Compact course notes

Dynamic systems

Fall 2011

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Contents

<u>*Note:*</u> For a more complete exposition, see the following textbooks, which were used in accompaniment with this course:

· *Introduction to the Modern Theory of Dynamical Systems,* Anatole Katok and Boris Hasselblatt

· *An Introduction to Dynamical Systems,* D.K. Arrowsmith and C.M. Place

1 Introduction

1.1 Definitons

Definition 1.1.1. Given a map $F : X \to X$,a point $x \in X$ is termed a fixed point of F iff $F(x) = x$. The orbit of x is the set $\text{Fix}(x) = \{F^n(x) \mid n \in \mathbb{Z}_{\geqslant 0}\}$.

Definition 1.1.2. Two maps $F : X \to X$ and $G : Y \to Y$ are termed conjugate if there exists a bijection $h : X \to Y$ such that $G \circ h = h \circ F$.

Proposition 1.1.3. If F and G are conjugated by h, then $h\big|_{\text{Fix}(F)} : \text{Fix}(F) \to \text{Fix}(G)$ is a bijection.

Proposition 1.1.4. Suppose $G \circ h = h \circ F$ and $F(x_0) = x_0$ with $h'(x_0) \neq 0$. Then $G'(h(x_0)) = F'(x_0)$.

Definition 1.1.5. Let X be a space and $F : X \to X$ a function. Then the fate of a point $x \in X$ is the sequence $\ldots, \omega_{-n}\omega_{-n+1} \ldots \omega_{-1}\omega_0\omega_1 \ldots \omega_n\omega_{n+1} \ldots$ with $\omega_i = k \iff F^i(x) \in R_k$ for R_k a predefined region of X.

Definition 1.1.6. A diffeomorphism between manifolds is an isomorphism that is both differentiable and has a differentiable inverse.

Definition 1.1.7. An Anosov diffeomorphism is a diffeomorphsim $f : M \to M$ from a C^1 manifold M to itself such that the tangent bundle of M is hyperbolic to f. An example is the set of matrices with unit length determinant.

Remark 1.1.8. Let $f : X \to f(X)$ be a map. Then $\det(f) = \frac{\text{Area}(f(X))}{\text{Area}(X)}$.

Moreover, if $A \in M_n$, then the number of fixed points of A^k is $|\det(A^k - I)|$ for $k \in \mathbb{Z}_{\geqslant 0}$.

Remark 1.1.9. Here on in, we use the notation $\mathbb{R}^2/\mathbb{Z} = T^2$ the real torus.

1.2 Markov

Definition 1.2.1. A Markov chain for k random variables is a $k \times k$ matrix A such that $A_{ij} = P(\omega_{n+1} = j \mid \omega_n = i)$ for any n and for all $1 \leqslant i, j \leqslant k$.

Theorem 1.2.2. [Perron, Frobenius]
Suppose the following hold:

- $A = P_{ij}$ is a map with $p_{ij} \geqslant 0$ for all i, j
- For all $k > 0$, the graph corresponding to A^k is strongly connected

Then there exists a unique set $\{p_1, \ldots, p_s\}$ with $p_i > 0$ and $(p_1 \ldots p_s)A = (p_1 \ldots p_s)$.
Moreover, if $p_{ij} > 0$, then for any initial set of probabilities $q_1, \ldots, q_s$ we have $(q_1 \ldots q_s)A = (p_1 \ldots p_s)$.

Definition 1.2.3. Let $f : S \to S'$ be a bijective map preserving orientation. Then $F : \mathbb{R} \to \mathbb{R}$ is termed a lifting of f if and only if $\pi \circ F = f \circ \pi$. Equivalently, when the following diagram commutes.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ S' & \xrightarrow{f} & S = \mathbb{R}/\mathbb{Z} \end{array}$$

In this case we also define the rotation number by

$$\rho(f) = \lim_{n \to \infty} \left[\frac{F^n(x) - x}{n} \right]$$

with the following properties:

1. $\rho(f) = 0 \iff f$ has a fixed point
2. $\rho(f^k) = k\rho(f)$
3. $f(x + 1) = f(x) + 1$

Theorem 1.2.4. [Denjoy]
Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a bijective C^2-map. Suppose that $\rho(f) \notin \mathbb{Q}$ and the second derivative is continuous. Then f is conjugated to $R_\rho(f) : X\ to$

Theorem 1.2.5. Let $\{x\}$ denote the fractional part of x and $\#S$ for S a set denote the cardinality of S. If I is an interval on $\mathbb{R}/\mathbb{Z}$ and α and angle, then for $m \in \mathbb{R}$,

$$\lim_{n\to\infty} \left[\frac{\#\{0 \leqslant m \leqslant n \mid \{m\alpha\} \in I\}}{n} \right] = |I|$$

Moreover, if $\alpha \notin \mathbb{Q}$, then for any function φ on $\mathbb{R}$,

$$\lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=0}^{n-1} \varphi(\{x + \alpha k\}) \right] = \int_0^1 \varphi(y)dy$$

2 Perturbations and structural stability

2.1 Stability

Definition 2.1.1. A map F is termed structurally stable if there exists $\epsilon > 0$ such that for every F' with $||F - F'||_{C^1} < \epsilon$, F' is conjugated to F.

