Compact course notes

Knot theory

Fall 2011

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December 23, 2011

Contents

Note: For a more complete exposition, see the following textbooks, which were used in accompaniment with this course:
· *Intuitive Topolgy,* V.V. Prasolov
· *Knots: Mathematics with a Twist,* A.B. Sossinsky
· *Knots, links, braids and 3-manifolds,* V.V. Prasolov and A.B. Sossinsky

1 Fundamentals

Definition 1.0.1. A knot K is a closed broken line without self-intersections (polygonal line) in 3-space. A knot is a 1-component link. The empty set is a knot.

Definition 1.0.2. A link L is one or more disjointly embedded knots in 3-space. An n-component link consists of n separate knots. The empty set is not a link.

Definition 1.0.3. A knot diagram is a 2 dimensional projection of the knot onto the plane.

Definition 1.0.4. A $\triangle$-move is the adding of two (or subtracting of one) and subtracting of one (or adding of two) line to (or from) the knot diagram so that all the lines being changed form a triangle.

Definition 1.0.5. Two knots K, K' are equivalent if there exists a sequence of triangular moves that take K to K'. This relationship is expressed $K \sim K'$ and $K \xrightarrow{\triangle\text{-move}} K'$. Note that $\triangle$-move $= \Omega_0$.
Equivalent knots are also termed isotopic or ambient isotopic.
Define $\mathscr{K} = \{\text{knot diagrams}\}$ and $\mathscr{L} = \{\text{link diagrams}\}$

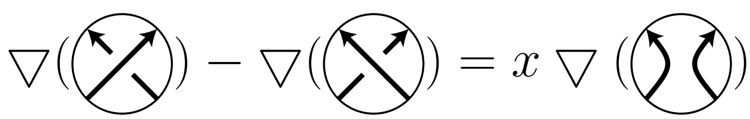
2 Knot topology

2.1 Homotopy

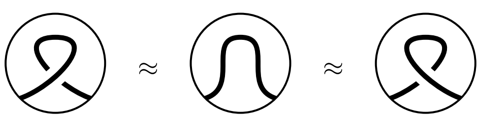
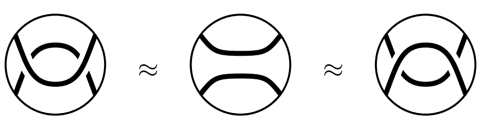
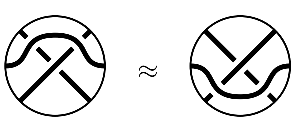
Definition 2.1.1. The Conway-Alexander polynomial is a knot invariant, or an assignment to every link (in particular, knot), that satisfies three actions:

1. $L \sim L' \implies \bigtriangledown_L(x) = \bigtriangledown_{L'}(x)$
2. $\bigtriangledown_0(x) = 1$
3. $\bigtriangledown_{L^+}(x) - \bigtriangledown_{L^-}(x) = x \cdot \bigtriangledown_{L^0}(x)$

The latter action may be summarized in the following manner:



Definition 2.1.2. There are 3 Reidemeister moves which formalize knot equivalence:

1. Ω_1-move: 
2. Ω_2-move: 
3. Ω_3-move: 

Theorem 2.1.3. [Riedemeister Lemma]
Given $K, K' \in \mathscr{K}$, $K \sim K' \iff K \xrightarrow{\{\Omega_i\}} K'$ for $i \in \{0, 1, 2, 3\}$.

Theorem 2.1.4. [Hass, Lagarias, Pippenger - 1998]
For all $n \in \mathbb{N}$, there exists $C(n) \in \mathcal{O}(2^n)$ such that if $K \sim K'$ have n crossing points or less, then $K \to K'$ in at most $C(n)$ Reidemeister moves.

Remark 2.1.5. There are several simplest invaraints attributable to knots:

1. *Stick number* - Least number of straight lines for embedding the knot in the plane
2. *Crossing number* - Least number of crossings of a diagram of a knot
3. *Unknotting number* - Least number of crossing changes needed to make the unknot

Definition 2.1.6. The genus $g(K)$ of a knot K is the least number g such that the knot is spanned by an orientable surface of genus g.

