later), and there is even a preferred connection, called the Levi-Civita connection.

Lemma 3.1.5. Let ∇ be a connection on the tangent bundle on M. If $X, \in \Gamma(TM)$, then $(\nabla_X Y)_p$ only depends on values of X and Y in a neighborhood of p. That is, it is a local operator.

<u>*Proof:*</u> Suppose $X = \tilde{X}$ on a neighborhood U of p. Then $X - \tilde{X} = 0$ on U. To show $(\nabla_X Y)_p = (\nabla_{\tilde{X}} Y)_p$, we can show that $(\nabla_{X-\tilde{X}} Y)_p = 0$ by the second condition of the definition above. It is enough to show if $X = 0$ on U, then $(\nabla_X Y)_p = 0$. Let φ be a smooth bump function such that $\varphi = 1$ at p and $\text{supp}(\varphi) \cong U$. Then, as $\varphi X = 0 = 0 \cdot \varphi X$, we have

$$\varphi(\nabla_X Y) = \nabla_{\varphi X} Y = \nabla_{0\cdot\varphi X} Y = 0 \cdot \nabla_{\varphi X} Y = 0$$

We evaluate this at p to get $\varphi(p)(\nabla_X Y)_p = 0_p$, so $(\nabla_X Y)_p = 0$. Similarly, if $Y = \tilde{Y}$ on U, then $(\nabla_X Y)_p = (\nabla_X \tilde{Y})_p = 0$, which occurs if and only if $(\nabla_X(Y - \tilde{Y}))_p = 0$. It is now enough to show that if $Y = 0$ on U, then $(\nabla_X Y)_p = 0$. It remains to show that if $Y = 0$ on U, then $(\nabla_X Y)_p = 0$.

To do this, we use the same φ, recalling that $\varphi Y = 0$ everywhere, and that $\varphi Y = 0 \cdot \varphi Y$. Then:

$$0 = 0 \cdot \nabla_X(\varphi Y) = \nabla_X(0 \cdot \varphi Y) = \nabla_X(\varphi Y) = (X\varphi)Y + \varphi\nabla_X Y = (X\varphi)_p Y_p + \varphi(p)(\nabla_X Y)_p = (\nabla_X Y)_p$$

This proves the desired claim, and proves the lemma. ■

Lemma 3.1.6. $(\nabla_X Y)_p$ depends only on X at p.

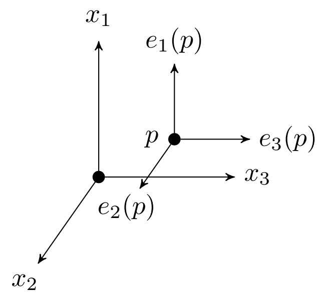
<u>*Proof:*</u> Let $X_p = \tilde{X}_p$. We would like to show that $(\nabla_X Y)_p = (\nabla_{\tilde{X}} Y)_p$. By the properties of ∇, it is enough to show that $(\nabla_X Y)_p = 0$ if $X_p = 0$. By the previous lemma, we may mork in local coordinates, so $X = a^i \frac{\partial}{\partial x^i}$ for a_i local smooth functions, and $a^i(p) = 0$ for all i. Then

$$(\nabla_X Y)_p = \left(\nabla_{\sum a_i \frac{\partial}{\partial x^i}} Y\right)_p = \sum \left(a_i \nabla_{\frac{\partial}{\partial x^i}} Y\right)_p = \sum a_i(p) \left(\nabla_{\frac{\partial}{\partial x^i}} Y\right)_p = 0$$

The second equality follows from linearity over $C^\infty(M)$. ■

Although this helps, we still have not seen that $\nabla_X Y$ is actually a derivative.

Example 3.1.7. Let $M = \mathbb{R}^n$. On $\mathbb{R}^n$, there exists a global frame, which is a set of n smooth vector fields that are everywhere linearly independent. We have that $e_1, \dots, e_n \in \Gamma(T\mathbb{R}^n)$ is a global frame, where $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.



Here, $e_i(p) = (p, e_i)$, which is the ith standard basis vector of $\mathbb{R}^n$. That is, $e_i = \frac{\partial}{\partial x^i}$ with $(x^1, \dots, x^n)$ the identity chart. We now define a connection ∇ on $\mathbb{R}^n$ as follows:

· Any $X \in \Gamma(T\mathbb{R}^n)$ can be written as $X = X^i e_i = X^i \frac{\partial}{\partial x^i}$
· Let $Y = Y^x e_k$, and define

$$\nabla_X Y = \nabla_X (Y^k e_k) = X(Y^k) e_k \in \Gamma(T\mathbb{R}^n)$$

We now must check that this actually defines a connection.

$$(f_1 X_1 + f_2 X_2)(h) = f_1(X_1 h) + f_2(X_2 h)$$
$$X(ch) = c(Xh)$$

The above holds for all $f_1, f_2 \in C^\infty(\mathbb{R}^n)$ and all $c \in \mathbb{R}$, so $\nabla_X Y$ is linear in X over $C^\infty(M)$ and linear in Y over $\mathbb{R}$.Now let $f \in C^\infty(\mathbb{R}^n)$, so $fY = (fY^k)e_k$. Then

$$\nabla_X(fY) = X(fY^k)e_k = (Xf)Y^k e_k + f(XY^k)e_k = (Xf)Y + f\nabla_X Y$$

So all three necessary conditions are satisfied, so ∇ is indeed a connection on $\mathbb{R}^n$, and is termed the *Euclidean* connection.

Remark 3.1.8. The above construction works whenever we have a global frame for our manifold. Such a manifold is termed *parallelizable*.

Remark 3.1.9. Let us consider what connections look like locally. Let $U \subset M$ be an open subset on which we have a local frame for M. That is, $E_1, \dots, E_n \in \Gamma(TU)$. So each $E_i : U \to TU$ is a map with $E_i(p) \in T_pU = T_pM$.

If $(x^1, \dots, x^n)$ are local coordinates, then $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ is a local frame, called a coordinate frame. However, not all coordinate frames are local frames.

Remark 3.1.10. Let ∇ be a connection on M, with $X, Y \in \Gamma(TU)$ such that $X = X^i E_i$ and $Y = Y^j E_j$, where $X^i, Y^j \in C^\infty(U)$ for all i, j. Then $\nabla_X Y$ is a vector field on U, which we may expand in terms of E_i:

$$\nabla_X Y = \underbrace{(\nabla_X Y)^k}_{\text{functions}} E_k = \nabla_{X^i E_i}(Y^j E_j) = X^i \nabla_{E_i}(Y^j E_j) = X^i(E_i(Y^j)E_j) + Y\nabla_{E_i} E_j = X(Y^j)E_j + X^i Y^j \nabla_{E_i} E_j$$

And $\nabla_{E_i} E_j = \Gamma^k_{ij} E_k$, where the Γ^k_{ij} are smooth functions on U (there are n^3 such functions). They are the Christoffel symbols of ∇ with respect to this local frame $\{E_1, \dots, E_n\}$, and are not components of a tensor.

Remark 3.1.11. Note that for the Euclidean connection in $\mathbb{R}^n$ in the standard global frame, $\Gamma^k_{ij} = 0$ for all i, j, k. However, in anoher local frame, The Christoffel symbols of the Euclidean connection need not vanish.

Consider $M = \mathbb{R}^2$ and ∇ the Euclidean connection. Then

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\} = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \quad \text{is the local frame} \qquad\qquad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0 \quad \text{for all } i, j$$

Now we choose another local frame, namely the polar coordinates. This gives

$$x = r\cos(\theta) \qquad\qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin(\theta) \qquad\qquad \theta = \arctan(y/x)$$

Therefore $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\}$ is a frame on $\mathbb{R} \setminus \{x = 0\}$. Then we have that

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} = \frac{\partial}{\partial r}$$

Next we consider the covariant derivative of the same field.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} &= \Gamma^1_{22}\frac{\partial}{\partial \theta} + \Gamma^2_{22}\frac{\partial}{\partial r} \\
&= \nabla_{\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}} \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} \right) \\
&= \frac{x}{r}\nabla_{\frac{\partial}{\partial x}} \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} \right) + \frac{y}{r}\nabla_{\frac{\partial}{\partial y}} \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} \right) \\
&= \frac{x}{r}\left(\frac{\partial}{\partial x}\left(\frac{x}{r}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\left(\frac{y}{r}\right)\frac{\partial}{\partial y} \right) + \frac{y}{r}\left(\frac{\partial}{\partial y}\left(\frac{x}{r}\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\left(\frac{y}{r}\right)\frac{\partial}{\partial y} \right)
\end{aligned}$$

We have the following simplifications:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{x}{r}\right) &= \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x}{r^2}\frac{x}{r} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3} \\
\frac{\partial}{\partial y}\left(\frac{y}{r}\right) &= \frac{x^2}{r^3} \\
\frac{\partial}{\partial x}\left(\frac{y}{r}\right) &= \frac{\partial}{\partial y}\left(\frac{x}{r}\right) = \frac{-xy}{r^3}
\end{aligned}$$

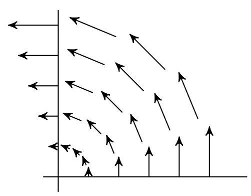
Hence

$$\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = \frac{x}{r}\left(\frac{y^2}{r^3}\frac{\partial}{\partial x} - \frac{xy}{r^2}\frac{\partial}{\partial y} \right) + \frac{y}{r}\left(\frac{-xy}{r^3}\frac{\partial}{\partial x} + \frac{x^2}{r^2}\frac{\partial}{\partial y} \right) = 0$$

This is not surprising, as the length and the direction of $\frac{\partial}{\partial r}$ do not change as we move in the $\frac{\partial}{\partial r}$ direction. However, $\frac{\partial}{\partial \theta}$ should not be zero, because the direction of $\frac{\partial}{\partial r}$ changes as we move in the $\frac{\partial}{\partial \theta}$ direction. Let us make sure of this:

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r} &= \nabla_{\frac{x}{r}\frac{\partial}{\partial x}+\frac{y}{r}\frac{\partial}{\partial y}}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\\
&= \frac{x}{r}\nabla_{\frac{\partial}{\partial x}}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+\frac{y}{r}\nabla_{\frac{\partial}{\partial y}}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\\
&= \frac{x}{r}\left(\frac{\partial}{\partial y}-0\right)+\frac{y}{r}\left(0-\frac{\partial}{\partial x}\right)\\
&= \frac{1}{r}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\\
&= \frac{1}{r}\frac{\partial}{\partial\theta}\\
&= \underbrace{\Gamma^{\theta}_{r\theta}}_{1/r}\frac{\partial}{\partial\theta}+\underbrace{\Gamma^{r}_{r\theta}}_{0}\frac{\partial}{\partial r}
\end{aligned}$$

Hence $\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial r}$ is never zero. The first quadrant of the field resembles the diagram below:



$$\frac{\partial}{\partial\theta} = (-y,x)$$

$$\left|\frac{\partial}{\partial\theta}\right| = \sqrt{x^2+y^2} = r$$

Next consider the vector field for which $\frac{1}{r}\frac{\partial}{\partial\theta} = -(\frac{y}{r},\frac{x}{r})$ has constant length and does not change as we move in the $\frac{\partial}{\partial r}$ direction. We may check this by noting that

$$\nabla_{\frac{\partial}{\partial r}}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\theta} = -\frac{1}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)=0$$

Proposition 3.1.12. Any smooth manifold has connections.

Proof: Cover M by coordinate charts $\{(U_\alpha,\varphi_\alpha)\}$. On each U_α, define a connection ∇^α by setting

$$\nabla^{\alpha}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = 0 \quad \forall\ i,j$$

Let $\{f_\alpha\}$ be a partition of unity subordinate to this open cover. Define a map

$$\nabla : \Gamma(TM)\times\Gamma(TM)\to\Gamma(TM) \quad \text{by} \quad \nabla_X Y = \sum_\alpha f_\alpha\left(\nabla^\alpha_X\right)\in\Gamma(TM)$$

To show that this is a connection, we need it to satisfy the three properties:

$$\nabla_{h_1X_2+h_2X_2}Y = \sum_\alpha f_\alpha\left(\nabla^\alpha_{h_1X_2+h_2X_2}Y\right) = \sum_\alpha f_\alpha\left(h_1\nabla^\alpha_{X_1}Y+h_2\nabla^\alpha_{X_2}\right) = h_1\nabla_{X_1}Y+h_2\nabla_{X_2}Y$$

$$\nabla_X\left(c_1Y_2+c_2Y_2\right) = \sum_\alpha f_\alpha\left(\nabla^\alpha_X\left(c_1Y_1+c_2Y_2\right)\right) = \sum_\alpha f_\alpha\left(c_1\nabla^\alpha_XY_1+c_2\nabla_XY_2\right) = c_1\nabla_XY_1+c_2\nabla_XY_2$$

$$\nabla_X(hY) = \sum_\alpha f_\alpha\nabla^\alpha_X(hY) = \sum_\alpha f_\alpha\left((Xh)Y+h\nabla^\alpha_XY\right) = (Xh)\left(\sum_\alpha f_\alpha\right)Y+h\left(\sum_\alpha f_\alpha\nabla^\alpha_XY\right) = (Xh)Y+h\nabla_XY$$

This is the desired result. ■

Note that the created connection is highly non-unique, and indeed, there are uncountably many connections on every manifold. So the natural question arises, what is (and is there) a best connection on TM? The answer depends on the context, on the manifold M. We will later see that for M a Riemannian manifold, there exists a natural connection, the Levi-Civita connection on X.

So far, we have shown that all manifolds have connections, but we still need to show that a connection is a differential derivative. So now we will show that for ∇ a connection on TM, here is an induced notion of covariant differentiation on any tensor bundles $T^k_\ell(TM)$.

Definition 3.1.13. Let $\alpha \in \Gamma(T^*M) = \Omega^1(M)$ and $X, Y \in \Gamma(TM)$. Define $\nabla_X \alpha$ to be the smooth 1-form satisfying

$$X(\alpha(Y)) = (\nabla_X \alpha)(Y) = \alpha(\nabla_X Y) \quad \implies \quad \nabla_X \alpha = X(\alpha(Y)) - \alpha(\nabla_X Y)$$

Further, given $\{E_1, \ldots, E_n\}$ a local form on M with $E_i \in \Gamma(TU)$, let $\{E^1, \ldots, E^n\}$ be the *dual coform* for M, so $E^i \in \Omega^1(U)$, and $E^i E_j = \delta^i_j$. Then

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k \quad \implies \quad \nabla_{E_i} E^k = C^k_{ij} E^j$$

This commes from the co-nature of the objects.

Proposition 3.1.14. For values as described above, $C^k_{ij} = -\Gamma^k_{ij}$.

<u>*Proof:*</u> Merely note that

$$\begin{aligned}
\left(\nabla_{E_i} E^k\right)(E_\ell) &= C^k_{ij} E^j E_\ell = C^k_{i\ell} \\
\left(\nabla_{E_i} E^k\right)(E_\ell) &= E_i(E^k(E_\ell) - E^k(\nabla_{E_i} E_\ell)) = 0 - E^k(\Gamma^j_{i\ell} E_j) = -\Gamma^k_{i\ell}
\end{aligned}$$

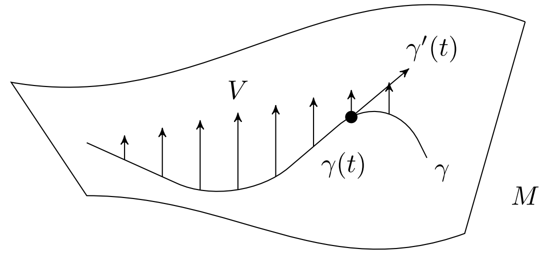
■

Remark 3.1.15. Let us generalize the previous definition. Let $\sigma \in \Gamma(T^k_\ell M)$ be a smooth (k,ℓ)-tensor with $X, Y_1, \ldots, Y_k \in \Gamma(TM)$ and $\alpha^1, \ldots, \alpha^\ell \in \Omega^1(M)$. Let $\nabla_X \sigma \in \Gamma(T^k_\ell M)$ be the (k,ℓ)-tensor defined by

$$\begin{aligned}
X(\sigma(Y_1, \ldots, Y_k, \alpha^1, \ldots, \alpha^\ell)) = (\nabla_X \sigma)(Y_1, \ldots, Y_k, \alpha^1, \ldots, \alpha^\ell) \\
+ \sum_i \sigma(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_k, \alpha^1, \ldots, \alpha^\ell) \\
+ \sum_j \sigma(Y_1, \ldots, Y_k, \alpha^1, \ldots, \nabla_X \alpha^j, \ldots, \alpha^\ell)
\end{aligned}$$

Let $\gamma : I \to M$ be a smooth curve on M, and let ∇ be a conection on M. Now we may define the "acceleration" of γ, which is the covariant derivative of the velocity $\gamma'(t)$ in the direction of $\gamma'(t)$.

Definition 3.1.16. Let $V \in \Gamma(TM)$. Then the *derivative* of V along the curve γ is $\nabla_{\gamma'(t)} V$, which is the covariant derivative of V in the direction of the tangent vector $\gamma'(t) \in T_{\gamma(0)} M$ at $\gamma(t) \in M$.



Notice that $\gamma'(t)$ is not a vector field for all of M. We get a tangent vector in $T_{\gamma(t)}M$ for all $t \in I$, only on the points of M that lie in the image on N.

Let $\gamma : I \to M$ be a smooth curve on M. Define $\gamma^*(TM)$ be the bundle of tangent spaces to M, over points on the image of φ. Then we have the following commutative diagram:

$$\begin{array}{ccc} \gamma^* * (TM) & & TM \\ \Big\downarrow \pi_{\gamma^*(TM)} & & \Big\downarrow \pi_{TM} \\ I & \xrightarrow{\ \gamma\ } & M \end{array} \qquad \begin{aligned} \gamma^*(TM) &= \bigsqcup_{t\in I} T_{\gamma(t)}M \\ \pi(t, X_{\gamma(t)}) &= t \end{aligned}$$

Definition 3.1.17. Define $\Gamma(\gamma^*(TM))$ to be the space of vector fields on M along γ. That is, $V \in \Gamma(\gamma^*(TM))$ is a smooth map $V : I \to \gamma^*(TM)$ such that $\pi \circ V = \mathrm{id}_I$ for $V_t \in T_{\gamma(T)}M$.

Lemma 3.1.18. Let $\gamma : I \to M$ be a curve on M. Then there exists a map $D_t : \Gamma(\gamma^*(TM)) \to \Gamma(\gamma^*(TM))$ called the *covariant derivative of vector fields* such that

- **a.** $D_t(aV + bW) = aD_tV + bD_tW$
- **b.** $D_t(fV) = \frac{df}{dt}V + fD_tV$
- **c.** If V is a restriction of γ of a vector field $\tilde{V} \in \Gamma(TM)$, then $(D_tV)(t) = (\nabla_{\gamma'(t)}\tilde{V}) \in T_{\gamma(t)}M$

The above holds for any $a, b \in \mathbb{R}$, $V, W \in \Gamma(\gamma^*(TM))$ and $f \in C^\infty(M)$.