Definition 2.1.2. For X a space, $F : X \to X$ is termed a contracting map if there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$, $d(F(x) - F(y)) \leqslant \lambda d(x, y)$.

Theorem 2.1.3. [Contracting map principle]
Let X be a complete metric space and $F : X \to X$ a contracting map. Then F has a unique fixed point. In other words, there exists unique $x \in X$ such that $F(x) = x$.

Note that the above implies both the inverse and implicit function theorems.

2.2 Newhouse phenomenon

Definition 2.2.1. Given a fixed point and its phase portrait, a curve with a transversal intersection with another curve is termed a separatrix.

Definition 2.2.2. Given a phase portrait of a dynamical system, if a separatrix intersects the same curve non-transversally that it separates, then the non-transversal intersection is termed a homoclinic tangency.

Theorem 2.2.3. [Newhouse phenomenon]
Consider a one-parameter family of dynamical systems with a set of fixed points and homoclinic tangencies of F_0 with F_ϵ, such that increasing the parameter changes a homoclinic tangency to two transversal intersections. Then there exist ”many” $\epsilon > 0$ such that F_ϵ has infinitely many stable periodic points.

Definition 2.2.4. Let U_i be an open, dense set. Then $\bigcap_{i=1}^{\infty} U_i$ is termed a residual set.

The set $S = \{\alpha \mid \forall\epsilon > 0, \forall N \in \mathbb{N},\ \text{there exists}\ \frac{p}{q} \in \mathbb{Q}, q > N \text{ with } |\alpha - \frac{p}{q}| < \frac{\epsilon}{q^3}\}$ is a residual set of Lebesgue measure 0.

2.3 Anosov diffeomorphisms

Theorem 2.3.1. Let $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ be a map $A : T^2 \to T^2$. Then for each $\epsilon > 0$ there exists a $\delta > 0$ such that there is a sequence $\{P_n\}_{n=-\infty}^{\infty}$ with $||AP_n - P_{n+1}|| < \delta$ and $p \in T^2$ such that $||A^n p - P_n|| < \epsilon$.

Definition 2.3.2. A sequence $\{P_n\}_{n=-\infty}^{\infty}$ is termed a δ-pseudo orbit of a torus map A if $||AP_n - P_{n+1}|| < \delta$.

Remark 2.3.3. Any map close enough to $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is topologically conjugated to it, and therefore is an Anosov diffeomorphism.

Theorem 2.3.4. Let F be a function on vectors with v_1, v_2 vector fields on $\mathbb{R}^n$. If

$$\left.\frac{dF}{dx}\right|_{v_1(x)}, \left.\frac{dF}{dx}\right|_{v_2(x)} \in \text{ the angle between } v_1(F(x)) \text{ and } v_2(F(x))$$

for all $x \in \mathbb{R}^n$, then F is an Anosov diffeomorphism.

Definition 2.3.5. A map $A : T^2 \to T^2$ is termed ergodic if for any continuous function φ, almost everywhere

$$\lim_{n\to\infty} \left[\frac{\sum_{k=0}^{n-1} \varphi(A^n(x,y))}{n} \right] = \int_{T^2} \varphi \, dx \, dy$$

3 Attractors

3.1 Defintions

Definition 3.1.1. An attractor, very generally, is a subset of the phase space such that all points except a set of measure 0 tend to the subset.

Definition 3.1.2. A maximal attractor for a map $F : U \to U$ for U open with $\overline{F(U)} \subset U$ and $\overline{F(U)}$ compact in U is the set $A_{max} := \bigcap_{n=0}^{\infty} F^n(U)$. Essentially, we have

$$\begin{array}{ccccc} U & \supset & F(U) & \supset & F^2(U) \supset \cdots \\ \cup & & \cup & & \cup \\ \overline{F(U)} & \supset & \overline{F^2(U)} & \supset & \overline{F^3(U)} \supset \cdots \end{array}$$

Definition 3.1.3. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map. A Milnor attractor is the minimal closed set A_M such that $\lim_{n\to\infty} [d(F^n(x), A_M)] = 0$ for all $x \in \mathbb{R}^n$ except possibly a set of measure 0.