2.2 Arithmetic of knots

Remark 2.2.1. [Equivalence of equivalences]
There are three main ways to decide whether two given knots are equivalent:

1. $K \sim K' \iff K \xrightarrow{\Delta\text{-moves}} K'$
2. $K \sim K'$ if there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h = \mathrm{Id}$ outside a ball containing K, K'
3. $K \sim K'$ if there exist homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0, 1]$ with $h_0 = \mathrm{Id}$ and $h_1(K) = K'$

Theorem 2.2.2. The above three equivalences are equivalent.

Definition 2.2.3. Given two knots $K, K' \in \mathscr{K}$, the connected sum of these two knots is $K\#K'$, and entails cutting the knots are tying one to the other.

Proposition 2.2.4. The # operation is well-defined and commutative.

Definition 2.2.5. A prime knot $P \in \mathscr{K}$ is a knot such that $P \nsim K\#K'$ for $K, K' \in \mathscr{K} \setminus \{\bigcirc\}$

Theorem 2.2.6. There are no inverse elements in $(\mathscr{K}, \#)$. That is, if $K \in \mathscr{K} \setminus \{\bigcirc\}$, then there does not exist $K' \in \mathscr{K}$ with $K\#K' = \bigcirc$.

Theorem 2.2.7. [Prime decomposition theorem]
For all $K \in \mathscr{K}$, there exists a unique set of prime knots $\{P_1, \dots, P_n\} \subset \mathscr{K}$ such that $K = P_1\# \cdots \#P_n$.

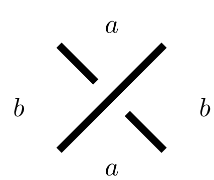
Theorem 2.2.8. [Gordon, Luecke - 1989]
Given $K, K' \in \mathscr{K}$, $K \sim K' \iff \mathbb{R}^3 \setminus K \sim \mathbb{R}^3 \setminus K'$ by a homeomorphism that is the identity outside a large enough ball.

3 Invariants

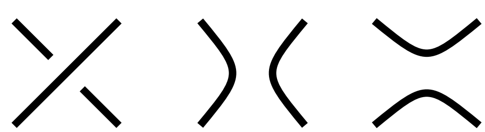
3.1 Kauffman bracket

Definition 3.1.1. Given a graph $G = (V, E)$ with edges and vertices (exactly 4 edges to 1 vertex), define the state of V to be a function $s : V \to \{\uparrow, \downarrow\}$ so that each vertex has a spin assigned to it. The set of all states is S.

Definition 3.1.2. Given a link L and an intersection of the link, define the a-angle and b-angle as follows:



Definition 3.1.3. With respect to the above definition, given an intersection in a link, define a-spin and b-spin by breaking the intersection and connecting the two a or b angles.



intersection a-spin b-spin

Definition 3.1.4. Given $L \in \mathscr{L}$ and S the set of states of L, define the Kauffman bracket of L to be

$$\langle L \rangle = \sum_{s \in S} a^{\alpha(s)} b^{\beta(s)} c^{\gamma(s)}$$

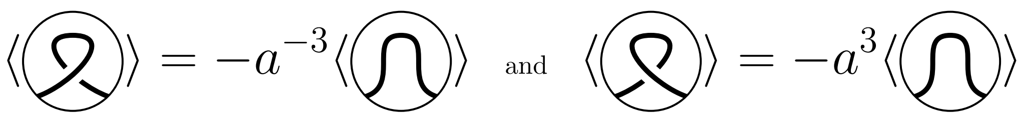
where $\alpha(s) =$ number of a-angles in s
$\beta(s) =$ number of b-angles in s
$\gamma(s) =$ (number of disjoint circles in s) $- 1$

Remark 3.1.5. The Kauffman bracket has the following properties:
1. $\langle \bigcirc \rangle = 1$
2. $\langle L \sqcup \bigcirc \rangle = c\langle L \rangle$
3. $\langle \times \rangle = a\langle)(\rangle + b\langle \asymp \rangle$

Remark 3.1.6. Set $c = -a^2 - a^{-2}$ and $b = a^{-1}$ so $\langle L \rangle$ is Ω_2 and Ω_3 invariant.