Proof: In local coordinates, we have $V = V^k(t)\frac{\partial}{\partial x^k}|_{\gamma(t)}$. By part **b.**, we have that

$$\begin{aligned}
(D_tV)(t_0) &= \left(D_t\left(V^t\frac{\partial}{\partial x^k}\right)\right)(t_0) \\
&= \frac{dV^k}{dt}\,\frac{\partial}{\partial x^k}\bigg|_{\gamma(t_0)} + \left(V^k\left(D_t\frac{\partial}{\partial x^k}\right)\right)(t_0) \\
&= \frac{dV^k}{dt}\,\frac{\partial}{\partial x^k}\bigg|_{\gamma(t_0)} + V^k(t_0)\nabla^k_{\frac{dx^i}{dt}(t_0)\,\frac{\partial}{\partial x^i}|_{\gamma(t_0)}} dx^k \\
&= \frac{dV^k}{dt}(t_0)\,\frac{\partial}{\partial x^k}\bigg|_{\gamma(t_0)} + V^k(t_0)\frac{dx^i}{dt}(t_0)\nabla_{\frac{\partial}{\partial x^k}|_{\gamma(t_0)}}\frac{\partial}{\partial x^k} \\
&= \frac{dV^k}{dt}(t_0)\,\frac{\partial}{\partial x^k}\bigg|_{\gamma(t_0)} + V^k(t_0)\frac{dx^i}{dt}(t_0)\Gamma^m_{ik}(\gamma(t_0))\,\frac{\partial}{\partial x^m}\bigg|_{\gamma(t_0)} \\
&= \left(\frac{dV^k}{dt}(t_0) + \Gamma^k_{ij}(\gamma(t_0))\frac{dx^i}{dt}(t_0)V^k(t_0)\right)\frac{\partial}{\partial x^m}\bigg|_{\gamma(t_0)}
\end{aligned}$$

From this we conclude that if such a map D_t exists, then it would not be unique. Above we only showed that $(\nabla_tV)(t_0)$ has to be this vector field. To prove existence, in any coordinate chart define D_t by

$$D_t(V) = \left(\frac{dV^k}{dt} + \Gamma^k_{ij}(\gamma(t))\frac{dx^i}{dx}V^j\right)\frac{\partial}{\partial x^k}$$

This is a smooth vector field along γ, Moreover, on overlapping charts, the definitions agree, because we already proved uniqueness. ■

3.2 Geodesics and parallel transports

Definition 3.2.1. Let (M, ∇) be a manifold with a connection on its tangent bundle. Let $\gamma : I \to M$ be a smooth curve. The *acceleration* of γ at $\gamma(t_0)$ is defined to be

$$(D_t\gamma')(t_0) = \nabla_{\gamma''(t_0)}\gamma' = (\nabla_{\gamma'}\gamma')(t_0) \in T_{\gamma(t_0)}M$$

A curve γ is called a *geodesic* if its acceleration is zero for all $t \in I$.

Example 3.2.2. In $\mathbb{R}^n$ with the Euclidean connection, the geodesics are constant speed parametrized straight lines.

Theorem 3.2.3. Let (M, ∇) be a manifold with a connection, and $p \in M$, $X_p \in T_pM$. Then there exists an open interval $(-\epsilon, \epsilon) \subset \mathbb{R}$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. In other words, γ is a geodesic.

That is, given an initial point and an initial velocity, there exists a unique geodesic with those initial conditions, at least for a small time interval.

Proof: In local coordinates, recall that if $V \in \Gamma(\gamma^*(TM))$, then

$$(D_tV) = \left(\frac{dV^k}{dt} + \Gamma^k_{ij}\frac{dx^i}{dt}V^j\right)\frac{\partial}{\partial x^k}$$

Let $V = \gamma'$, the velocity vector field of γ. Then $V = \frac{dx^i}{dt}\frac{\partial}{\partial x^i}$ and $V^i = \frac{dx^i}{dt}$, and we get

$$D_t\gamma' = \left(\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}(\gamma(t))\frac{dx^i}{dt}\frac{dx^j}{dt}\right)\frac{\partial}{\partial x^k}$$

If γ is a geodesic, then in coordinates $\gamma(t) = (x^1(t), \ldots, x^n(t))$ has to satisfy the geodesic equations:

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma^k_{ij}(x^1(t), \ldots, x^n(t))\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \quad \forall\ k = 1, \ldots, n$$

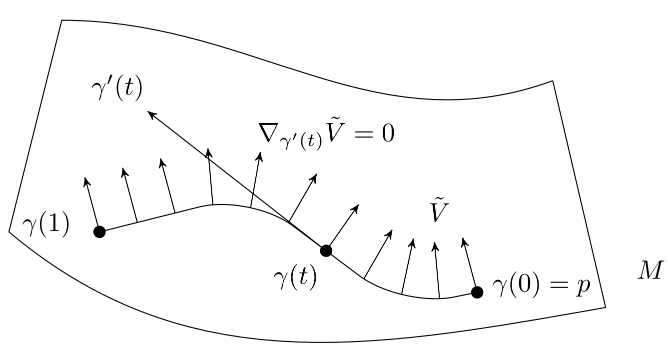
So $(x^1(t), \ldots, x^n(t))$ satisfy a system of non-linear 2nd order ODEs. If $\gamma(0) = p$, then we have conditions on $x^1(0)$ for all i, and if $\gamma'(0) = X_p$, then we have conditions on $\frac{dx^i}{dt}(0)$ for all i as well. By the ODE theorem, there exists a unique solution at least for $t \in (-\epsilon, \epsilon)$. ■

Note that to change a 2nd order ODE to a 1st order ODE, define $y^i(t) = \frac{dx^i}{dt}$, so the system becomes $\frac{dx^k}{dt} = y^k$, and $\frac{dx^k}{dt} = -\Gamma^k_{ij}(x^1(t), \ldots, x^n(t))y^i(t)y^j(t)$, which is a 1st order system for $(x^1(t), \ldots, x^n(t), y^1(t), \ldots, y^n(t))$.

Remark 3.2.4. Any two geodesics, by uniqueness, agree on their common domain. Hence, given $(p, X_p) \in T_pM$, there exists a unique minimal geodesic $\gamma : I \to M$ with $0 \in I$ such that $\gamma(0) = p$ and $\gamma'(p) = X_p$.

Definition 3.2.5. Let (M, ∇) be a manifold with a connection. Let $V_p \in T_pM$, and $\gamma : [0, 1] \to M$ be a smooth curve such that $\gamma(0) = p$. A vector field $\tilde{V}$ along γ (so $\tilde{V} \in \Gamma(\gamma^*(TM))$) is called a *parallel transport* of $V_p \in T_pM$ along γ iff:

$$\tilde{V}(0) = V_p \in T_{\gamma(0)}M \qquad \text{and} \qquad (D_t\tilde{V})(t) = 0 \ \ \forall\ t$$



Proposition 3.2.6. There exists a parallel transport, and given V_p and γ, it is unique.

Proof: We want to have

$$D_t\tilde{V} = \left(\frac{\tilde{V}^k}{dt} + \Gamma^k_{ij}(\gamma(t))\frac{dx^i}{dt}\tilde{V}^j\right)\frac{\partial}{\partial x^k} = 0$$

for $\tilde{V} = \tilde{V}^k\frac{\partial}{\partial x^k}$. The curve γ is fixed, and the unknowns are $V^i(t)$ for $i = 1, \ldots, n$. We then need

$$\frac{d\tilde{V}^k}{dt} = -\Gamma^k_{ij}(x^1(t), \ldots, x^n(t))\frac{dx^i}{dt}V^j$$

for all $k = 1, \ldots, n$, where $x^i = \gamma(t)^i$. This is a system of first order linear ODEs, so by the ODE theorem, the solution exists for all $t \in \mathbb{R}$, given $\tilde{V}(0) = V_p$. Hence there exists a unique solution to the parallel transport equations for all $t \in [0, 1]$. ■

Definition 3.2.7. For structures $V_p \in T_pM$, $\gamma : [0, 1] \to M$ with $p = \gamma(0)$ as above, define the parallel transport of V_p along γ to be a map $\Pi_\gamma : T_pM \to T_qM$ with

$$\Pi_\gamma(V_p) = \tilde{V}(1) \in T_{\gamma(1)}M = T_qM$$

Remark 3.2.8. Let ∇ be the Euclidean connection on $\mathbb{R}^n$, so $\Gamma^k_{ij} = 0$ in the standard chart. The parallel transport equations then will be $\frac{dV^k}{dt} = 0$, with solutions $V^k(t) = V^k(0) = \gamma(0)^k$ for all t and for any γ. So we say that a vector field V is *parallel* along a curve γ if $D_tV = 0$. More generally, any tensor σ on M is called *parallel* if $\nabla_{X_p}\sigma = 0$ for all $(p, X_p) \in TM$.

Theorem 3.2.9. For any γ, $\Pi_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$ is a linear isomorphism of vector spaces.

Proof: Let $U_p, W_p \in T_pM$ with $\gamma(0) = p$. Let $\tilde{U}, \tilde{W}$ be parallel transports of U_p and W_p, respectively, along γ. Let $V_p = aU_p + bW_p$. We claim that $a\tilde{U} + b\tilde{W}$ is the parallel transport of V along γ. This follows as

$$(a\tilde{U} + b\tilde{W})(0) = a\tilde{U}(0) + b\tilde{W}(0) = aU_p + bW_p = V_p$$

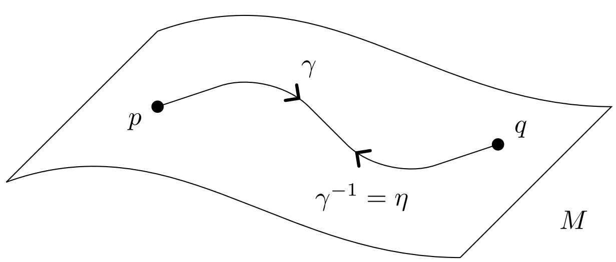
Next, note that

$$D_t(a\tilde{U} + b\tilde{W}) = aD_t\tilde{U} + bD_t\tilde{W} = 0 + 0 = 0$$

So $a\tilde{U} + b\tilde{W}$ is $\tilde{V}$, the unique parallel transport of V_p along γ. So

$$\Pi_\gamma(V_p) = \tilde{V}(1) = a\tilde{U}(1) + b\tilde{W}(1) = a\Pi_\gamma(U_p) + b\Pi_\gamma(W_p)$$

Therefore Π_γ is linear. Next, consider the curve η, given by



$$\begin{aligned} \eta : [0,1] &\to M & \eta(0) &= q \\ \eta(t) &= \gamma(-t) & \eta(1) &= p \end{aligned}$$

We call $\eta = \gamma^{-1}$. By uniqueness, $\Pi_{\gamma^{-1}}(\tilde{V}(1)) = \nabla(0) = V_p$, as parallel transports are unique. Hence $(\Pi_\gamma)^{-1} = \Pi_{\gamma^{-1}}$, so Π_γ is an invertible linear map, i.e. an isomorphism. ■

Now we may use the parallel transport to show that the covariant derivative is really a directional derivative (that is, a limit of difference quotients), so $(\nabla_XY)_p$ depends only on X_p.

Proposition 3.2.10. Let γ be a curve on M and V a vector field along γ. Then

$$(D_t V)_{\gamma(t_0)} = \lim_{t\to 0}\left[\frac{\Pi_\gamma^{-1}(V(t)) - V(t_0)}{t - t_0}\right] = (\nabla_{\gamma'(t_0)}V)_{\gamma(t_0)}$$

The proposition says that covariant diferentiation is a directional derivative.

Proof:

It now follows that $(\nabla_X Y)_p$ only depends on X_p. To compute it, choose any curve γ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Let P_t be the parallel transport of X_p alonf γ from p to $\gamma(t)$, so

$$(\nabla_{X_p} Y) = \lim_{t\to 0}\left[\frac{P_t^{-1}(Y|_{\gamma(t)}) - Y_p}{t}\right].$$

However, parallel transport along a closed loop may fail to bring you back to the same vector. This is called the *holonomy* if the connection. We will see that holonomy depends on the curvature.

Definition 3.2.11. Let ∇ be a connection on TM. Define the *torsion* of ∇ to be the function

$$T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \qquad \text{given by} \qquad T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = -T(Y,X).$$

On the previous assignment we showed that $T(fX,Y) = T(X,fY) = fT(X,Y)$ for all $f \in C^\infty(M)$, so T is a $(2,1)$-tensor on M.

Example 3.2.12. Let $M = \mathbb{R}^n$ and ∇ be the Euclidean connection. The set $\{E_1,\dots,E_n\}$ is called the *standard global frame* if any $Y \in \Gamma(T\mathbb{R}^n)$ can be expressed as $Y = Y^i E_i$, where the Y^is are global smooth functions on $\mathbb{R}^n$. Further, we then have that $\nabla_X Y = (\nabla_X Y)^i E_i$, where $(\nabla_X Y)^i = XY^i$, which only holds because we have the Euclidean connection in the specified frame.

Remark 3.2.13. What does the torsion measure? By comparing paths along different geodesics, we find that when $T = 0$, then the order of geodesic paths taken does not matter, i.e. the result will be the same.

3.3 Riemannian metrics

Definition 3.3.1. A *Riemannian metric* g on a smooth manifold M is a smooth (2,0) tensor field with

1. $g(X,Y) = g(Y,X)$ for all $X,Y \in \Gamma(TM)$
2. $g(X,Y) \geqslant 0$, with equality iff $X = 0$

This describes g as a symmetric, positive-definite tensor.

Remark 3.3.2. So $g \in \Gamma(T_0^2 M)$. Given $X,Y \in \Gamma(TM)$, $g(X,Y) \in C^\infty(M)$, and $(g(X,Y))_p = g_p(X_p,Y_p)$. This shows that g_p gives a positive definite inner product on T_pM. In general, a Riemannian metric is a smoothly varying family of positive definite inner products on the tangent space of M. That is, if $(x^1,\dots,x^n)$ are local coordinates, then $g = g_{ij}dx^i \otimes dx^j$, where the g_{ij} are smooth functions on the domain U of the coordinate chart, with

$$\begin{aligned} g_p &= g_{ij}dx^i|_p \otimes dx^j|_p \\ g_{ij} &= g\left(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}\right) \end{aligned} \qquad \text{and} \qquad \begin{aligned} V_p &= V^i_{(p)} \tfrac{\partial}{\partial x^i}\Big|_p \\ W_p &= W^j_{(p)} \tfrac{\partial}{\partial x^j}\Big|_p \end{aligned} \quad \Longrightarrow \; g_p(V_p,W_p) = g_{ij}(p)V^i(p)W^j(p)$$

Definition 3.3.3. A *pseudo-Riemannian metric* g is a (2,0) tensor such condition **2.** in the definition of Riemannian metric is replaced by the non-degeneracy condition:

2. if $g(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$, then $X = 0$

Note that positive-definiteness implies non-degeneracy.

In particular, we may say that g is pseudo-Riemannian with index $(1, n-1)$, equivalently 1 time-like dimension, and $(n-1)$ space-like dimensions. This is a generalization of the Lorenztian metric, which always has only one time-like dimension.

Remark 3.3.4. A metric in local coordinates looks like

$$g = g_{ij} dx^i \otimes dx^j \qquad \text{for} \qquad g_{ij} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = g_{ji}$$

where the g_{ij} are smooth functions on the domain U of the chart. That is, $[g_{ij}]$ is an $n \times n$ positive-definite symmetric matrix of smooth functions.

Definition 3.3.5. Let $\alpha, \beta \in \Omega^1(M)$. Then as a shorthand, we write

$$\alpha\beta = \frac{\alpha \otimes \beta + \beta \otimes \alpha}{2}$$

This is a (2,0) tensor, with the property that $(\alpha\beta)(X,Y) = (\alpha\beta)(Y,X) = (\beta\alpha)(X,Y)$. The expression $\alpha\beta$ is called the *symmetric product* of α and β. The *skew-symmetric product* is given by

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$$

For a metric g, it then follows that

$$\begin{aligned} g &= g_{ij} dx^i \otimes dx^j \\ &= \frac{1}{2} g_{ij} dx^i \otimes dx^j + \frac{1}{2} g_{ji} dx^j \otimes dx^i \\ &= \frac{1}{2} g_{ij} dx^i \otimes dx^j + \frac{1}{2} g_{ij} dx^i \otimes dx^j \\ &= g_{ij} dx^i dx^j \end{aligned}$$

Example 3.3.6. Consider $\mathbb{R}^n$ with the standard Euclidean metric $\bar{g}$. Then $\bar{g} = g_{ij} dx^i dx^j$ with $g_{ij} = \delta_{ij}$ (in the standard coordinate chart). That is,

$$\begin{array}{rcl} v &=& v^i \frac{\partial}{\partial x^i} \\ w &=& w^j \frac{\partial}{\partial x^j} \end{array} \qquad \Longrightarrow \qquad \begin{array}{rcl} g(v,w) &=& g\left(v^i \frac{\partial}{\partial x^i}, w^j \frac{\partial}{\partial x^j}\right) \\ &=& v^i w^j g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &=& v^i w^j g_{ij} \end{array}$$

For $\bar{g}$ on $\mathbb{R}^n$, $\bar{g}(v,w) = \sum_{i=1}^n v^i w^i = \langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}$, which is the usual dot product on $\mathbb{R}^n$. However, as soon as we change the coordinate system, $\bar{g}$ will have a different expression in local coordinates. In rectangular coordinates:

$$\bar{g} = (dx)^2 + (dy)^2$$

In polar coordinates:

$$\begin{array}{lll} x = r\cos(\theta) & dx = \cos(\theta)dr - r\sin(\theta)d\theta & (dx)^2 = \cos^2(\theta)(dr)^2 + (r^2\sin^2(\theta))(d\theta)^2 - 2r\sin(\theta)\cos(\theta)drd\theta \\ y = r\sin(\theta) & dy = \sin(\theta)dr + r\cos(\theta)d\theta & (dy)^2 = \sin^2(\theta)(dr)^2 + (r^2\cos^2(\theta))(d\theta)^2 + 2r\sin(\theta)\cos(\theta)drd\theta \end{array}$$

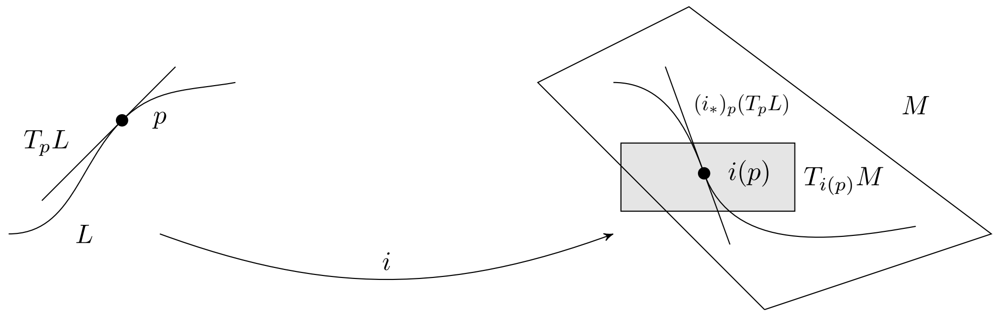
So $\bar{g} = (dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2$, and

$$\bar{g}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1 \qquad \bar{g}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = r^2 \qquad \bar{g}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = 0$$

Definition 3.3.7. Let M^n, L^k be smooth manifolds. A smooth map $i : L^k \to M^n$ is called an *injective immersion* if the following conditions are satisfied:

1. i is injective (that is, $p \neq q \implies i(p) \neq i(q)$)
2. i is an immersion (that is, $(i_*)_p : T_pL \to T_{i(p)}M$ is injective for all $p \in M$)

If i is an immersion, we may use $(i_*)_p$ to identify each T_pL with a subspace of $T_{i(p)}M$.