Definition 3.1.4. Let $F : M \to M$ be a map with attractor $A \subset M$. Then F is termed Lyapunov stable if for fixed $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in M$ with $d(x, A) < \delta$, $d(F^n(x), A) < \epsilon$ for all n.

In other words, if we start in an ϵ-neighborhood of x, then we never leave a δ-neighborhood of x.

Open problem 3.1.5. How can it be determined for a generic dynamical system that A_M is Lyapunov stable?

Theorem 3.1.6. Let $F : M \to M$ be a map with attractor $A \subset M$ and $x \in M$. Then $x \in A_M \iff$ for all open $U \ni x$, measure$(S) > 0$ for $S = \{y \mid \text{for all } N \in \mathbb{N}, \text{ there exists } n > N \text{ such that } F^n(y) \in U\}$.

Definition 3.1.7. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map with $x \in \mathbb{R}^n$ a Lyapunov stable point. Then x is termed asymptotically stable if there exists $\epsilon > 0$ such that for all $y \in \mathbb{R}^n$, $d(x, y) < \epsilon \implies \lim_{n\to\infty} [F^n(y)] = x$.

Theorem 3.1.8. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map in C^1 with $x_0 \in \mathbb{R}^n$ such that $F(x_0) = x_0$. Let $A = F'(x_0)$ with $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then

· $|\lambda_i| < 1$ for all $i \implies x_0$ is asymptotically stable

· $|\lambda_i| > 1$ for at least one $i \implies x_0$ is not asymptotically stable

Proposition 3.1.9. The Milnor attractor is invariant under forward and backward applications of F.

3.2 Fixed point classification

A map $F : \mathbb{R}^2 \to \mathbb{R}^2$ may have several types of fixed points.

To determine the type of fixed point at $(x, y) \in \mathbb{R}^2$ for $F(x, y) = (u(x), v(y))$, let

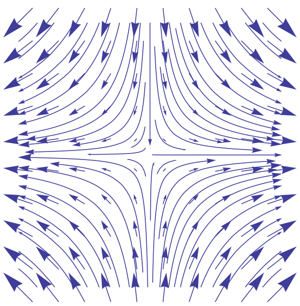
$$J = \begin{pmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{pmatrix}$$

be the Jacobian of F. Evaluate $J(x, y)$ and identify it with one of the matrix types below.

Saddle point

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

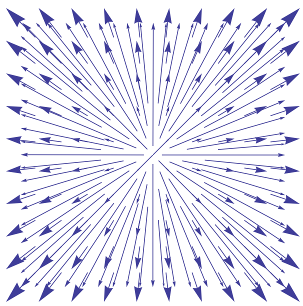
Stable if $ad > 0$
Unstable if $ad < 0$



Node

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

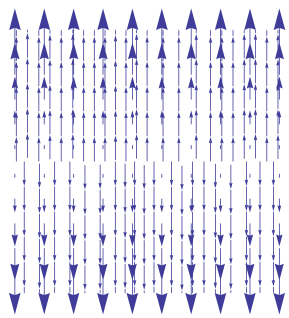
Stable if $a = d < 0$
Unstable if $a = d > 0$



Axis of fixed points

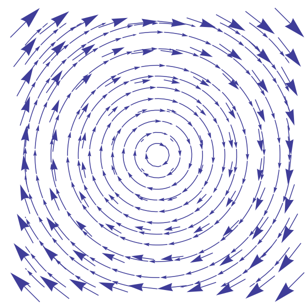
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Stable if $d < 0$
Unstable if $d > 0$



Center

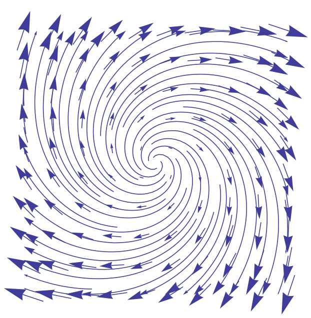
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



Focus

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

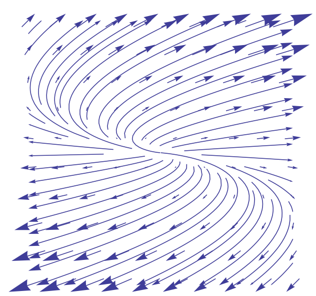
Stable if $bc > 0$
Unstable if $bc < 0$



Jordan cell

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Stable if $a, d < 0$
Unstable if $a, d > 0$



The variables above refer to the general matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.