3.2 Jones polynomial

Remark 3.2.1. Note that $\langle L \rangle$ is not Ω_1 invariant. Indeed, we have



Definition 3.2.2. Define the writhe number of an oriented link $\vec{L}$ to be

$$w(\vec{L}) = \sum_{\text{crossings i}} E(i) \quad \text{with} \quad \begin{array}{l} E(\bigcirc) = 0 \\ E(\times) = 1 \\ E(\times) = -1 \end{array}$$

Definition 3.2.3. Define the X-polynomial of an oriented link in the following way:

$$X(\vec{L}) = (-a)^{-3w(\vec{L})}\langle L \rangle$$

Proposition 3.2.4. The polynomial $X(\vec{L})$ is an invariant of knot equivalence, and has these properties:
1. $X(\bigcirc) = 1$
2. $X(\vec{L} \sqcup \bigcirc) = (-a^2 - a^{-2})X(\vec{L})$
3. $a^4 X(\times) - a^{-4}X(\times) = (a^{-2} - a^2)X()()$

Remark 3.2.5. The X polynomial also has the following property, for all $K, K' \in \mathscr{K}$:

$$X(K \# K') = X(K)X(K')$$
$$X(K \sqcup K') = -(a^2 + a^{-2})X(K)X(K')$$

Definition 3.2.6. Define the Jones polynomial $V(\vec{L}) : \mathscr{L} \to \mathbb{Z}[\sqrt{q}, \frac{1}{\sqrt{q}}]$ with the above three properties:

$$V(\vec{L}) = X(\vec{L})\big|_{a = q^{-1/4}}$$

Theorem 3.2.7. There exists a unique V such that the following properties are satisfied:
1. V is invariant
2. $V(\bigcirc) = 1$
3. $V(L \sqcup \bigcirc) = (-q^{1/2} - q^{-1/2})V(L)$
4. $q^{-1}V(\times) - qV(\times) = (q^{1/2} - q^{-1/2})V()()$

Proposition 3.2.8. Any link may be trivialized by crossing changes.

Remark 3.2.9. Some other properties of the Jones polynomial:

1. $V(K\#K') = V(K)V(K')$
2. $V(L \sqcup L') = (-q^{-1/2} - q^{1/2})V(L)V(L')$
3. V does not in general distinguish K from $\mathrm{mir}(K)$, the chiral knot of K

Proposition 3.2.10. [Tait conjecture]
The crossing number of an alternating knot is equal to the number of crossings of any alternating diagram of this knot without loops.

3.3 Vassiliev invariants

Definition 3.3.1. [Thom, Arnold, Vassiliev]
Let $N = \{\text{nice objects}\}$ and $S = \{\text{degenerate objects}\}$. Consider $N \cup S = L$ a linear space. Then S is termed the discriminant of L.

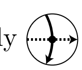
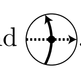
Definition 3.3.2. Let L be a space as above with N disconnected. Let $P_+, P_- \in N$ such that both are not in the same component. Depending on the nature of L, different components may be assigned "positive" and "negative" value.

Then on the path from P_+ to P_- there exists $P_0 \in S$ such that

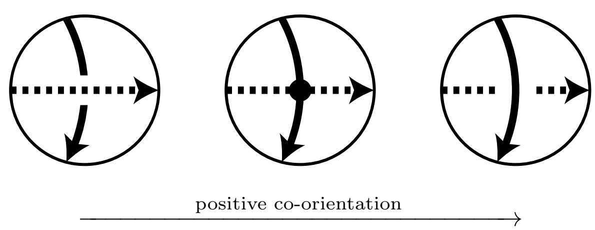
$$\nu(P_+) - \nu(P_-) = \nu(P_0)$$

This is termed the Vassiliev relation.