If $i : L \to M$ is an injective immersion, then (L, i) is called an *immersed submanifold* on M.

Remark 3.3.8. Note that
· this often occurs when $L \subset M$ is a subset which is also a smooth manifold, and i is the inclusion
· the requirement of injectivity is sometimes dropped
· there exists a stronger notion, that of *embedded manifold*, even as any immersed manifold is locally embedded

Lemma 3.3.9. Let $i : L \to M$ be an immersed submanifold, and let g_M be a Riemannian metric on M. Then $i^*(g_M)$ is a Riemannian metric on L. This metric is called the *induced* or *pullback metric* on L from M.

Proof: Note that $i^*(g_M)$ is a smooth $(2, 0)$ tensor on L (by the properties of the pullback), with

$$(i^*(g_M))_p(Y_p, Z_p) := (g_M)_{i(p)}((i_*)_p(Y_p), (i_*)_p(Z_p)) = (i^* g_M)_p(Z_p, Y_p)$$

Further, if $(i^* g_M)_p(X_p, X_p) = 0$, then

$$(g_M)_{i(p)}((i_*)_p X_p, (i_*)_p X_p) = 0 \implies (i_*)_p(X_p) = 0 \implies X_p = 0$$

The first implication is from the positive definiteness of g_M, and the second is from the fact that i is an immersion. This concludes the proof. ■

Example 3.3.10. Consider the n-sphere $\mathbb{S}^n$. There exists a map $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$, the inclusion map, such that i is smooth, injective, and an immersion. Hence $i^*(\bar{g})$ is a Riemannian metric on $\mathbb{S}^n$, called the *round metric*. This metric looks differently in different charts.

Example 3.3.11. If $p \in U_+^{n+1} = \{p \in \mathbb{S}^{n+1} \ : \ x^{n+1}(p) > 0\}$, then we have graph coordinates $x^1, \ldots, x^n$ and $x^{n+1} = \sqrt{1 - (x^1)^2 - \cdots - (x^n)^2}$. In these coordinates,

$$\begin{aligned} &i : \left\{(u^1, \ldots, u^n) \in \mathbb{R}^n \ : \ \textstyle\sum_{i=1}^n (u^i)^2 < 1\right\} \to U_+^{n+1} \\ &(u^1, \ldots, u^n) \mapsto \left(u^1, \ldots, u^n, \sqrt{1 - (u^1)^2 - \cdots - (u^n)^2}\right) \end{aligned} \qquad (i_*)_{(u^1,\ldots,u^n)} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \\ * & * & * & * & * & * \end{bmatrix}$$

As this matrix is rank n, the map is injective. Let us write down the round metric in these coordinates:

$$\begin{aligned} \bar{g} &= (dx^1)^2 + \cdots + (dx^{n+1})^2 \quad \text{on } \mathbb{R}^{n+1} \\ g_{\text{round}} &= i^*(\bar{g}) \\ &= (d(x^1 \circ i))^2 + \cdots + (d(x^n \circ i))^2 \end{aligned} \qquad \begin{aligned} x^i(u^1, \ldots, u^n) &= u^i \quad \text{for } i = 1, \ldots, n \\ x^{n+1}(u^1, \ldots, u^n) &= \sqrt{1 - (u^1)^2 - \cdots - (u^n)^2} \\ dx^i &= du^i \quad \text{for } i = 1, \ldots, n \\ dx^{n+1} &= \frac{-1}{2\sqrt{1 - \sum(u^i)^2}}(2u^1 du^1 + \cdots + 2u^n du^n) \end{aligned}$$

So the metric may be expressed in local coordinates as

$$\begin{aligned} g_{\text{round}} &= (du^1)^2 + \cdots + (du^n)^2 + \frac{1}{1-|u|^2}(u^1 du^1 + \cdots + u^n du^n) \qquad \text{for } |u|^2 = \sum_{i=1}^n (u^i)^2 \\ &= \frac{1-|u|^2+(u^1)^2}{1-|u|^2}(du^1)^2 + \cdots + \frac{1-|u|^2+(u^n)^2}{1-|u|^2}(du^n)^2 + \sum_{\substack{i,j=1 \\ i \neq j}} \frac{2}{1-|u|^2} u^i u^j du^i du^j \end{aligned}$$

$$\text{with} \qquad g_{ii} = \frac{1-|u|^2+(u^i)^2}{1-|u|^2} \qquad g_{ij} = \frac{u^i u^j}{1-|u|^2}$$

It is left as an exercise to check that in spherical coordinates,

$$\begin{aligned} x &= r\cos(\theta)\sin(\varphi) \\ y &= r\sin(\theta)\sin(\varphi) \\ z &= r\cos(\theta) \end{aligned} \qquad \begin{aligned} \bar{g} &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= dr^2 + r^2\sin^2(\varphi)(d\theta)^2 + r^2(d\varphi)^2 \\ g_{\text{round}} &= i^*(\bar{g}) \\ &= \sin^2(\varphi)(d\theta)^2 + (d\varphi)^2 \end{aligned}$$

So we see that a Riemannian metric gives you a measure of how much a manifold is curving.

Definition 3.3.12. Let (M, g_M), (N, g_N) be Riemannian manifolds. A map $f : M \to N$ is called an *isometry* if it is a diffeomorphism and $f^*(g_N) = g_M$.

Two Riemannian manifolds are termed *isometric* if there exists an isometry between them.

Further, M is *locally isometric* to N if for all $p \in M$, there exists $U \ni p$ open with a map $g : U \to N$ such that $g(U) = V$ open, and $g : U \to V$ is an isometry. Here, the metric on U is $g_M|_U = i^*(g_N)$, where $i : U \to M$ is the inclusion.

Note that if (M, g_M) is locally isometric to (N, g_N), it does not necessarily follow that (N, g_N) is locally isometric on (M, g_M).

Definition 3.3.13. Let (M, g_M), (N, g_N) be Riemannian manifolds. A smooth map $f : M \to N$ is called a *local isometry* if for all $p \in M$, there exists $U \ni p$ open such that $f(U)$ is open in N, and $f|_U : U \to f(U)$ is an isometry.

Definition 3.3.14. Let (M, g_M) be a Riemannian manifold. Then (M, g_M) is called *flat* if it is locally isometric to $(\mathbb{R}^n, \bar{g})$. That is, for all $p \in M$ there exists $U \ni p$ open and a map $f : U \to f(U)$ for $f(U)$ open in $\mathbb{R}^n$ such that $f^*(\bar{g}|_{f(U)}) = g|_U$.

Proposition 3.3.15. Any smooth manifold M admits a Riemannian metric.

Proof: Cover M by coordinate charts $(U_\alpha, \varphi_\alpha)$ for $\alpha \in A$. Define a Riemannian metric g_α on U_α by

$$g_\alpha = \delta_{ij} dx^i dx^j = (\varphi_\alpha)^*(\bar{g}|_{\varphi_\alpha(U_\alpha)})$$

Let $\{f_\alpha \ : \ \alpha \in A\}$ be a partition of unity subordinate to this open cover. Define $g = \sum_\alpha f_\alpha g_\alpha$, which is a finite sum for all $p \in M$. Then g is a smooth $(2,0)$ symmetric tensor. It remains to check positive definiteness, so suppose that $g(X, X) = 0$. Then

$$0 = g_p(X_p, X_p) = \sum_{\alpha \in A} f_\alpha(p) g_\alpha(X_p, X_p) \quad \text{as } 0 \leqslant f_\alpha \leqslant 1 \text{ and at least one } f_{\alpha_0}(p) > 0$$

This implies that $X_p = 0$. Since g_{α_0} is positive definite, g is positive definite. ■

Note that this will not work for pseudo-Riemannian metrics. For example, not every manifold admits a Lorentzian metric, as there exist topological obstructions.

Remark 3.3.16. For covariant derivatives ∇, we have that

$$\nabla_X f = Xf \qquad \text{and} \qquad \nabla_X(\alpha \otimes \beta) = (\nabla_X \alpha) \otimes \beta + \alpha \otimes (\nabla_X \beta)$$

3.4 Elementary constructions with Riemannian metrics

Proposition 3.4.1. Let (M, g) be a Riemannian manifold. There exists a canonical isomorphism between tangent vectors at $p \in M$ and cotangent vectors at p. This is termed the *musical isomorphism*, and given

- · V finite dimensional real vector space, V^* its dual, and
- · $B : V \times V \to \mathbb{R}$ a non-degenerate bilinear form (i.e. $B(v, w) = 0$ for all w implies $v = 0$),

define $\flat : V \to V^*$ by setting $\flat(v) \in V^*$ to be the linear functional such that $(\flat(v))(w) = B(v, w)$. This is the proposed isomorphism.

Proof: Suppose $v \in \ker(\flat)$, so $\flat(v) = 0_{V^*}$. Then $(\flat(v))(w) = B(v, w) = 0$ for all $w \in V$, meaning that $v = 0$. Hence $\ker(\flat) = \{0\}$. Since $\dim(V) = \dim(V^*)$, $\flat$ is an isomorphism. ■

Remark 3.4.2. Apply this approach to the tangent space of M at p with a non-degenerate bilinear form g_p. Given $X_p \in T_pM$, we let $X_p^\flat = \flat(X_p) \in T_p^*M$ with

$$X_p^\flat(Y_p) = g_p(X_p, Y_p) = g_p\left(X^i \frac{\partial}{\partial x^i}\bigg|_p, Y^j \frac{\partial}{\partial x^j}\bigg|_p\right) = \left(\alpha_k dx^k|_p\right)\left(Y^j \frac{\partial}{\partial x^j}\bigg|_p\right) = \alpha_k Y \delta_j^k = \alpha_k Y^k$$

for $X_p^\flat = \alpha_k dx^k|_p$. Further, for all $Y_p \in T_pM$, we have that $X_p^\flat(Y_p) = g_p(X_p, Y_p) = X^j Y^i g_{ij}(p) = Y^i \alpha_i$. Hence $\alpha_i = g_{ij}(p) X^j$. If $X = X^i \frac{\partial}{\partial x^i}\big|_p \in T_pM$, then $X^\flat = X_k dx^k|_p \in T_p^*M$ for $X_k = g_{kj}(p) X^j$. More generally,

$$\begin{aligned} &\flat : \Gamma(TM) \to \Gamma(T^*M) \\ &(\flat(X))(Y) = g(X, Y) \in C^\infty(M) \end{aligned}$$

So $(\flat)_p : T_pM \xrightarrow{\cong} T_p^*M$, so it has an inverse $(\#)_p : T_p^*M \to T_pM$. If $\alpha \in \Gamma(T^*M)$, in local coordinates $\alpha = \alpha_k dx^k$, and $\alpha^\# \in \Gamma(TM)$.

So we conclude that given g on M, we get a positive-definite inner product g_p on the space T_p^*M, varying smoothly for all p. It is defined by

$$g_p(\alpha_p, \beta_p) = g_p(a_p^\#, \beta_p^\#)$$

This follows by demanding that $\flat_p$ be an isometry of inner product spaces.

Definition 3.4.3. Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Define the *gradient* ∇f of f to be the smooth vector field on M given by

$$\nabla f = (df)^\# = \left(\frac{\partial f}{\partial x^i} dx^i\right)^\# = \left(g^{ij} \frac{\partial f}{\partial x^i}\right) \frac{\partial}{\partial x^j}$$

where the first factor in the last expression on the right is the component of ∇f in the $\frac{\partial}{\partial x^j}$ direction. Note that ∇f is the vector field canonically associated to $df \in \Omega^1(M)$.

Example 3.4.4. Consider the metric in the following situations.

· in $\mathbb{R}^2$ with polar coordinates. We can use our previous knowledge to describe $g = dr^2 + r^2 d\theta^2$.

$$\begin{matrix} g_{rr} = 1 & g_{r\theta} = 0 \\ g_{\theta r} = 0 & g_{\theta\theta} = r^2 \end{matrix} \qquad \nabla f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}$$

· in $\mathbb{S}^2$ with the round metric and polar coordinates. Here we have $g = d\varphi^2 + \sin^2(\varphi)d\theta^2$.

$$\begin{matrix} g_{\varphi\varphi} = 1 & g_{\varphi\theta} = 1 \\ g_{\theta\varphi} = 1 & g_{\theta\theta} = \sin^2(\varphi) \end{matrix} \qquad \nabla f = \frac{\partial f}{\partial \varphi}\frac{\partial}{\partial \varphi} + \frac{1}{\sin^2(\varphi)}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}$$

Definition 3.4.5. Let (M, g) and (N, h) be two Riemannian manifolds, and $f : M \to N$ a map. Consider the following definitions:

1. f is a *conformal diffeomorphism* with respect to g, h iff f is a diffeomorphism and f is conformal, i.e. $f^*(h) = \lambda g$ for some positive function λ on M. If $\lambda = 1$, then f is an isometry.

2. f is a *local conformal diffeomorphism* iff for all $p \in M$ there exists $U \subset M$ open with $U \ni p$ such that $f(U) = V$ is open in N, and $f|_U : U \to V$ is a conformal diffeomorphism.

3. f is *locally conformal to* N iff for all $p \in M$, there exists $U \subset M$ open with $U \ni p$ and a map $f : U \to N$ such that $f(U) = V$ is open in N and ${}^p f : U \to V$ is a conformal diffeomorphism.

Note that if $f : M \to N$ is a conformal diffeomorphism, then the angle between X_p and Y_p with respect to g_p is the same as the angle between $(f_*)_p(X_p)$ and $(f_*)_p(Y_p)$ in $T_{f(p)}M$, with respect to $h_{f(p)}$. Also, **1**. $\implies$ **2**. $\implies$ **3**.

Remark 3.4.6. There are several important examples of geometric spaces.

$(\mathbb{R}^n, \bar{g})$, Euclidean space	zero curvature
$(\mathbb{S}^n, g_{round})$, the round n-sphere	constant positive curvature
(H^n, g_{hyp}), n-dimensional hyperbolic space	constant negative curvature

We have seen the first two so far. There are two models of hyperbolic geometry. For the first, we define an open ball of radius 1 with respect to g.

$$M = \left\{ (u^1, \dots, u^n) \in \mathbb{R}^n \ : \ \sum_{i=1}^n (u^i)^2 < 1 \right\}$$
$$g_M = \frac{4}{(1 - |u|_{\bar{g}}^2)^2}\left((du^1)^2 + \dots + (du^n)^2\right)$$

In local coordinates, $g_{ij} = 0$ when $i \neq j$, and $g_{ij} = 4/(1 - |u|_{\bar{g}}^2)^2$ when $i = j$. This is called the *ball model* of hyperbolic space. It is clear that (M, g_M) is conformally diffeomorphic to $(M, \bar{g}|_M)$ by the identity map. Forthe second madel, we consider half of a whole space.

$$N = \left\{ (x^1, \dots, x^{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \ : \ y > 0 \right\}$$
$$g_N = \frac{1}{y^2}\left((dx^1)^2 + \dots + (dx^{n-1})^2 + (dy)^2\right)$$

This is called the *upper half space model* of hyperbolic space. Again, (N, g_N) is conformally diffeomorphic to $(N, \bar{g}|_N)$. Note that M and N are not isometric to subsets of Euclidean space with the Euclidean metric.

Proposition 3.4.7. The spaces (M, g_M) and (N, g_N) are isometric.

<u>*Proof:*</u> Express the spaces as $M = \{(u,v) \in \mathbb{R}^{n-1} \times \mathbb{R} \ : \ |u|_{\bar{g}}^2 + v^2 < 1\}$ and $N = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \ : \ y > 0\}$. Define a map between them by

$$\begin{array}{rccl} f: & M & \to & N \\ & (u,v) & \mapsto & \left(\frac{2u}{|u|^2+(v-1)^2}, \frac{1-|u|^2-v^2}{|u|^2+(v-1)^2} \right) = (x,y). \end{array}$$

This map is smooth and maps into N. To show that f is invertible and ha an inverse, note that

$$f^{-1}(x,y) = \left(\frac{2x}{|x|^2+(y+1)^2}, \frac{|x|^2+y^2-1}{|x|^2+(y+1)^2} \right) \qquad \text{and} \qquad v + |u|^2 = \frac{4|x|^2 + (|x|+y^2-1)^2}{(|x|^2+(y+1)^2)^2} < 1.$$

It remains to check the above and that $f^*(g_N) = g_M$, which implies that f is an isometry. ■

3.5 The Riemannian connection / the Levi-Civita connection

For (M,g) a Riemannian manifold, we will see that g determines a unique connection ∇ on the tangent bundle.

Definition 3.5.1. A connection ∇ on TM of a Riemannian manifold (M,g) is *compatible* with the metric g if $\nabla_X g = 0$ for all $X \in \Gamma(TM)$.

Remark 3.5.2. What does compatible mean? First, recall that for all $X,Y,Z \in \Gamma(TM)$ and all $g \in \Gamma(T_0^2 M)$,

$$(\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

So ∇ compatible is equivalent to g being parallel with respect to ∇.

Proposition 3.5.3. Let ∇ be a connection on a Riemannian manifold (M,g). Then equivalently:

1. ∇ is compatible with g
2. $\nabla_X g = 0$ for all $X \in \Gamma(TM)$
3. $X(g(Y,Z)) = g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$ for all $X,Y,Z \in \Gamma(TM)$
4. the parallel transport $\Pi_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ is an isometry for all t_0, t_1 and curves γ

This is natural from the Euclidean connection $\bar{g}$. Let $X = X^i e_i$, $Y = Y^j e_j$ and $Z = z^k e_k$ with $\{e_1, \dots, e_n\}$ a globas frame and $\bar{g}(e_i, e_j) = \delta_{ij}$. In the system $(\mathbb{R}^n, \bar{g}, \nabla^{n+1})$, we have that

$$X(\bar{g}(Y,Z)) = \bar{g}(\nabla_X Y, Z) - \bar{g}(Y, \nabla_X Z)$$

so in $\mathbb{R}^n$ compatibility comes naturally.

Remark 3.5.4. Recall from Assignment 3 that a g-compatible connection ∇ means the parallel transport with respect to ∇ preserves the inner product defined by g between tangent vectors. That is, if γ is a curve from p to q, then

$$g_p(X_p, Y_p) = g_q(\Pi_\gamma X_p, \Pi_\gamma Y_p)$$

However, if $\nabla_X g = 0$ for all X, it does not follow that the component functions g_{ij} of g with respect to a chart are constants. In fact,

$$(\nabla_X g)_{ij} = \frac{\partial}{\partial x^k} g_{ij} - \Gamma_{ki}^m g_{mj} - \Gamma_{kj}^m g_{im}$$

Definition 3.5.5. A connection ∇ on TM is *torsion-free* if $T(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$ for $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

Recall that the Euclidean connection on $\mathbb{R}^n$ is torsion-free.

Theorem 3.5.6. [Fundamental theorem of Riemannian geometry]
Let (M, g) be a Riemannian manifold. There exists a unique connection ∇ that is g-compatible and torsion-free, called the *Riemannian connection*, or *Levi-Civita connection*.

Proof: Uniqueness comes from the fact that $g(\nabla_X Y, Z)$ being determined for all X, Y, Z means that $\nabla_X Y$ is determined for all X, Y.

For existence, we use a formula to define ∇_j. To see that ∇ is a torsion-free g-compatible connection, consider in local coordinates

$$X = \frac{\partial}{\partial x^i},\ Y = \frac{\partial}{\partial x^j},\ Z = \frac{\partial}{\partial x^k} \quad \text{for which} \quad \frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g'_{ij} = 2g\left(\nabla_i \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$$

From calculations, we conclude that

$$\Gamma_{ij}^m = \Gamma_{ij}^\ell \delta_\ell^m = \Gamma_{ij}^\ell g_{\ell k} g^{\ell m} = \frac{g^{km}}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) = \Gamma_{ij}^m$$

We define a connection on TM by demanding that its Christoffel symbols are given by the equation above in local coordinates. It remains to show that this connection is torsion-free and g-compatible.

For torsion-free, note that $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ij}^\ell = 0$. For compatibility, we need to check that $(\nabla_k g)_{ij} = 0$ for all m, i, j, which we leave as an exercise. ■

From now on, (M, g) is always a Riemannian manifold and ∇ is always the Levi-Civita connection of g. A geodesic with respect to the Levi-Civita connection is called a *Riemannian geodesic*.

Proposition 3.5.7. Let γ be a Riemannian geodesic. The *speed* $s(t) = \sqrt{g(\gamma'(t), \gamma'(t))}$ of $\gamma : I \to M$ is constant for all $t \in I$.

Proof: Observe that

$$\frac{d}{dt} s^2(t) = \frac{d}{dt} g(\gamma', \gamma') = D_t(g(\gamma', \gamma')) = \nabla_{\gamma'}(g(\gamma', \gamma')) = g\left(\nabla_{\gamma'}\gamma', \gamma'\right) + g\left(\gamma', \nabla_{\gamma'}\gamma'\right) = 0 + 0 = 0$$

■

Let's compute this in an example.

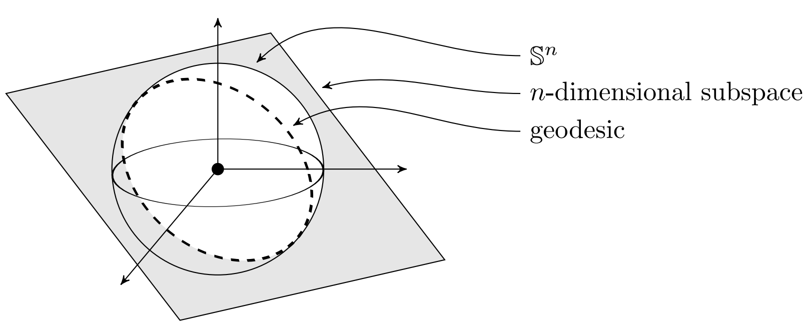
Example 3.5.8. Consider $(\mathbb{R}^n, \bar{g})$, where the geodesics are constant-speed straight lines. So $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ is a local frame with

$$\gamma^1 = \frac{dx^i}{dt} e_i \qquad \text{and} \qquad \nabla_{\gamma'}\gamma' = D_t \gamma' = \frac{d}{dt}\left(\frac{d\gamma^i}{dt}\right) e_i + \frac{d\gamma^i}{dt} D_t e_i = \frac{d^2\gamma^i}{dt^2} e_i$$

Next, we want $\nabla_{\gamma'}\gamma' = 0$ so then $\frac{d^2\gamma^i}{dt^2} = 0$ for all i, and

$$\gamma^i(t) = a_i t + b_i \qquad\qquad \gamma(t) = \vec{a}t + \vec{b}$$

Example 3.5.9. Now consider $(\mathbb{S}^n, g_{\text{round}})$. The geodesics on $\mathbb{S}^n$ are the arcs of great circles, i.e. intersections with $\mathbb{S}^n$ of n-dimensional vector subspaces of $\mathbb{R}^{n+1}$.



Example 3.5.10. Next, consider (H^2, g) the hyperbolic 2-space with the upper half-space model, so $H^2 = \{x, y \in \mathbb{R}^2 \ : \ y > 0\}$ and $g = \frac{1}{y^2}((dx)^2 + (dy)^2)$. The Christoffel symbols are given by $\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}\left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right)$, where $x^1 = x$ and $x^2 = y$. So specifically,

$$\Gamma^1_{11} = \frac{1}{2}g^{1\ell}\left(\frac{\partial g_{\ell 1}}{\partial x^1} + \frac{\partial g_{\ell 1}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\ell}\right) = \frac{1}{2}y^2\left(2\frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{dx}\right) = 0$$
$$\Gamma^2_{11} = \frac{1}{2}g^{2\ell}\left(\frac{\partial g_{\ell 1}}{\partial x^1} + \frac{\partial g_{\ell 1}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\ell}\right) = \frac{1}{2}y^2\left(-\frac{\partial g_{11}}{\partial y}\right) = 0\frac{-y^2}{2} \cdot \frac{-2}{y^3} = \frac{1}{y}$$
$$\Gamma^1_{12} = \frac{1}{2}g^{1\ell}\left(\frac{\partial g_{\ell 1}}{\partial x^2} + \frac{\partial g_{\ell 2}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\ell}\right) = \frac{1}{2}y^2\left(\frac{\partial g_{11}}{\partial y}\right) = \frac{y^2}{2} \cdot \frac{-2}{y^3} = \frac{-1}{y}$$
$$\Gamma^2_{12} = 0$$
$$\Gamma^1_{22} = 0$$
$$\Gamma^2_{22} = \frac{1}{y}$$

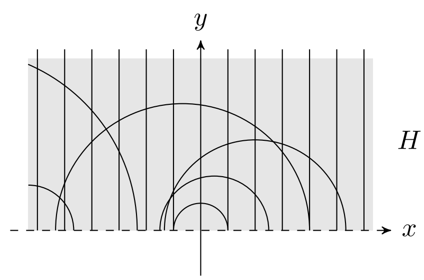
Recall the geodesic equation in general, which was $\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$. Then

$$\frac{d^2x}{dt^2} + \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0 \qquad\qquad \frac{d^2y}{dt^2} + \frac{1}{y}\left(\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right) = 0$$

Now let's try to solve these equations. Suppose that $x = x_0$ a constant. Then $\frac{dx}{dt} = 0$ and $\frac{d^2x}{dt} = 0$. Also,

$$y'' - \frac{1}{y}(y')^2 = 0 \implies \frac{y''}{y'} = \frac{y'}{y} \implies \begin{array}{rcl} \log(y') & = & \log(y) + c \\ y' & = & \tilde{c}y \end{array} \implies \frac{y}{y'} = \tilde{c} \implies y = ce^{kt}$$

To find the rest, suppose that $x(t) = x_0 + \lambda\cos(f(t))$ and $y(t) = \lambda\sin(f(t))$ for some $f(t)$. This reduces, in both geodesic equations, to $f''(t)/(f''(t))^2 = \cot(f(t))$, which tells us that circles centered on the x-axis are also geodesics.



4 Digressions and distances

We take a small detour to talk about two related topics. The first is that of volume forms.

4.1 Digression one - volume forms

Recall that a manifold M is termed *orientable* if there exists a smooth n-form $\mu \in \Omega^n(M)$ such that $\mu_p \neq 0$ for all $p \in M$, for $\mu_p \in \Lambda^n(T^*_pM)$. Such an n-form, which may or may not exist, is termed a *nowhere-vanishing n-form*.

Proposition 4.1.1. Suppose that M is orientable and connected. There exist exactly two equivalence classes of orientation forms, where $\mu \sim \mu'$ off $\mu/f\mu'$ for some $f \in \sigma(\mu)$, $f > 0$. The orientation class of μ is then denoted by $[f]$.

Definition 4.1.2. An *orientation* on M (for M orientable) is a choice of orientation class. An *oriented manifold* is an orientable manifold together with a choice of orientation class.

Remark 4.1.3. Suppose that M is compact and oriented. Then we can integrate n-forms on M. Let $\omega \in \Omega^n(M)$ and $\int_M \omega \in \mathbb{R}_{>0}$. Then Stokes' theorem tells us that

$$\partial M = \emptyset \quad \Longrightarrow \quad \int_M d\omega = 0$$

So if $f : M^n \to N^n$ is an orientation-preserving diffeomorphism,

$$\int_{f(M)} \omega = \int_M f^*\omega$$

Definition 4.1.4. Let M be orientable. A *volume form* on M is a choice μ of a representative of the given orientation class. The *volume* of M (for M compact) is defined to be

$$\mathrm{Vol}(M) = \int_M \mu$$

Clealy the volume depends on the chosen volume form. Next note that if (M, μ) is a compact manifold with a volume form, then we can define the integration of smooth functions over M as:

$$\int_M f = \int_M f\mu \quad \text{for } f \in C^\infty(M)$$

Remark 4.1.5. Suppose (M, g) is an oriented Riemannian manifold. Then there exists a canonical choice of volume form μ in the given orientation. It has the property that

$$M_p = (e_1, \ldots, e_n) = \pm 1$$

for $e_1, \ldots, e_n$ an oriented orthonormal basis of T_pM. This M is termed the *Riemannian volume form* associted to g and the appropriate orientation class. Note that in local coordinates with an oriented chart, we have that

$$\mu = f dx^1 \wedge \cdots \wedge dx^n$$

for some $f \in C^\infty(U)$. We can relate the local orientable orthonormal frame $\{e_1, \ldots, e_n\}$ to the oriented coordinate frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ by

$$\frac{\partial}{\partial x^i} = \sum_j P_{ij} e_j \qquad \text{and} \qquad e_j = \sum_k Q_{jk} \frac{\partial}{\partial x^k} \qquad \text{and} \qquad S_{ik} = \sum_j P_{ij} Q_{jk}$$

Plugging this into the definition of μ, we get that

$$1 = \mu(e_1, \ldots, e_n) = \mu\left(\sum_i Q_{1k_i} \frac{\partial}{\partial x^{k_i}}, \ldots, \sum_i Q_{nk_i} \frac{\partial}{\partial x^{k_n}}\right) = \det(Q)\mu\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = \det(Q) f$$

The associated metric g is then given as

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) = g\left(\sum_k P_{ik} e_k, \sum_\ell S_{i\ell} e_\ell\right) = \sum_{k,\ell} P_{k\ell} P_{j\ell} \underbrace{g(e_k, e_\ell)}_{=\delta_{k\ell}} = \sum_k P_{ik} P_{jk} = (PP^T)_{ij}$$

And $\det(g) = \det(PP^T) = \det(P)^2 = 1/\det(Q)^2$, so $f = \sqrt{\det(g)}$.

Definition 4.1.6. Let (M,μ) be a manifold with a volume form. Define $\operatorname{div} : \Gamma(TM) \to C^\infty(M)$ by, for $X \in \Gamma(TM)$,

$$\operatorname{div}(X)\mu = d(X \lrcorner \mu) = d(X \lrcorner \mu) + \underbrace{X \lrcorner \mu}_{=0} = \mathcal{L}_X \mu$$

This makes it clear that div is $\mathbb{R}$-linear, and that $\operatorname{div}(fX) = f\operatorname{div}(X)$.

Example 4.1.7. Consider $(\mathbb{R}^n, \bar{g})$. Let $\bar{\mu}$ be the Riemannian volume form of $\bar{g}$, with $\bar{\mu} = dx^1 \wedge \cdots \wedge dx^n$. Then

$$\operatorname{div}(X)\mu = d(X \lrcorner \mu) = \left(\sum_i \frac{\partial X^i}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n$$

Theorem 4.1.8. [Divergence theorem]
Suppose (M,μ) is a compact manifold with a volume form. Then $\int_M \operatorname{div}(X) = 0$.

Proof: Note that

$$\int_M \operatorname{div}(X) = \int_M \operatorname{div}(X)\mu = \int_M d(X \lrcorner \mu) = 0$$

The first equality follows from the definition of an integral of a function, and the last follows by Stokes' theorem. ■

Remark 4.1.9. Now suppose that $\operatorname{div}(X) = 0$. Let Θ_t be the (global) flow of X, so $\Theta_t : M \to M$ is a diffeomorphism. Then

$$\operatorname{Vol}(\Theta_t(M)) = \int_{\Theta_t(M)} \mu = \int_M \Theta_t^* \mu = \int_M \mu = \operatorname{Vol}(M)$$

The second-last equality follows as $\frac{d}{dt}\Theta_t^*\mu = -\mathcal{L}_X\mu = -\operatorname{div}(X)\mu = 0$. Hence $\Theta_t^*\mu = \Theta_0^*\mu = \mu$, so μ is invariant under Θ_t for all t.

In summary, we now have that if X in divergence-free (i.e. $\operatorname{div}(X) = 0$), then the flow $\Theta_t : M \to M$ preserves the volume, or $\operatorname{Vol}(\Theta_t(M)) = \operatorname{Vol}(M)$. We say that a diffeomorphism of M that is the flow of a divergence-free vector field is called a *volume-preserving* diffeomorphism. As an aside, note that M has at least as many divergence-free vector fields as the first Betti number of M.

4.2 Digression two - Lie groups

In this section, we will show that every compact Lie group has a bi-invariant Riemann metric. Later, we will compute the geodesics and curvature on compact Lie groups with respect to a bi-invariant metric.

Remark 4.2.1. Let G be a Lie group. For all $a \in G$,

$$\begin{array}{rccc} L_a : & G & \to & G \\ & g & \mapsto & ag \end{array} \qquad \qquad \begin{array}{rccc} R_a : & G & \to & G \\ & g & \mapsto & ga \end{array}$$

are both diffeomorphisms of G with $(L_a)^{-1} = L_{a^{-1}}$ and $(R_a)^{-1} = R_{a^{-1}}$. As $(ag)b = a(gb)$, it follows that $R_b \circ L_a = L_a \circ R_b$, so L_a and R_b commute for all a, b. Define *conjugation* by a to be $I_a = L_a \circ R_{a^{-1}}$, so $I_a(g) = aga^{-1}$, so it too is a diffeomorphism. Note that $I_a(e) = e$ and $I_a(g)I_a(h) = I_a(gh)$, so $I_a : G \to G$ is a Lie group automorphism. As it is a diffeomorphism, we can use it to push forward vector fields.

Proposition 4.2.2. The map $(I_a)^*$ maps the space of left-invariant vector fields to itself, i.e. $(I_a)^* : \mathfrak{g} \to \mathfrak{g}$.

Proof: Let X be left-invariant, so $(L_a)_*X = X$, and

$$(L_a)_*((R_b)_*X) = (R_b)_*((L_a)_*X) = (R_b)_*X \quad \text{and} \quad (I_a)_*X = (L_a)_*(R_{a^{-1}})_*X = (R_{a^{-1}})_*X$$

So it indeed takes left-invariant vector fields to left-invariant vector fields. ■

Proposition 4.2.3. For $a \in G$, let $(I_a)_* = \text{Ad}(a) : \mathfrak{g} \to \mathfrak{g}$ be the *adjoint* map. Then

1. $\text{Ad}(a)$ is an automorphism of $\mathfrak{g}$
2. $\text{Ad} : G \to GL(\mathfrak{g})$ is a homomorphism of groups
3. Ad is smooth

Proof: **1.** For $x, y \in \mathfrak{g}$, we have that

$$\text{Ad}(a)[X, Y] = (I_a)_*[X, Y] = [(I_a)_* X, (I_a)_* Y] = [\text{Ad}(a)X, \text{Ad}(a)Y]$$

Note that $\text{Ad}(a) = (I_a)_*$ is linear with $(\text{Ad}(a))^{-1} = \text{Ad}(a^{-1})$. So the map is an automorphism of the vector space $\mathfrak{g}$ preserving Lie brackets.

2. For this, observe that

$$I_{ab}(g) = (ab)g(ab)^{-1} = (ab)g(b^{-1}a^{-1}) = a(bgb^{-1}a^{-1} = I_a(I_b(g))$$

So $I_{ab} = I_a \circ I_b$, hence $(I_{ab})_* = (I_a)_* \circ (I_b)_*$, or $\text{Ad}(ab) = \text{Ad}(a)\text{Ad}(b)$, so Ad is a group homomorphism.

3. Consider the map $G \times G \to G$ by $(g, h) \mapsto ghg^{-1}$. This is smooth in g, h, so by fixing g, $\text{Ad}(g) = (I_g)_*$ is the Jacobian of this map at (g, e), in local coordinates. Hence it is also smooth. ■

Definition 4.2.4. Let σ be a smooth $(k, 0)$-tensor on G. Then for all $a, g \in G$,

$$\begin{aligned} \sigma \text{ is left-invariant} &\iff (L_a)_* \sigma_{ag} = \sigma_g \\ \sigma \text{ is right-invariant} &\iff (R_a)^* \sigma_{ga} = \sigma_g \\ \sigma \text{ is bi-invariant} &\iff \sigma \text{ is left- and right-invariant} \end{aligned}$$

Proposition 4.2.5. Let $\sigma : G \to T_0^k(TG)$ be a section of $T_0^k(TG)$, so $\pi \circ \sigma = \text{id}_M$, i.e. $\sigma_g \in T_0^k(T_g G)$. If σ is left- or right-invariant, then σ is smooth.