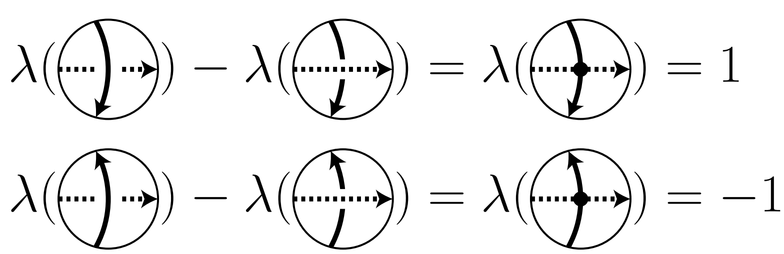
3.4 Gauss number

Given any oriented link embedded in 3-space, there are two different singular points, namely  and .

Definition 3.4.1. Given an intersection of the type above, define the co-orientation of the intersections as below, with positive co-orientation from left to right:



Proposition 3.4.2. For 2-component links L, define the Gauss linking number $\lambda(L)$ to describe links as below. For such L, there are 2 Vassiliev relations:



with the base case given by

$$\lambda(K \sqcup K') = 0$$

Remark 3.4.3. For a link L, $\lambda(L) = 0$ does not imply that L may be unlinked.

4 Vassiliev invariants

4.1 Motivation

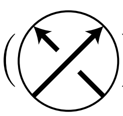
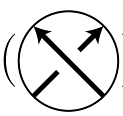
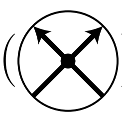
Definition 4.1.1. Consider $\sigma : \mathbb{S}^1 \to \mathbb{R}^3$ such that $\text{Im}(\sigma)$ has $n \geqslant 0$ transveral intersections, such as . Thus there are n points on $\mathbb{S}^1$ where σ is not 1-1. Then define $\Sigma_n = \{\sigma\}$ for σ as above.

Moreover, define the set of all knots to be $\Sigma_\infty = \mathscr{K} \cup \Sigma_1 \cup \Sigma_2 \cup \cdots$, which is an infinite dimensional space with knots as points.

Further, $\Sigma_\infty \setminus \mathscr{K}$, the set of all singular knots, is termed the discriminant of Σ_∞.

Definition 4.1.2. Given an object X in a linear space Y, denote the codimension of X to be the difference in dimension between X and the ambient space Y. In other words, $\text{codim}(X) = \dim(Y) - \dim(X)$.

Definition 4.1.3. Define a Vassiliev invariant of order n to be a function $v : \Sigma_\infty \to \mathbb{F}$ such that $\text{char}(\mathbb{F}) = 0$ and $v|_{\Sigma_k} = 0$ if $k > n$. Most often $\mathbb{F} = \mathbb{C}$ is used. Also, the following relation is satisfied:

$v($  $) - v($  $) = v($  $)$

Definition 4.1.4. A Vassiliev invariant is said to be of exactly order n if the order of v is n, but not $n-1$.

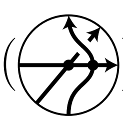
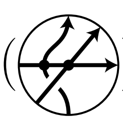
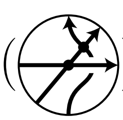
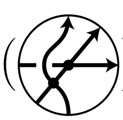
Remark 4.1.5. Vassiliev invariants of order n form a linear space over $\mathbb{F}$, denoted V_n. Here n denotes the codimension of knots in V_n.

Remark 4.1.6. There is only one Vassiliev invariant of order 0 and 1. In other words, $V_0 \simeq V_1 \simeq \mathbb{C}$.

4.2 1-term and 4-term relations

Theorem 4.2.1. [1-TERM RELATION]
For any Vassiliev invariant, $v($  $) = 0$. More generally, if a knot $K \in \Sigma_\infty$ can be separated completely in two separate parts with only a singular point joining the two, then $v(K) = 0$.

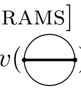
Theorem 4.2.2. [4-TERM RELATION]
For any Vassiliev invariant v and a knot $K \in \Sigma_{n \geqslant 2}$, the following holds:

$v($  $) - v($  $) + v($  $) - v($  $) = 0$

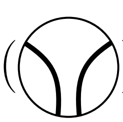
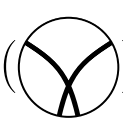
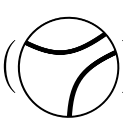
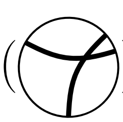
Remark 4.2.3. The 4-term relation is equivalent to the Jacobi identity.