Proof: Let $E_1, \ldots, E_n$ be a local frame for a smooth manifold M (that is, $E_1, \ldots, E_n$ are smooth vector fields on some $U \subset M$ and a basis of $T_p M$ for all $p \in U$). Suppose that σ is a section of $T_0^k(TM)$. Then σ is smooth iff $\sigma(E_1, \ldots, E_k)$ is a smooth function on U for all i. Let $n = \dim(\sigma)$ and $E_1, \ldots, E_n$ be a global frame of left-invariant vector fields. As

$$\begin{aligned} \sigma_{ag}\left(E_{i_1}|_{ag}, \ldots, E_{i_k}|_{ag}\right) &= \sigma_{ag}\left((L_a)^*\ E_{i_1}|_{ag}, \ldots, (L_a)^*\ E_{i_k}|_{ag}\right) \\ &= ((L_a)^* \sigma_{ag})\left(E_{i_1}|_{ag}, \ldots, E_{i_k}|_{ag}\right) \\ &= \sigma_{ag}\left(E_{i_1}|_{ag}, \ldots, E_{i_k}|_{ag}\right), \end{aligned}$$

the function $\sigma_{i_1 \cdots i_k} = \sigma(E_1, \ldots, E_k)$ is constant (thus smooth), so σ is smooth. The same approach works for right-invariance. ■

Lemma 4.2.6. Let $\sigma_e \in T_0^k(T_e G)$, for e the identity element. Then:

1. there exists a unique left-invariant $(k, 0)$-tensor α on G such that $\alpha_e = \sigma_e$, and a unique right-invariant $(k, 0)$-tensor β on G such that $\beta_e = \sigma_e$.

2. $\alpha = \beta$ on G (that is, σ_e determines a bi-invariant $(k, 0)$-tensor on G) iff $(\text{Ad}(g))^* \sigma_e = \sigma_e$ for all $g \in G$.

Proof: **1.** As $L_g : G \to G$ with $L_g(e) = ge = g$, we have that $(L_g)_* : T_e G \to T_e G$, so $(L_{g^{-1}})_* : T_g G \to T_e G$. Hence $(L_{g^{-1}})^*$ takes $(k, 0)$-tensors at e to $(k, 0)$-tensors at g. Next define $\alpha_g = (L_{g^{-1}})^* \sigma_e$. To see that α is left-invariant, observe that

$$\begin{aligned} (L_a)^* \alpha_{ag} &= (L_a)^*((L_{(ag)^{-1}})^* \sigma_e) \\ &= (L_a)^*((L_{g^{-1}a^{-1}})^* \sigma_e) \\ &= (L_a)^*(L_{a^{-1}})^*(L_{g^{-1}})^* \sigma_e \\ &= (L_{g^{-1}})^* \sigma_e \\ &= \alpha_g. \end{aligned}$$

By the previous result, α is an isomorphism. Uniqueness is clear. Right-invariance is done similarly.

2. Suppose that α, β are both such tensors. Then by left-invariance,

$$(\mathrm{Ad}(g))^* \sigma_e = (L_g \circ R_{g^{-1}})^* \sigma_e = (R_{g^{-1}})^*(L_g)^* \sigma_e = \sigma_e.$$

If we suppose that $(\mathrm{Ad}(g))^* \sigma_e = \sigma_e$, then

$$(R_{g^{-1}})^* \sigma_e = (L_g^{-1})^* \sigma_e = (L_{g^{-1}})^* \sigma_e,$$

so $\alpha_g = \beta_g$ for all g. ■

Corollary 4.2.7. Every Lie group G admits a left- and right-invariant Riemannian metric, and a left- and right-invariant volume form.

In particular, every Lie group is orientable, as it has a volume form.

Theorem 4.2.8. A compact, connected Lie group has a unique bi-invariant volume form μ such that $\mathrm{vol}(G, \mu) = 1$.

The proof is omitted

Corollary 4.2.9. A compact, connected Lie group admits a bi-invariant Riemann metric.

The proof is omitted

This ends the digression on Lie groups. We will soon compute geodesics and curvature for Lie groups with a bi-invariant metric.

4.3 The exponential map and normal coordinates

On (M, g), we can use geodesics to get a canonical chart containing p for every $p \in M$. Let $X_p \in T_pM$. We know there exists $\epsilon > 0$ and a geodesic $\gamma_{X_p} : (-\epsilon, \epsilon) \to M$ such that $\gamma_{X_p}(0) = p$ and $(\gamma_{X_p})'(0) = X_p$.

Definition 4.3.1. Define the space $\mathcal{E} = \{(p, X_p) \in TM, \gamma_{X_p}$ is defined on an interval containing $[0, 1]\}$. That is, $(p, X_p) \in \mathcal{E}$ iff γ_{X_p} is defined.

So $\mathcal{E}$ is the set of all points (p, X_p) in the tangent bundle for which the geodesic γ_{X_p} is defined for at least $t = 1$.

Definition 4.3.2. The map $\exp : \mathcal{E} \to M$ is defined by $\exp((p, X_p)) = \gamma_{X_p}(1)$, and is termed the *exponential map*. For $p \in M$, let $\mathcal{E}_p = \mathcal{E} \cap T_pM$. For $(p, X_p) \in \mathcal{E}$, define $\exp_p : \mathcal{E}_p \to M$ by $\exp_p(X_p) = \exp((p, X_p))$. This is termed the *restricted exponential map.*

Proposition 4.3.3.
a. $\mathcal{E}$ is an open subset of TM containing the zero section (i.e. $(p, 0_p) \in \mathcal{E}$ for all $p \in M$), and each $\mathcal{E}_p$ is star-shaped (convex from a point) with respect to $0_p \in T_pM$
b. if $(p, X_p) \in T_pM$, then $\exp_p(tX_p) = \gamma_{X_p}(t)$ for all t such that either side is defined
c. exp is smooth

Before we can prove this proposition, we need the folowing lemma:

Lemma 4.3.4. [Rescaling lemma]
Let $V_p \in T_pM$ and $c, t \in \mathbb{R}$. Then $\gamma_{cV_p}(t) = \gamma_{V_p}(ct)$ whenever either side is defined.

Proof: We will show if $\gamma_{cV_p}(t)$ exists, then so does $\gamma_{V_p}(ct)$ and they are equal. The other direction follows by setting $V_p \to cV_p$, $t \to ct$, and $c \to 1/c$.

So let $\gamma(t) = \gamma_{V_p}(t)$. Define $\tilde{\gamma} = \gamma(ct)$. Let I be the domain of γ, so the domain of $\tilde{\gamma}$ is $\{t \ : \ ct \in I\}$. We want to show that $\tilde{\gamma}$ is a geodesic, with initial point p and initial velocity cV_p. Note that in local coordinates,

$$\gamma(t) = \gamma^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \qquad \tilde{\gamma}(t) = \gamma^i(ct) \left.\frac{\partial}{\partial x^i}\right|_{\tilde{\gamma}(t)}$$

Evaluating at $t = 0$, we have

$$\tilde{\gamma}(0) = \gamma(c \cdot 0) = \gamma(0) = p \qquad \tilde{\gamma}'(0) = \frac{d\tilde{\gamma}}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_{\tilde{\gamma}(0)} = c\frac{d\gamma^i}{dt}(0) \left.\frac{\partial}{\partial x^i}\right|_{p} = c\gamma'(0) = cV_p$$

It remains to check that $\tilde{\gamma}$ is a geodesic. As γ is a geodesic,

$$\frac{d^2\tilde{\gamma}^k}{dt^2} + \Gamma^k_{ij}\frac{d\tilde{\gamma}^i}{dt}\frac{d\tilde{\gamma}^j}{dt} = c^2\left(\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\right) = 0$$

■

We now move to the proof of Proposition 4.3.3.

Proof: The rescaling lemma with $t = 1$ says that $\gamma_{cV_p}(1) = \gamma_{V_p}(c)$ whenever either side is defined. So $\gamma_{tV_p}(1) = \gamma_{V_p}(t)$ for all t when both sides are defined, so $\exp_p(tV_p) = \gamma_{V_p}(t)$, proving **b.**.

Next let $V_p \in \mathcal{E}_p$p Then γ_{V_p} is defined on at least $[0,1]$, so for $0 \leqslant t \leqslant 1$, the rescaling lemma says that

$$\exp_p(tV_p) = \gamma_{tV_p}(1) = \gamma_{V_p}(t)$$

is defined for all $t \in [0,1]$, so $tV_p \in \mathcal{E}_p$ for all $t \in [0,1]$. Hence $\mathcal{E}_p$ is star-shaped with respect to 0_p, proving the second part of **a.** It remains to show that exp is smooth and that $\mathcal{E}_p$ is open. ■

Proposition 4.3.5. Let $f \in C^\infty(TM)$. Then

$$(Gf)_{(p,X_p)} = \left.\frac{d}{dt}\right|_{t=0} f(\gamma_{X_p}(t), (\gamma_{X_p})'(t)).$$

The proof is omitted

Proposition 4.3.6. The integral curves of G satisfy the equations

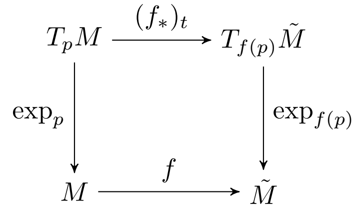
$$\frac{dx^i}{dt} = y^i \qquad \text{and} \qquad \frac{dy^i}{dt} = -Y^iY^j\Gamma_{ij}k(x^1(t), \ldots, x^n(t))$$

implying that

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}((x^1(t), \ldots, x^n(t))\frac{dx^i}{dt}\frac{dx^j}{dt}.$$

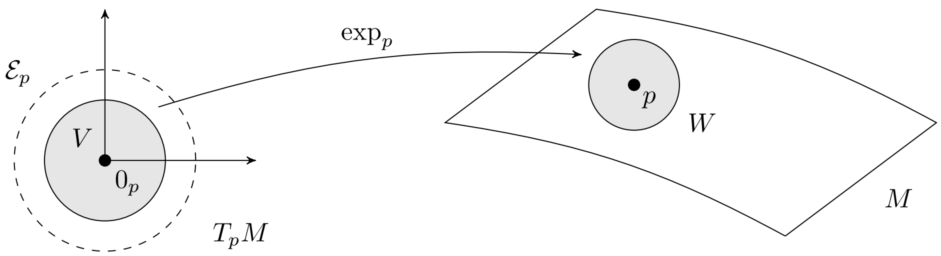
The proof is omitted

Proposition 4.3.7. [Naturality of the exponential map]
Let $f : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometry. Then for any $p \in M$, the following diagram commutes:



Proof: The proof follows immediately from Assignment 4, question 2b, which says that if γ_{X_p} is a geodesic on (M, g) with initial data (p, X_p), then $f \circ X_p$ is a geodesic on $(\tilde{M}, \tilde{g})$ with initial data $(f(p), (f_*)_p X_p)$. ■

Lemma 4.3.8. Let $p \in M$ and consider the smooth map $\exp_p : \mathcal{E}_p \to M$, with $\mathcal{E}_p$ open in T_pM. There exists an open neighborhood $V \subset \mathcal{E}_p$ of 0_p in T_pM and an open neighborhood W of $p \in M$ such that $\exp_p : V \to W$ is a diffeomorphism:



Proof: This follows from the inverse function theorem. We start by noting that T_pM is a vector space, so $T_{0_p}(T_pM) \cong T_pM$ canonically. We also have that $\exp_p : T_PM \to M$ with

$$\begin{array}{rrcl} ((\exp_p)_*)_{0_p} : & T_{0_p}(T_pM) & \to & T_{\exp_p(0_p)}M \\ & T_pM & \to & T_pM \end{array}$$

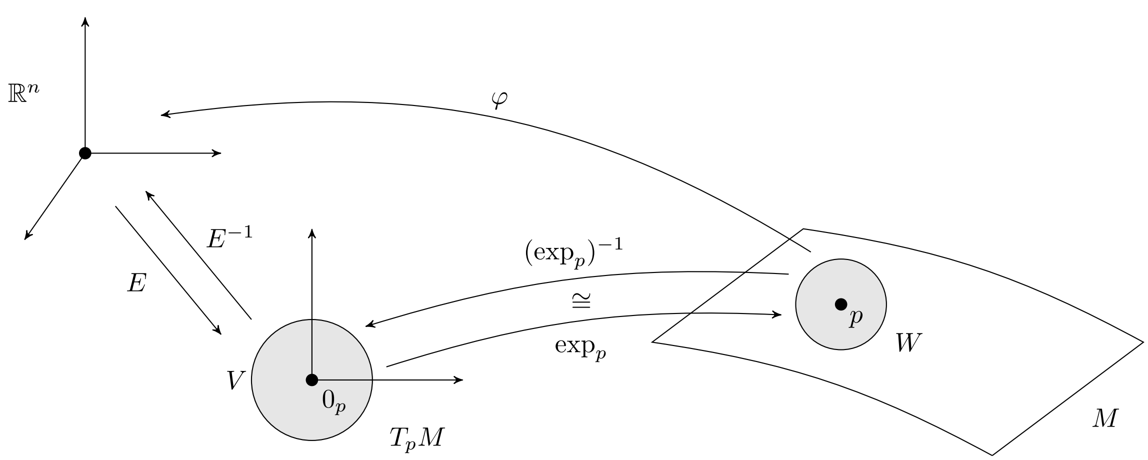
Let $V_p \in T_pM$. To find what $((\exp_p)_*)_{0_p}(V_p)$ is, we need a smooth curve $\sigma(t)$ in T_pM with $\sigma'(t) = V_p$. Then $((\exp_p)_*)_{0_p}(V_p) = (\exp_p \circ \sigma)'(0)$. So take $\sigma(t) = tV_p \in T_pM$ for all t, with $\sigma'(t) = V_p$ for all t, and $\sigma'(0) = V_p$. Then

$$((\exp_p)_*)_{0_p}(V_p)\ \frac{d}{dt}\bigg|_{t=0} \exp_p(tV_p) = \frac{d}{dt}\bigg|_{t=0} \sigma_{V_p}(t) = V_p \quad \Longrightarrow \quad ((\exp_p)_*)_{0_p} = \mathrm{id}_{T_pM}$$

So the map is invertible, and the inverse function theorem completes the proof. ■

This lemma allows us to prove the existence of so-called "normal coordinates."

Definition 4.3.9. Let $E_1, \ldots, E_n$ be any orthonormal basis of T_pM with respect to g_p. This gives an isomorphism $E : \mathbb{R}^n \to T_pM$, with $E(x^1, \ldots, x^n) = x^i f_i$. Then we let $\varphi = E^{-1} \circ (\exp_p)^{-1}$, and φ becomes a diffeomorphism from $W \ni p$ to $\varphi(W) \subset \mathbb{R}^n$, so it is a chart.



Then φ is a smooth chart for M centered at p, with $\varphi(p) = (0, 0, \ldots, 0) \in \mathbb{R}^n$. This is called a *normal coordinate chart* centered at p. Note that it is not unique.

Proposition 4.3.10. Let (W, φ) be a normal coordinate chart centered at p. Then:

a. for any $V_p = V^k \frac{\partial}{\partial x^k}|_p \in T_pM$, the geodesic γ_{V_p} of M with initial data (p, V_p) is represented in these coordinates by $\gamma_{V_p}(t) = (tV^1, \ldots, tV^k)$ as long as γ_{V_p} stays in W.

b. the coordinates of p are $(0, 0, \ldots, 0)$

c. the components of the metric g_{ij} at p are $g_{ij}(p) = \delta_{ij}$

d. the Christoffel symbols vanish at p, i.e. $\Gamma^k_{ij}(p) = 0$

The proof is omitted

Definition 4.3.11. Let $\epsilon > 0$ be such that $\exp_p$ is a diffeomorphism on $B_\epsilon(0) \subset T_pM$, where $B_\epsilon(0) = \{X_p \in T_pM \ : \ |X_p|_{g_p} < \epsilon\}$, and $|v|_{g_p} = \sqrt{g_p(v,v)}$. Then we say that $\exp_p(B_\epsilon(0))$ is a *geodesic ball* centered at p. Similarly, $\exp_p(\overline{B_\epsilon(0)})$ is the *closed geodesic ball* centered at p if $\overline{B_\epsilon(0)}$ lies in the domain of $\exp_p$. Continuing the analogy, we call $\partial B_\epsilon(0)$ the *geodesic sphere* centered at p.

Let (U, φ) be a normal coordinate chart centered at p. Define the *radial distance function* r given by

$$r(x^1, \ldots, x^n) = \sqrt{\sum_{i=1}^n (x^i)^2}.$$

Define the *unit radial vector field* on $U \setminus \{p\}$ to be

$$\frac{\partial}{\partial r} = \sum_{i=1}^n \frac{x^i}{r} \frac{d}{dx^i}$$

Lemma 4.3.12. At any point $q \in U \setminus \{p\}$, $\frac{\partial}{\partial r}$ is the velocity vector of the unit speed geodesic function from p to q, and hence has unit length with respect to g.

Proof: Let $q \in U \setminus \{p\}$. In normal coordinates, $\varphi(q) = (x^1, \ldots, x^n)$. Consider $\widehat{\gamma}(t) = (tx^1, \ldots, tx^n)$, which is a smooth curve with $\widehat{\gamma}(0) = 0$ (so $\gamma(0) = p$) and $\widehat{\gamma}(1) = (x^1, \ldots, x^n)$ (so $\gamma(1) = q$). From part **d.** of the previous proposition, γ is a geodesic strating at p. Then we have that

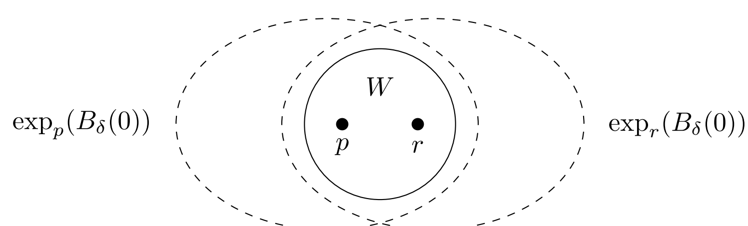
$$\gamma'(t) = \frac{d\widehat{\gamma}^i}{dt}\frac{\partial}{\partial x^i} = x^i \frac{\partial}{\partial x^i} = r\frac{\partial}{\partial r},$$
$$|\gamma'(t)|^2 = g(\gamma'(t), \gamma'(t)) = g\left(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right) = r^2 \left|\frac{\partial}{\partial r}\right|_g^2.$$

Recall that geodesics have constant speed, so

$$|\gamma'(t)|^2 = g(\gamma'(t), \gamma'(t)) = g\left(x^i \frac{\partial}{\partial x^i}, x^j \frac{\partial}{\partial x^j}\right) = x^i x^j g_{ij}.$$

At $t = 0$, $g_{ij}(p) = \delta_{ij}$. Hence $r^2|\frac{\partial}{\partial r}|^2_g = r^2$, so $|\frac{\partial}{\partial r}|_g = 1$, and $|\gamma'(0)| = x^i x^j \delta_{ij} = r^2$. ■

Definition 4.3.13. Let $W \subset M$ be open. Then W is called a *uniformly normal* (or *totally normal*) subset of M iff there exists $\delta > 0$ such that $W \subset \exp_q(B_\delta(0))$ for all $q \in W$.



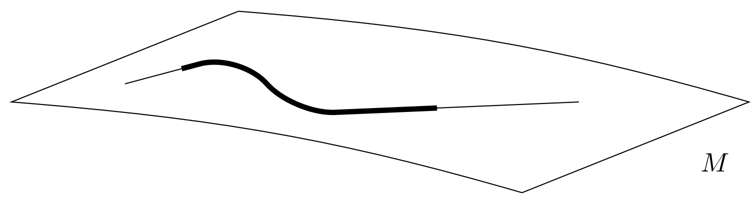
So W is uniformly normal if it is contained in a geodesic ball of radius δ around each of its points.

Lemma 4.3.14. Given $p \in M$ and any open neighborhood $U \ni p$, there exists a uniformly normal neighborhood W of p such that $W \subset U$.

The proof is omitted

4.4 Distances and parametrization

Definition 4.4.1. Let (M, g) be a Riemannian manifold. A *smooth curve segment* on M is a smooth map $\gamma : [a, b] \to M$. That is, it is the restriction of a smooth curve on M to a closed bounded interval.



Definition 4.4.2. Let $\gamma[a, b]$ be a smooth curve segment. The *length* of γ, denoted by $L(\gamma)$, is defined to be

$$L(\gamma) = \int_a^b |\gamma'(s)|_{g_{\gamma(0)}} \, ds = (\text{integral over } [a, b] \text{ of the speed of } \gamma)$$

Note that $L(\gamma)$ depends on the metric g on M. Clearly $L(\gamma) \geqslant 0$.

Definition 4.4.3. Let $\varphi : [c, d] \to [a, b]$ be a smooth map with a smooth inverse. We say that φ is a *forward reparametrization* (or *backward reparamaterization*) if it is orientation preserving (or reversing), i.e. $(\varphi_*)_t > 0$ for all $t \in [c, d]$ (or $(\varphi_*)_t < 0$ for all $t \in [c, d]$).

Let $\gamma : [a, b] \to M$ be a smooth curve segment. Let $\varphi : [c, d] \to [a, b]$ be a parametrization. Then $\tilde{\gamma} = \gamma \circ \varphi : [c, d] \to M$ is called a *reparametrization of* γ.

Lemma 4.4.4. The length of a curve is independent of parametrization, i.e. $L(\gamma) = L(\tilde{\gamma})$.

Proof: Let $t = \varphi(s)$ and $\tilde{\gamma} = \gamma(\varphi(s)) = \gamma(t)$. Then

$$\tilde{\gamma}'(s) = \frac{d\varphi}{ds}\gamma'(\varphi(s))$$
$$|\tilde{\gamma}'(s)| = \left|\frac{d\varphi}{ds}\right| |\gamma'(\varphi(s))|_g$$

Putting this together, we have that

$$L(\tilde{\gamma}) = \int_c^d |\tilde{\gamma}'(s)| \, ds = \int_c^d |\gamma'(\varphi(s))|_g \frac{d\varphi}{ds} = \int_a^b |\gamma'(t)| \, dt = L(\gamma)$$

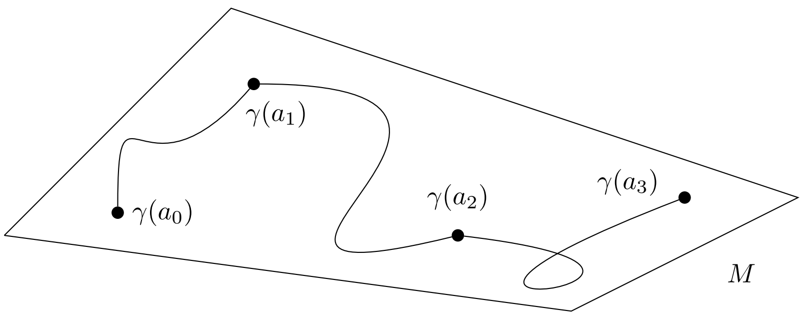
■

Definition 4.4.5. A smooth curve $\gamma : I \to M$ is termed a *regular curve* if $\gamma'(t) \neq 0_{\gamma(t)}$ for all $t \in I$, where $0_{\gamma(t)} \in T_{\gamma(t)}M$.

Remark 4.4.6. A curve γ is regular iff it is an immersion, as $(\gamma_*)_t \left(\frac{d}{dt}\big|_t\right) = \gamma'(t)$, where $(\gamma_*)_t : T_*I \to T_{\gamma(t)}M$.

Also note that all non-constant geodesics are regular (by the constant speed criterion).

Definition 4.4.7. An *admissible curve* on M is a piecewise-regular curve segment. That is, it is a continuous map $\gamma : [a,b] \to M$ such that there exists a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ of $[a,b]$ and $\varphi|_{[a_{i-1},a_i]} : [a_{i-1}, a_i] \to M$ is a regular smooth curve segment.



$$\gamma'(a_i^-) = \lim_{t \to a_i^-} [\gamma'(t)] \quad = \quad \text{velocity from the left}$$

$$\gamma'(a_i^+) = \lim_{t \to a_i^+} [\gamma'(t)] \quad = \quad \text{velocity from the right}$$

This definition implies that the curve has a well-defined non-zero one-sided velocity vector when approaching $\gamma(a_i)$ from either side (but they may not be equal), as indicated to the left of the diagram.

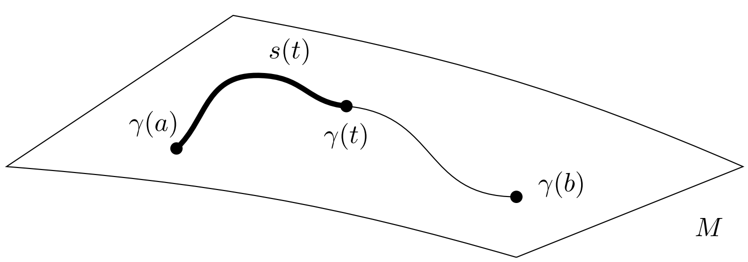
We also allow trivial (constant) curves to be admissible.

Definition 4.4.8. Let γ be an admissible curve. The *length* of γ is $L(\gamma) = \sum_{i=1}^{k} L(\gamma|_{[a_{i-1},a_i]})$.

Definition 4.4.9. A *reparametrization of an admissible curve* $\gamma : [a,b] \to M$ is a homeomorphism $\varphi : [c,d] \to [a,b]$ with a subdivision $c = c_0 < c_1 < \cdots < c_k = d$ such that $\varphi|_{[c_{i-1},c_i]} \to [a_{i-1}, a_i]$ is a reparametrization of the previous sets.

Note that the length of admissible curves is invariant under reparametrizations.

Definition 4.4.10. Let $\gamma : [a,b] \to M$ be an admissible curve. Define a function $s : [a,b] \to \mathbb{R}$ by $s(t) = L(\gamma|_{[a,t]}) = \int_a^t |\gamma'(t)|\, dt$. This is termed the *arc length* of γ from 0 to t.



The fundamental theorem of calculus says that $\frac{ds}{dt} = |\gamma'(s)|_g$, which is the speed of γ as $\gamma(s)$.

Lemma 4.4.11. Let $\gamma : [a,b] \to M$ be an admissible curve. Let $\ell = L(\gamma)$. Then

a. there exists a unique reparametrization $\tilde{\gamma} : [0,\ell] \to M$ of γ such that $\tilde{\gamma}$ is a unit speed curve, i.e. $|\tilde{\gamma}(s)| = 1$ for all s

b. if $\tilde{\gamma}$ is any unit speed curve whose domain is of the form $[0,\ell]$, then $s(t) = t$ for $\tilde{\gamma}$

Hence unit speed curves are said to be parametrized by length.

The proof for the above lemma follows by noting that $\frac{ds}{dt} > 0$, so $s(t)$ is invertible. Finding the inverse $t = t(s)$ will give the required parametrization.

Remark 4.4.12. Let $\gamma : [a,b] \to M$ be an admissible curve and $f \in C^\infty(M)$. Consider the integral of f over γ with respect to length. We denote this by

$$\int_\gamma f\ ds = \int_a^b f(t)|\gamma'(t)|\ dt$$

Lemma 4.4.13. Let $\varphi : [c, d] \to [a, b]$ be a reparametrization of γ. Then

$$\int_a^b f(t)|\gamma'(t)|\ dt = \int_c^d \tilde{f}(u)|\tilde{\gamma}'(u)|\ du \qquad \begin{array}{rcl} \tilde{f} &=& f \circ \varphi \\ \tilde{\gamma} &=& \gamma \circ \varphi \end{array}$$

Proof: Same as above. ■

Definition 4.4.14. A continuous map $V : [a, b] \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for all t is called a piecewise smooth vector field along γ if there exists a (possibly finer) subdivision $a = \tilde{a}_0 < \tilde{a}_1 < \cdots < \tilde{a}_k = b$ such that V is smooth on each $[\tilde{a}_{i-1}, \tilde{a}_i]$.

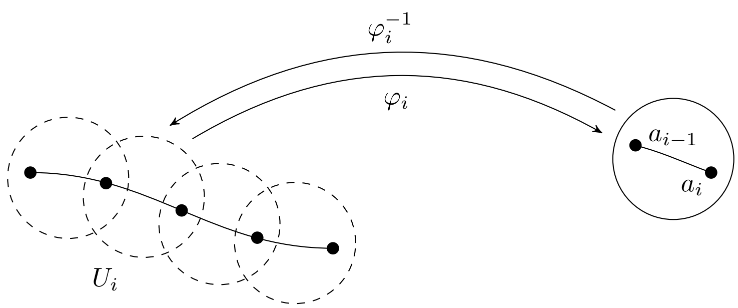
Given an admissible curve γ and $V_a \in T_{\gamma(a)}M$, we can parallel transport V_a along γ to get a piecewise smooth vector field V along γ with $V(a) = V$.

Definition 4.4.15. Let (M, g) be connected, and $p, q \in M$. Define the *distance* $d(p, q)$ between p and q by

$$d(p, q) = \inf_{\text{all admissible curves } \gamma \text{ from } p \text{ to } q} \{L(\gamma)\}$$

We need to show that this is well defined (i.e. there exists an admissible curve between p and q). This is clear by noting that as M is connected, there exists a continuous path $\alpha : [a, b] \to M$ with $\alpha(a) = p$, $\alpha(b) = q$.

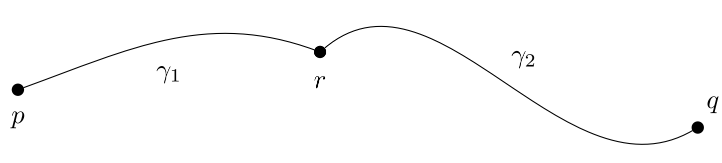
Further, $\alpha([a, b])$ is compact in M, so there exists a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that $\alpha([a_{i-1}, a_i])$ is in a domain (coordinate ball) of a single chart.



Then replace each $\alpha|_{[a_{i-1}, a_i]}$ by a smooth path in coordinates to get an admissible curve γ. So there exists an admissible curve γ from p to q, hence $d(p, q)$ is well-defined. Equivalently, $d(p, q) \geqslant 0$, and $d(p, q) = d(q, p)$.

Theorem 4.4.16. In the sense of metric spaces, d is a metric. The metric space topology determined by d is the same as the original manifold topology.

Proof: We need to show that $d(p, q) \leqslant d(p, r) + d(r, q)$ for all $p, q \in M$ and $d(p, q) = 0$ iff $p = q$. Consider:

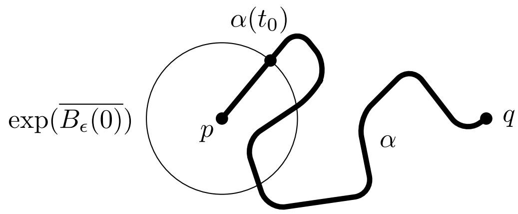


Here γ_1 is an admissible curve from p to r and γ_2 is an admissible curve from r to q. Moreover, it is clear that $L(\gamma_2 \cdot \gamma_1) = L(\gamma_2) + L(\gamma_1)$ and $d(p, q) \leqslant L(\gamma_2) + L(\gamma_1)$. Taking the infimum avor all γ_1 from p to r and the infimum over all γ_2 from r to q, we get that $d(p, q) \leqslant d(p, r) + d(r, q)$, as desired.

Next we show that if $p \neq q$, then $d(p,q) > 0$. The idea is to compare Riemannian distance to Euclidean distance in cordinate balls. So let $p \in M$, and let $(x^1, \ldots, x^n)$ be normal coordinates centered at p. As in the proof of the uniformly normal neighborhood lemma, there exists a closed geodesic ball $\exp(\overline{B_\epsilon(0)})$ of radius ϵ centered at p, and $c, C > 0$ such that

$$c|V_q|_{\bar{g}} \geqslant |V_q|_g \leqslant C|V_q|_{\bar{g}} \quad \forall\ V_q \in T_qM,\ q \in \exp(\overline{B_\epsilon(0)})$$

From the definition of length, for any admissible curve γ whose image is in $\exp(\overline{B_\epsilon(0)})$, we have that $cL_{\bar{g}}(\gamma(\leqslant L_g(\gamma) \leqslant CL_{\bar{g}}(\gamma)$.



If $p \neq q$, by shrinking ϵ if necessary, $q \notin \exp(\overline{B_\epsilon(0)})$. Next, let $\alpha : [a,b]$ be any admissible curve from p to q. Then α must intersect the geodosic sphere $\exp(B_\epsilon(0))$, as the complement of $\exp(\partial B_\epsilon(0))$ is disconnected, and p, q lie in different components, as shown above. Let $t_0 \in [a,b]$ be the first time in $[a,b]$ where $\alpha(t_0) \in \exp(\partial B_\epsilon(0))$. Then

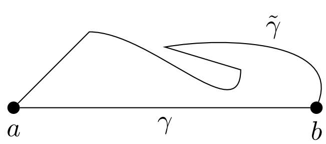
$$L_g(\alpha) \geqslant L_g(\alpha|_{[a,t_0]}) \geqslant cL_g(\alpha|_{[a,t_0]}) = cd_{\bar{g}}(p, \alpha(t_0)) = c\epsilon > 0$$

So $d(p,q) > \epsilon$, hence $p \neq q$ implies that $d(p,q) \neq 0$, so we do indeed have a metric.

Finally, a basis for the manifold topology is given by small "Euclidean balls" in open sets of the form $\exp(B_\delta(0))$. The metric topology of d is generated by small metric balls. This shows that the topologies are the same, so each open set in one topology is an open set in the other. ■

Remark 4.4.17. Every smooth manifold is metrizable.

Definition 4.4.18. Let (M,g) be a Riemannian manifold. An admissible curve γ on M is called *minimizing* if $L(\gamma) \leqslant L(\tilde{\gamma})$ for all admissible curves $\tilde{\gamma}$ with the same endpoints as γ.

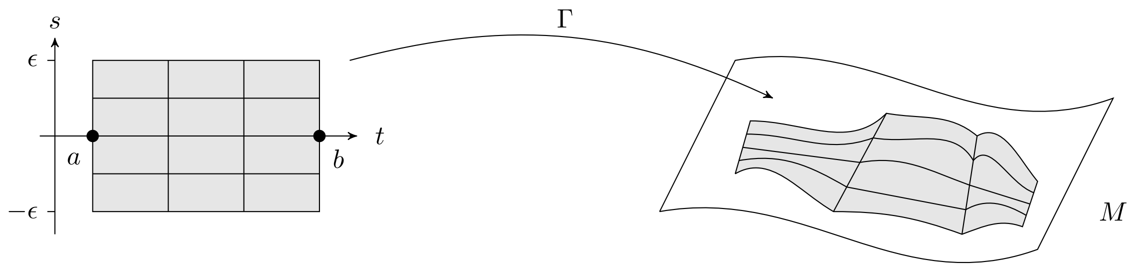


That is, γ is minimizing iff $L(\gamma) = d(\gamma(a), \gamma(b))$, which follows from the definition of distance. Note there need not exist a minimizing curve. If it exists, it may not be unique.

We would like to prove that a minimizing curve is a geodesic. The idea is to use the calculus of variations. If we consider Γ_s a family of curves between p and q such that $\gamma = \Gamma_0$ is minimizing, then we should have that $\frac{d}{ds}|_{s=0} L(\Gamma_s) = 0$. Let's formalize this approach.

Definition 4.4.19. An *admissible family of curves* is a continuous map $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M$ such that there exists a finite subdivision $a = a_0 < \cdots < a_k = b$ such that Γ is smooth on $(-\epsilon, \epsilon) \times [a_{i-1}, a_i]$ and

$\Gamma_s(t) = \Gamma(s,t)$ is an admissible curve on M for each fixed $s \in (-\epsilon, \epsilon)$.



If Γ is an admissible family, a *vector field along* Γ is a continuous map $V : (-\epsilon,\epsilon) \times [a,b] \to TM$ such that $V(s,t) \in T_{\Gamma(s,t)}M$ for all s,t, and $V|_{(-\epsilon,\epsilon)\times[\tilde{a}_{i-1},\tilde{a}_i]}$ is smooth for some (possibly finer) subdivision $\tilde{a}_i$ of $[a,b]$. An admissible family Γ defines twe collections of curves:

$$\begin{aligned} \Gamma_s(t) &= \Gamma(s,t) \quad s \text{ is constant on the } \textit{main curves} \\ \Gamma^t(s) &= \Gamma(s,t) \quad t \text{ is constant on the } \textit{transverse curves} \end{aligned}$$

The main curves are piecewise smooth on $[a,b]$, and the transverse curves are smooth on $(-\epsilon,\epsilon)\times[a,b]$ for all $t \in [a,b]$.

Remark 4.4.20. Further on, we will use some shorthand notation:

$$\begin{aligned} (\partial_s\Gamma)(s,t) &= \frac{d}{dt}\Gamma(s,t) = \frac{d}{dt}\Gamma_s(t) = T(s,t) && \text{(the velocity vector field of the main curves)} \\ (\partial_t\Gamma)(s,t) &= \frac{d}{ds}\Gamma(s,t) = \frac{d}{ds}\Gamma^t(s) = S(s,t) && \text{(the velocity vector field of the transverse curves)} \end{aligned}$$

Note that S is continuous on $(-\epsilon,\epsilon)\times[a,b]$, so it is always a vector field along Γ. However, T is not always continuous at $t = a_i$, so it is only a vector field along Γ at points where Γ is smooth. Next we will denote:

$$\begin{aligned} D_sV &= \text{the covariant derivative of } V \text{ along the transvrse curves } \Gamma^t s) \\ D_tV &= \text{the covariant derivative of } V \text{ along the main curves } \Gamma_s(t) \text{ when } \Gamma \text{ is smooth} \end{aligned}$$

Lemma 4.4.21. [Symmetry lemma] - *Lemma 6.3 in [Lee97]*
Let $\Gamma : (-\epsilon,\epsilon)\times[a,b] \to M$ be an admissible family of curves. On any rectangle $(-\epsilon,\epsilon)\times[a_{i-1},a_i]$ where Γ is smooth,

$$D_s(\partial_t\Gamma) = D_t(\partial_s\Gamma)$$

That is, $D_s(T) = D_t(S)$.

Proof: In local coordinates, we have

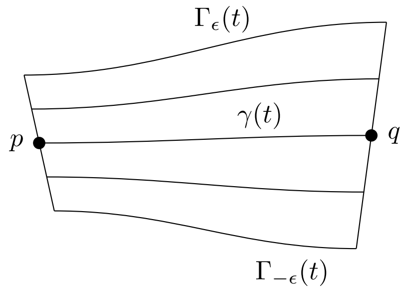
$$T = \partial_t\Gamma = \frac{\partial\Gamma^i}{\partial t}(s,t)\,\frac{\partial}{\partial x^i}\bigg|_{\Gamma(s,t)} \qquad S = \partial_s\Gamma = \frac{\partial\Gamma^i}{\partial s}(s,t)\,\frac{\partial}{\partial x^i}\bigg|_{\Gamma(s,t)}$$

By the properties of the covariant derivative, we then have

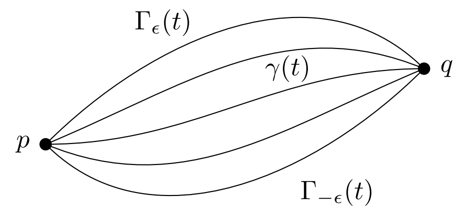
$$\begin{aligned} D_s(T) &= D_s\left(\frac{\partial\Gamma^i}{\partial t}\frac{\partial}{\partial x^i}\right) = \frac{\partial^2\Gamma^i}{\partial s\partial t}\frac{\partial}{\partial x^i} + \frac{\partial\Gamma^i}{\partial t}\frac{\partial\Gamma^j}{\partial s}\Gamma^k_{ji}\frac{\partial}{\partial x^k} \\ D_t(S) &= D_t\left(\frac{\partial\Gamma^i}{\partial s}\frac{\partial}{\partial x^i}\right) = \frac{\partial^2\Gamma^i}{\partial t\partial s}\frac{\partial}{\partial x^i} + \frac{\partial\Gamma^i}{\partial s}\frac{\partial\Gamma^j}{\partial t}\Gamma^k_{ji}\frac{\partial}{\partial x^k} \end{aligned}$$

The two Christoffel symbols are the same if the connection is torsion-free, which it is in this case. ■

Definition 4.4.22. Let $\gamma : [a,b] \to M$ be an admissible curve. A *variation* of γ is an admissible family of curves $\Gamma : (-\epsilon, \epsilon) \times [a,b] \to M$ such that $\gamma(t) = \Gamma_0(t) = \Gamma(0,t)$. It is termed a *proper variation* (or a *fixed point variation*) if $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all $s \in (-\epsilon, \epsilon)$.



a variation of γ



a proper variation of γ

The *variation field* V of a variation Γ is $V = \partial_s \Gamma = S$, the velocity vector field of the transverse curves. A velocity vector field along γ is called *proper* if $V(a) = 0 \in T_{\gamma(a)}M$ and $V(b) = 0 \in T_{\gamma(b)}M$. It is clear that the variation of a proper variation is a proper vector field along γ.

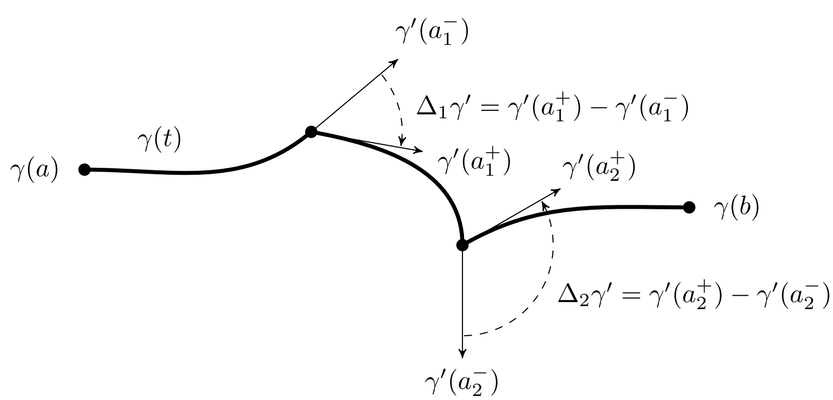
Lemma 4.4.23. - *Lemma 6.4 in [Lee97]*
If γ is admissible and V is any vector field along γ, then V is the vector field for some (non-unique) variation Γ of γ. Moreover, if V is proper, the variation Γ can be taken to be proper as well.

Proof: Define $\Gamma(s,t) = \exp(sV(t))$, i.e. we follom the geodesic starting at $\gamma(t)$ with initial velocity $v(t) \in T_{\gamma(t)}M$ for a time s. Since $[a,b]$ is compact, there exists an $\epsilon > 0$ such that Γ is defined on $(-\epsilon,\epsilon) \times [a,b]$. And Γ is smooth on $(-\epsilon,\epsilon) \times [a_{i-1}, a_i]$ by the properties of the exponentiol map, for each interval $[a_{i-1}, a_i]$ on which V is smooth. Further, Γ is continuous on the whole domain. By further properties of the exponential map,

$$(\partial_s \Gamma)(0,t) = \frac{d}{ds}\bigg|_{s=0} \exp(sV(t)) = V(t)$$

So the variation field of Γ is V. Moreover, if V is proper, then $V(a) = 0$ and $V(b) = 0$ imply that $\Gamma(s,a) = \gamma(a)$ and $\Gamma(s,b) = \gamma(b)$ for all $s \in (-\epsilon,\epsilon)$. Hence Γ is proper. ■

Theorem 4.4.24. [FIRST VARIATION FORMULA OF THE LENGTH FUNCTIONAL] - *Proposition 6.5 in [Lee97]*
Let $\gamma : [a,b] \to M$ be any unit speed admissible curve, Γ a proper variation of γ, and V its variation field.



Then $\frac{d}{ds}\big|_{s=0} L(\Gamma_s) = -\int_a^b g(V, D_t\gamma')\, dt - \sum_{i=1}^{k-1} g(V(a_i), \Delta_i\gamma')$.

Proof: We first note that the length on every subinterval is given by

$$L\left(\Gamma_s|_{[a_{i-1},a_i]}\right) = \int_{a_{i-1}}^{a_i} \left|\frac{\partial \Gamma_s}{\partial t}\right|_g \, dt = \int_{a_{i-1}}^{a_i} |T|_g \, dt$$

We may integrate on any interval $[a_{i-1}, a_i]$ where Γ is smooth:

$$\frac{d}{ds} L\left(\Gamma_s|_{[a_{i-1},a_i]}\right) = \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s}|T|_g \, dt = \int_{a_{i-1}}^{a_i} \frac{1}{2|T|_g}\left(g(D_s(T),T) + g(T, D_s(T))\right) \, dt = \int_{a_{i-1}}^{a_i} \frac{g(D_t(S),T)}{|T|_g} \, dt$$

Before we proceed, consider the following decomposition:

$$\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s) = \sum_{i=1}^{k} \left.\frac{d}{ds}\right|_{s=0} L\left(\Gamma_s|_{[a_{i-1},a_i]}\right) = -\sum_{i=1}^{k-1} g(V(a_i), \Delta_i\gamma') - \int_a^b g(V, D_t\gamma') \, dt$$

Now set $s = 0$ and recall that $S(0,t) = V(t)$ and $T(0,t) = \gamma'(t)$. Apply the above for:

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} L(\Gamma_s|_{[a_{i-1},a_i]}) &= \int_{a_{i-1}}^{a_i} g(D_t(V), \gamma') \, dt \\
&= \int_{a_{i-1}}^{a_i} \left(\frac{d}{dt} g(V, \gamma') - g(V, D_t\gamma')\right) \, dt \\
&= g(V(a_i), \gamma'(a_i^-)) - g(V(a_{i-1}), \gamma'(a_{i-1}^+)) - \int_{a_{i-1}}^{a_i} g(V, D_t\gamma') \, dt
\end{aligned}$$

Summing over all i with the observation that $V(a_0) = V(a_k) = 0$, the desired formula appears. ■

Theorem 4.4.25. - *Theorem 6.6 in [Lee97]*
Suppose that γ is a minimizing curve between two points $p, q \in M$. Without loss of generality, γ has unit speed parametrization, and is thus a geodesic.

Theorem 4.4.26. [Gauss lemma] - *Theorem 6.8 in [Lee97]*
Let $U = \exp_p(B_\delta(0))$ be a geodesic ball centered at $p \in M$. The unit radial vector field $\frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i}$ is orthogonal to the geodesic sphere $\exp_p(\partial B_\delta(0))$ for all $\delta < \epsilon$.

Proof:

Corollary 4.4.27. - *Corollary 6.9 in [Lee97]*
Let $(x^1, \dots, x^n)$ be normal coordinates ceneterd at p, on some geodesic ball $U = \exp_p(B_\epsilon(0))$. Let $r = \sqrt{\sum_{i=1}^n (x^i)^2}$, which is smooth on $U \setminus \{p\}$. Then $\mathrm{grad}(r) = \nabla r = \frac{\partial}{\partial r}$ on $U \setminus \{p\}$.

The proof is omitted

Proposition 4.4.28. - *Proposition 6.10 in [Lee97]*
Let $p \in M$, and q be contained in a geodesic ball centered at p. Then up to reparametrization, the radial geodesic from p to q is the unique minimizing curve from p to q.

The proof is omitted

Corollary 4.4.29. - *Corollary 6.11 in [Lee97]*
Within a geodesic ball centered at $p \in M$, the function $r(x) = \sqrt{\sum_{i=1}^n (x^i)^2}$ is the distance function from p to x, i.e. $d(p,x) = r(x)$.

The proof is omitted

So far, we have shown that if q lies in a geodesic ball centered at p (i.e. if q is in the image of $\exp_p$), then there exists a unique minimizing curve γ from p to q, and this curve is a radial geodesic, and $r(q) = d(p,q) = L(\gamma)$.

Definition 4.4.30. An admissible curve γ is *locally minimizinig* if for any $t_0 \in I$, there exists a neighborhood U of t_0 in I such that $\gamma|_U$ is minimizing between any two points on $\gamma|_U$.

Theorem 4.4.31. - *Theorem 6.12 in [Lee97]*
Every geodesic is locally minimizing.

The proof is omitted

Definition 4.4.32. Let (M, g) be a Riemannian manifold. Then (M, g) is *geodesically complete* if every geodesic is defined for all $t \in \mathbb{R}$. This is equivalent to saying that the geodesic vector field is a complete vector field.

Theorem 4.4.33. [Hopf, Rinow] - *Corollary 6.11 in [Lee97]*
Let (M, g) be a connected Riemannian manifold. Then (M, g) is geodesically complete iff it is complete as a metric space.

The proof is omitted

Definition 4.4.34. Let $\gamma : [0, b] \to M$ be a geodesic. We say that γ *aims at* q if:

i. γ is minimizing from $\gamma(0)$ to $\gamma(b)$
ii. $d(\gamma(0), q) = d(\gamma(0), \gamma(b)) + d(\gamma(b), q)$

Note that if γ were the initial segment of a minimizing geodesic from $\gamma(0)$ to q, then γ aims at q.

Corollary 4.4.35. - *Corollaries 6.14-6.16 in [Lee97]*
The following are corollaries to the Hopf–Rinow theorem:

1. If there is $p \in M$ such that $\exp_p$ is defined on all of T_pM, then M is complete
2. M is complete iff any two points in M can be joined by a minimizing geodesic
3. If M is compact, then every geodesic can be defined for all $t \in \mathbb{R}$

The proof is omitted

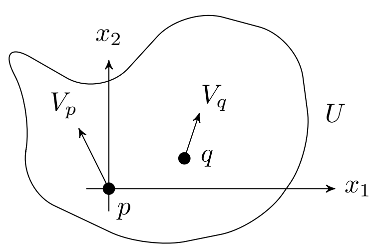
5 Curvature

5.1 Flatness and curvature

Recall that (M, g) is flat iff it is locally isometric to $(\mathbb{R}^n, \overline{g})$. We proved that this is equivalent to the existence of local coordinates such that the coordinate frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is orthonormal. We are now looking for a coordinate-free ("geometric") way to characterize flatness.

Remark 5.1.1. In $(\mathbb{R}^n, \bar{g})$, any tangent vector $V_p \in T_p\mathbb{R}^n$ may be extended uniquely to a parallel vector field $V \in \Gamma(T\mathbb{R}^n)$. Let $\{E_1, \ldots, E_n\}$ be the standard global orthonormal frame, so $V_p = a^i E_i|_p$ for $a_i \in \mathbb{R}$. So V is a smooth vector field in $\mathbb{R}^n$.

Let (M, g) be a Riemannian manifold with $p \in M$, $V_p \in T_pM$. Does there exist a smooth vector field V in a neighborhood V of p such that $V|_p = V_p$ and $\nabla_X V = 0$ for all vector fields X on U? For $(x^1, \ldots, x^n)$ local coordinates centered at p, the situtation looks as below.



Here $p = (0,0)$ and $q = (a_1, a_2)$. If such a V did exist, then $V|_q = V_q$ would be the result of parallel transport of V_p along every path in U from p to q. Using this idea, define $\tilde{V}$, a smooth vector field on U by

$$\tilde{V}|_q = \text{parallel transport of } V_p \text{ from } (0,0) \text{ to } (a_1, 0) \text{ and then to } (a_1, a_2).$$

The field $\tilde{V}$ is smooth because the parallel transport of it is an ODE, so it has smooth dependence on initial conditions. If there exists a parallel extension V of V_p to a neighborhood of p, it would have to agree with this $\tilde{V}$ we have just defined, i.e. $\tilde{V}$ would have to be parallel. By construction, $\nabla_{\frac{\partial}{\partial x^2}} \tilde{V} = 0$ at every point in U, and $\nabla_{\frac{\partial}{\partial x^1}} \tilde{V} = 0$ on the x-axis. For $\tilde{V}$ to be parallel, we need $\nabla_{\frac{\partial}{\partial x^1}} \tilde{V} = 0$ at every point (this will imply that $\nabla_X \tilde{V} = 0$ for any X). So let $W = \nabla_{\frac{\partial}{\partial x^1}} \tilde{V}$, so W is a smooth vector field on U. Note that $W|_{(a_1,0)} = 0$, so by the uniqueness of the parallel transport, $W = 0$ is an eigenvalue. Finally observe that

$$\nabla_{\frac{\partial}{\partial x^2}} W = \nabla_{\frac{\partial}{\partial x^2}} \left(\nabla_{\frac{\partial}{\partial x^2}} \tilde{V} \right) = \nabla_{\frac{\partial}{\partial x^1}} \left(\nabla_{\frac{\partial}{\partial x^2}} \tilde{V} \right) = \nabla_{\frac{\partial}{\partial x^1}} (0) = 0.$$

Hence if $\nabla_{\frac{\partial}{\partial x^1}}$ and $\nabla_{\frac{\partial}{\partial x^2}}$ commute, we can find such a parallel extension. But where is it true?
On $(\mathbb{R}^n, \bar{g})$ with $V = V^i E_i \in \gamma(T\mathbb{R}^n)$, we have that

$$\overline{\nabla}_Y V = Y(V^i) E_i \qquad \text{and} \qquad \begin{aligned} \overline{\nabla}_X \left(\overline{\nabla}_Y V \right) &= X(Y(V^i)) E_i \\ \overline{\nabla}_X \left(\overline{\nabla}_Y V \right) - \overline{\nabla}_Y \left(\overline{\nabla}_X V \right) &= X(Y(V^i)) E_i - Y(X(V^i)) E_i \\ &= ([X,Y] V^i) E_i \\ &= \overline{\nabla}_{[X,Y]} V \end{aligned} .$$

This shows that for the Euclidean connection $\overline{\nabla}$ of $\bar{g}$ and all vector fields X, Y, Z on $\mathbb{R}^n$,

$$\overline{\nabla}_X \left(\overline{\nabla}_Y Z \right) - \overline{\nabla}_Y \left(\overline{\nabla}_X Z \right) - \overline{\nabla}_{[X,Y]} Z = 0.$$

This motivates the following definition.

Definition 5.1.2. Let (M, g) be a Riemannian manifold. The *Riemann curvature tensor* R is a type $(3,1)$ tensor on M, such that if $X, Y, Z \in \Gamma(TM)$, then $R(X, Y, Z) = R(X, Y)Z \in \Gamma(TM)$, where

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

Remark 5.1.3. In local coordinates, we have that

$$\begin{aligned} R &= R_{ijk\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell \\ R_{ijk\ell} &= g \left(R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) = R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \end{aligned}$$

The symmetries of the tensor are then given by

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$$

Theorem 5.1.4. - *Theorem 7.3 in [Lee97]*
The Riemannian manifold (M, g) is flat iff $R = 0$.

The proof is omitted

Remark 5.1.5. One may define the curvature R_∇ of any connection on TM (by the same formula). We proved above that $R_\nabla = 0$ iff every point $p \in M$ lies in a neighborhood U on which there exists a parallel frame $\{E_1, \ldots, E_n\}$, and $E_i \in \Gamma(TU)$ is linearly independent at every point, and $\nabla_X E_i = 0$ for all $X \in \Gamma(TU)$.

With the theorem, we get more - if ∇ is metric-compatible, this frame can be takn to be orthonormal. If ∇ is torsion-free, this frame can be taken to be a coordinate frame.

Proposition 5.1.6. - *Proposition 7.4 in [Lee97]*
Let (M,g) be a Riemannian manifold and R the $(4,0)$ curvature tensor of g. Let $X,Y,Z \in \Gamma(TM)$. Then

a. $R(X,Y,Z,W) = -R(Y,X,Z,W)$
b. $R(X,Y,Z,W) = -R(X,Y,W,Z)$
d. $R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,X,Z,W) = 0$
c. $R(X,Y,Z,W) = R(Z,W,X,Y)$

The proof is omitted

Note that **a.** is true for all connections, i.e. the (3,1)-curvature tensor is always skew-symmetric in its first two arguments. Further, **b.** is true for any connection compotible with the metric.

Remark 5.1.7. The symmetries of the tensor are then given by

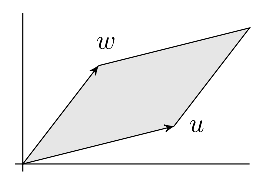
$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$$

It follows from **c.** above that if we fix any of the four arguments and cyclically permute the other three, the sum is zero. We then conclude that there are far fewer than n^4 independent components.

5.2 Sectional curvature

Suppose (V,g) is a finite-dimensional real, positive-definite inner-product space. Given $v,w \in V$, define

$$|u \wedge v|^2 = |u|^2|w|^2 - g(u,w)^2 = \begin{array}{c}\text{area of the}\\ \text{parallelogram}\\ \text{spanned by } u,w\end{array} \rightarrow$$



$$= \det \begin{bmatrix} |u|^2 & g(u,w) \\ g(u,w) & |w|^2 \end{bmatrix}$$

Definition 5.2.1. Let L_p be a 2-dimensional subspace of T_pM. Let X_p, Y_p be any basis of L_p. Define the *sectional curvature* of (M,g) at the 2-plane T_pM to be

$$K(L_p) = \frac{R(X_p,Y_p,Y_p,X_p)}{|X_p \wedge Y_p|^2}$$

To show that this is well-defined, suppose that $\tilde{X}_p, \tilde{Y}_p$ is another basis of L_p. We need to show that

$$\frac{R(X_p,Y_p,Y_p,X_p)}{|X_p \wedge Y_p|^2} = \frac{R(\tilde{X}_p,\tilde{Y}_p,\tilde{Y}_p,\tilde{X}_p)}{|\tilde{X}_p \wedge \tilde{Y}_p|^2}$$

From linear algebra, we know that any two bases are related by a finite sequence of the form

$$(X,Y) \to (Y,X)$$
$$(X,Y) \to (\lambda X, Y) \quad \lambda \neq 0$$
$$(X,Y) \to (X + \lambda X, Y) \quad \lambda \in \mathbb{R}$$

The Riemann curvature determines all the sectional curvatures. We will prove the converse, that knowing all K_p determines R_p.

Lemma 5.2.2. Let $(V, \langle \cdot\,,\, \cdot \rangle)$ be a finite-dimensional vector space. Let R, R' be two tri-linear maps $V^{\times 3} \to V$ with $(X,Y,Z,W) = \langle R(X,Y,Z), W\rangle$ and $(X,Y,Z,W)' = \langle R'(X,Y,Z), W\rangle$ such that (X,Y,Z,W) is skew-symmetric in X,Y and Z,W, with

$$(X,Y,Z,W) = (Z,W,X,Y) \qquad \text{and} \qquad (X,Y,Z,W) + (Z,X,Y,W) + (Y,Z,X,W) = 0$$

and similarly for $(X,Y,Z,W)'$. For X,Y linearly independent and $\sigma = \mathrm{span}\{X,Y\}$ with

$$K(\sigma) = \frac{(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X,Y\rangle^2} \qquad \text{and} \qquad K'(\sigma) = \frac{(X,Y,Y,X)'}{|X|^2|Y|^2 - \langle X,Y\rangle^2},$$

if $K = K'$ for all σ, then $R = R'$.

The proof is omitted

Corollary 5.2.3. If we know all the sectional curvatures $K_p(\sigma_p)$ for all $\sigma_p \in T_pM$, then we know R_p.

Lemma 5.2.4. [2nd Bianchi identity]
Let $U,V,W,X,Y,Z \in \Gamma(TM)$. Then

$$(\nabla_U R)(X,Y,V,W) + (\nabla_V R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,V) = 0.$$

The proof is omitted

Proposition 5.2.5. The trace $\mathrm{trace}(A) = g^{ij}\langle Ae_i, e_j\rangle$, where $g^{ij} = \langle e^i, e^j\rangle$, and $\{e^1,\dots,e^n\}$ is a dual basis of V^*.

The proof is omitted

Definition 5.2.6. The Ricci tensor Ric is a $(2,0)$-tensor on M, defined by $(\mathrm{Ric})_p(X_p,Y_p) = \mathrm{trace}(A_p)$.

Proposition 5.2.7. The Ricci tensor is symmetric, i.e. $R_{jk} = R_{kj}$.

The proof is omitted

Definition 5.2.8. The *scalar curvature* is the trace of the Ricci curvature.

5.3 Einstein manifolds

Definition 5.3.1. A Riemannian manifold (M,g) is called *Einstein* if $\mathrm{Ric} = fg$ for $f \in C^\infty(M)$.

Then $R_{jk} = fg_{jk}$, so $R = R_{jk}g^{jk} = fg_{jk}g^{jk} = f\delta^k_k = nf$. Hence if (M,g) is Einstein, then this function f must be R/n. This gives the *Einstein equation*:

$$\mathrm{Ric} = \frac{R}{n}g.$$

Remark 5.3.2. This notation was introduced by Einstein in general relativity. The equations of general relativity say that $\mathrm{Ric} - \frac{R}{2}g = T$, for T the stress-energy tensor, which measures the matter in the universe. In a vacuum, $T = 0$, so $\mathrm{Ric} = \frac{R}{2}g = fg$, hence $\frac{R}{2} = \frac{R}{n}$. So if $n \neq 2$, then $R = 0$. Therefore any solution to the stress-energy tensor equation in a vacuum must have $\mathrm{Ric} = 0$. This situation is called *Ricci-flat*.

Remark 5.3.3. We can apply the above observations to the 2nd Bianchi identity. First contract with $g^{i\ell}$ and use the fact that ∇ is g-compatible.

$$\begin{aligned} \nabla_m(R_{ijk\ell}g^{i\ell}) + \nabla_k(R_{ij\ell m}g^{i\ell}) + \nabla_\ell(R_{ijmk}g^{i\ell}) &= 0 \\ \nabla_m(R_{jk}) - \nabla_k(R_{jm}) + \nabla_\ell(g^{i\ell}R_{ijm\ell}) &= 0 \end{aligned}$$

Next contract with g^{jk}.

$$\begin{aligned} \nabla_m R - g^{jk}\nabla_k R_{jm} - g^{i\ell}\nabla_\ell R_{im} &= 0 \\ g^{pq}\nabla_p R_{qk} &= \tfrac{1}{2}\nabla_k R \end{aligned}$$

The last expression is known as the *twice contracted 2nd Bianchi identity*. It follows that $\mathrm{div}(\mathrm{Ric}) = \frac{1}{2}\nabla R$, where ∇ is the gradient operator.

Proposition 5.3.4. If (M^n, g) is Einstein and $n \geqslant 3$, then R is constant (i.e. the scalar curvature is constant).

Proof: So we have that $R_{ij} = \frac{R}{n} g_{ij}$ and $\nabla_k R_{ij} = \frac{1}{n}(\nabla_k R) g_{ij}$, so

$$\begin{aligned} g^{ik}\nabla_k R_{ij} &= \tfrac{1}{n}(\nabla_k K) g_{ij} g^{ik} \\ = \tfrac{1}{2}\nabla_j R \quad &= \tfrac{1}{n}\nabla_j R. \end{aligned}$$

Then if $n \neq 2$,

$$\begin{aligned} \nabla_j R = 0 \;\forall\; j &\text{ iff } \nabla R = 0 \\ &\text{ iff } dR = 0 \\ &\text{ iff } R = \text{ constant.} \end{aligned}$$

■

Remark 5.3.5. Note that when $N = 2$, R may be non-constant. In fact, every such Riemannian manifold is Einstein. For $K : M \to R$, we have that

$$\begin{aligned} R'(X, Y, Z, W) &= g(X, W) g(Y, Z) - g(X, Z) g(Y, W) \\ R(X, Y, Y, X) &= K(|X|^2 |Y|^2 - g(X, Y)^2) \\ &= K R(X, Y, Y, X). \end{aligned}$$

Hence if $n = 2$, $R_{ijk\ell} = K(g_{i\ell} g_{jk} - g_{ik} g_{j\ell})$, where K is the sectional curvature function. In the case of the Ricci curvature, we have that $R_{jk} = R_{ijk\ell} g^{k\ell} = K(2g_{jk} - g_{jk}) = K g_{jk}$. So any Riemannian 2-manifoldis Einstein with $\mathrm{Ric} = Kg$, for K the sectional curvature.

Remark 5.3.6. Consider some other remarks about Einstein manifolds.

· It is still a wide open question on which manifolds admit Einstein metrics.

· There exists a variational characterization of Einstein manifolds, called the Einstein–Hilbert functional

$$\mathcal{H} : \left\{ \begin{array}{c} \text{space of Riemannian} \\ \text{metrics on } M \end{array} \right\} \to \mathbb{R},$$

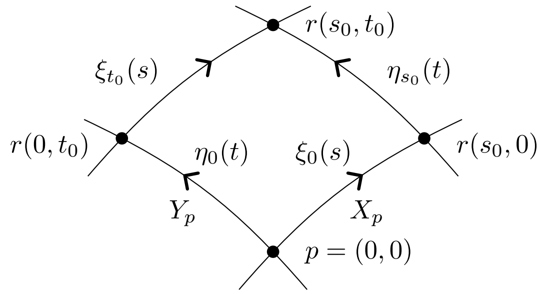
where $\mathcal{H}(g) = \int_g R_g \mu_g$ is called the *total scalar curvature*.

· Fixing some metric g_0 on M, we can talk about the *conformal class* of g_0, denoted by $[g_0]$, which is the set of all metrics of the form $e^{2f} g_0$ for some $f \in C^\infty(M)$. When restricted to a fixed conformal class, critical points of $\mathcal{H}$ are the constant scalar curvature metrics $\nabla R = 0$.

The Yamabe conjecture asked, given g_0 on M, if there exists a metric $g = e^{2f} g_0$ (i.e. conformal to g_0) such that g_0 has constant scalar curvature. It was solved in the affirmative for dimension 2, but counterexamples have been constructed for dimensions $\geqslant 3$.

5.4 Geometric interpretations of the Riemannian curvature

Remark 5.4.1. Let $(x^1, \ldots, x^n)$ be local coordinates centered at p. Let $X_p, Y_p \in T_p M$ be linearly independent. Define $r(s,t) = \exp_p(sX_p + tY_p)$. This is a diffeomorphism of an open rectangle in (s,t) from the plane containing $(0,0)$ onto an open neighborhood of $p \in M$. This may be visualized as below.



Let ρ denote the path from p to $r(s_0, 0)$ then to $r(s_0, t_0)$ by the lines indicated, and $\tilde{\rho}$ the path from p to $r(0, t_0)$ then to $r(s_0, t_0)$ also by the lines indicated. Fix some $Z_p = Z^k \frac{\partial}{\partial x^k}|_p \in T_pM$, and let $Z(s,t)$ be the parallel transport of Z_p to $r(s_0, t_0)$ along ρ and $\tilde{Z}(s,t)$ the parallel transport of Z_p along $\tilde{\rho}$.

$$\Pi_{\rho_1|_{[s,0]}}(Z_p) = Z^k(s) \left.\frac{\partial}{\partial x^k}\right|_{r(s,0)} \qquad Z(s,t) = \Pi_{\rho_2}(Z(s,0)) = Z^k(s,t) \left.\frac{\partial}{\partial x^k}\right|_{r(s,t)}$$

$$\frac{dZ^k}{ds} = -\Gamma^k_{ij} \frac{d\xi^i_0}{ds} Z^j \qquad \frac{dZ^k}{dt} = -\Gamma^k_{ij} \frac{d\xi^i_{s_0}}{dt} Z^j(s,t)$$

$$\Pi_{\tilde{\rho}_1|_{[0,t]}}(Z_p) = \tilde{Z}^k(t) \left.\frac{\partial}{\partial x^k}\right|_{r(0,t)} \qquad \tilde{Z}(s,t) = \Pi_{\tilde{\rho}_2}(\tilde{Z}(0,t)) = \tilde{Z}^k(s,t) \left.\frac{\partial}{\partial x^k}\right|_{r(s,t)}$$

$$\frac{d\tilde{Z}^k}{dt} = -\Gamma^k_{ij} \frac{d\xi^i_0}{ds} \tilde{Z}^j \qquad \frac{d\tilde{Z}^k}{dt} = -\Gamma^k_{ij} \frac{d\xi^i_{s_0}}{dt} \tilde{Z}^j(s,t)$$

Above, ρ_1 denotes the path from p to $r(s_0, 0)$ and ρ_2 the path from $r(s_0, 0)$ to $r(s_0, t_0)$. The decomposition is analogous for $\tilde{\rho}$. After some more calculation, we find that

$$\tilde{Z}^k(s,t) - Z^k(s,t) = 0 + 0 - (R(X_p, Y_p)Z_p)^k + \mathcal{O}(3),$$

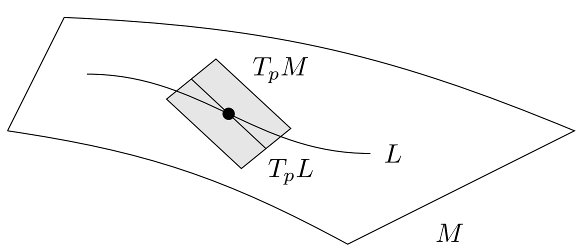
where the first two zeros are the 0th and 1st order terms in the Taylor expansion. So in general,

$$\tilde{Z}(s,t) - Z(s,t) = -R(X_p, Y_p)Z_p(s,t) + \mathcal{O}(3).$$

In other words, the curvature is a local obstruction to the path independence of parallel transport. This may be formulated in a more precise manner, as below.

Theorem 5.4.2. [Ambrose, Singer]
Let M^n be a connected and simply-connected manifold with ∇ the Levi–Civita connection on TM. Then the homolonomy group H of ∇ at $p \in M$ is a compact Lie group. The Lie algebra $\mathfrak{h}$ is generated as a vector space by the curvature operator $R(X_p, Y_p)$ for all $X_p, Y_p \in T_pM$.

Remark 5.4.3. There is another interpretation of curvature, specific to the Riemann curvature tensor.Let $f : L^k \to M^n$ be an immersion, so $(f_*)_p : T_pL \to T_{f(p)}M$ is injective for all p. Then further, $(f_*)_p(T_pL) \subset T_{f(p)}M$.



Now we may view L as a subset of M with $i : L \to M$ the inclusion map, which is a smooh 1-1 immersion. If g is a metric on M, then $i^*g = g|_L$ is a metric on L. Note if $p \in L$, then $T_pM = T_pL \oplus (T_pL)^\perp$, where the second term is the orthogonal complement with respect to g. Using this structure we can make some new definitions.

Definition 5.4.4. Let $N_pL = (T_pL)^\perp$ be the *normal space* to L at p. Let $NL = \bigsqcup_{p\in L} N_pL$ be the *normal bundle* of L in M, essentially then bundle of $(n-k)$-dimensional vector spaces over L.

Remark 5.4.5. Given $p \in L$, we can find a local frame $\{E_1, \dots, E_n\}$ over some $U \ni p$ such that when restricted to $U \cap L$, $\{E_1, \dots, E_k\}$ is a frame for TL, so then $\{E_{k+1}, \dots, E_n\}$ is a frame for NL. So consider a vector field X along L, meaning that $X \in \Gamma(i^*(TM)) = \Gamma(TM|_L) = \Gamma(TL \oplus NL)$. Then given $X_p \in T_pL$, we can decompose it as

$$X_p = \underset{T_pL}{\underset{\Cap}{X_p^T}} + \underset{N_pL}{\underset{\Cap}{X_p^N}}.$$

Let ∇^M be the Levi–Civita connection of g on M. Let X, Y be vector fields along L. Then

$$\nabla_X^M Y = \left(\nabla_X^M Y\right)^T + \left(\nabla_X^M Y\right)^N.$$

Proposition 5.4.6. With reference to the notation above,

i. $(X, Y) \to (\nabla_X^M Y)^T$ is a connection on TL and is metric-compatible and torsion-free. By uniqueness, $\left(\nabla_X^M Y\right)^T = \nabla_X^L Y$.

ii. $\left(\nabla_X^M Y\right)^N = B(X, Y) = B(Y, X)$ is symmetric in X and Y

iii. For $V \in \Gamma(NL)$, $g\left(\nabla_X^M V, Y\right) = -g(V, B(X, Y))$. This is called the Weingarten equation.

<u>*Proof:*</u> Only a sketch of **i.** is presented. To show that it is torsion-free, note that

$$(\nabla_X^m Y)^T - \left(\nabla_Y^M X\right)^T = \left(\nabla_X^M Y - \nabla_Y^M X\right)^T = ([X, Y])^T = [X, Y].$$

For metric compatibility, where X, Y, Z are all tangent to L, observe that

$$X(g(Y, Z)) = g\left(\nabla_X^M Y, Z\right) + g\left(Y, \nabla_X^M Z\right) = g\left(\left(\nabla_X^M Y\right)^T, Z\right) + g\left(Y, \left(\nabla_X^M Z\right)^T\right).$$

■

Corollary 5.4.7. [Gauss]
For vector fields X, Y, Z, W along L,

$$R^M(X, Y, Z, W) = R^L(X, Y, Z, W) - g(B(X, W), B(Y, Z)) + g(B(X, Z), B(Y, W)).$$

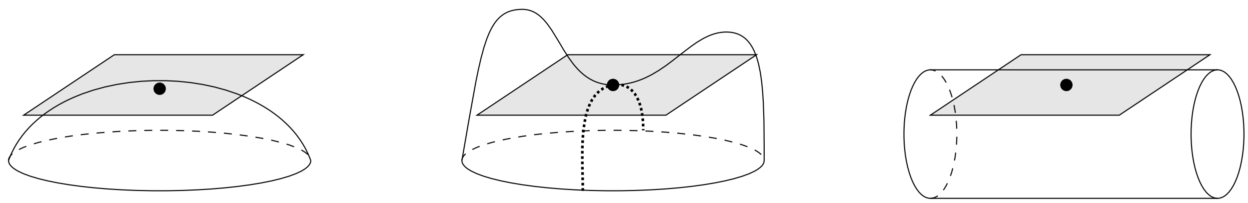
This is called the *Gauss equation.*

Remark 5.4.8. Let (M, g) be a Riemannian manifold with $p \in M$ and $\sigma_p \subset T_pM$ a 2-dimensional subspace. Let $V \subset T_pM$ on which $\exp_p : V \to M$. Define $S = \exp_p(V \cap \sigma_p)$, where $V \cap \sigma_p$ is an open neighborhood of 0_p. Then S is a 2-dimensional immersed submanifold of M. More specifically, it is the collection of all points on geodesics starting at p with initial velocities in σ_p. Now apply the Gauss equation to $L = S$.

Let $X_p \in \sigma_p$ with $\gamma(t) = \gamma_{X_p}(t)$ geodesics, so $0 = D_t^M\gamma' = D_t^L\gamma' + \beta(\gamma', \gamma')$, where β is the normal component. Hence $\beta(\gamma', \gamma') = 0$ at $t = 0$, so $\beta(X_p, X_p) = 0$ for all $X_p \in \sigma_p$. By polarization (β is symmetric), $\beta(X_p, Y_p) = 0$ for all $X_p, Y_p \in \sigma_p$. Hence the Gauss equation gives that

$$R^M(X, Y, Y, X) = R^L(X, Y, Y, X)$$

for all X, Y tangent to S. This may be equivalently written as $K^M(\sigma_p) = K^L(\sigma_p)$ for $K^L : S \to \mathbb{R}$. Hence if $p \in M$ and $\sigma_p \subset T_pM$, S is 2-dimensional, and the sectional curvature $K_p^M(\sigma_p)$ of g equals the sectional curvature $K^L(\sigma_p)$ of $(S, g|_S)$. Consider this in the following situations:



$K > 0$ $\qquad$ $K < 0$ $\qquad$ $K = 0$

Hence the sign of $K_p(\sigma_p)$ describes the qualitative behavior of geodesics.

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References

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