Theorem 4.2.4. [CROSSING CHANGE LEMMA]
Let $K, K' \in \Sigma_n$ and $v \in V_n$. If $K \xrightarrow{\text{crossing changes}} K'$, then $v(K) = v(K')$.

Proposition 4.2.5. Any finite knot $K \in \Sigma_\infty$ may be represented by a Gauss diagram.

Theorem 4.2.6. [1-TERM RELATION WITH GAUSS DIAGRAMS]
Let $v \in V_n$. If no chords intersect the shown chord, then $v($  $) = 0$.

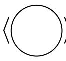
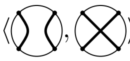
Theorem 4.2.7. [4-TERM RELATION WITH GAUSS DIAGRAMS]
Let $v \in V_n$. If no chords end in each smallest space separating every two chords below, then

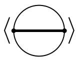
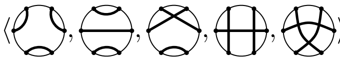
$v($  $) - v($  $) - v($  $) + v($  $) = 0$

4.3 The algebra of Gauss diagrams

Definition 4.3.1. Define $G_n :=$ (the linear space over $\mathbb{C}$ of Gauss diagrams of knots exactly in Σ_n).

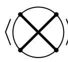
Proposition 4.3.2. The following are bases for the given spaces:

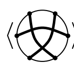
$G_0 = \langle$  $\rangle$ $\qquad$ $G_2 = \langle$  $\rangle$

$G_1 = \langle$  $\rangle$ $\qquad$ $G_3 = \langle$  $\rangle$

Definition 4.3.3. Define $\mathcal{G}_n := G_n \big/ \left(\substack{\text{1-term relation} \\ \text{4-term relation}} \right)$ That is, in $\mathcal{G}$, $K \sim K'$ if $K \xrightarrow[\text{4-term relation}]{\text{1-term relation}} K'$.

Proposition 4.3.4. The following are dimensions and bases for the given spaces:

$\dim(\mathcal{G}_0) = 0$ $\qquad$ $\dim(\mathcal{G}_2) = 1$ with $\mathcal{G}_2 = \langle$  $\rangle$

$\dim(\mathcal{G}_1) = 0$ $\qquad$ $\dim(\mathcal{G}_3) = 1$ with $\mathcal{G}_3 = \langle$  $\rangle$

Proposition 4.3.5. Multiplication of Gauss diagrams in G_n is well defined.

Theorem 4.3.6. [Kontsevich]
As above, $\mathcal{G}_n \simeq V_n$ as graded algebras (or linear spaces).

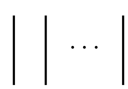
5 Braids

5.1 Group structure

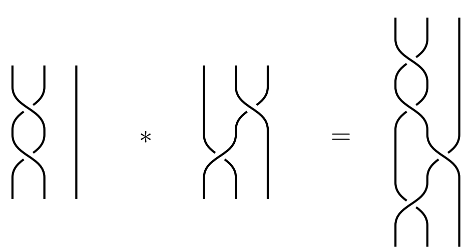
Definition 5.1.1. A braid in n strands is the image of a smooth injective map $f : I^n \to \mathbb{R}^3$ such that $f(0, \ldots, 0)$ and $f(1, \ldots, 1)$ each lie on non-coincidental axes parallel to each other.

Theorem 5.1.2. [Artin]
Braids, in contrast with knots, form a group, termed the braid group B_n.

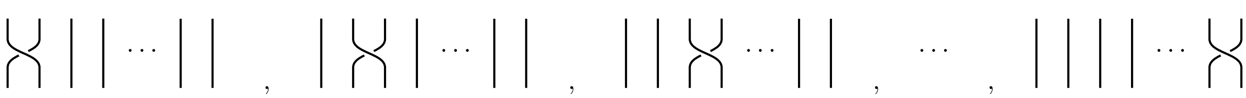
· The identity element is a braid homotopic in each variable to n parallel lines



· The group operation for $b_1, b_2 \in B_n$, is demonstrated through example by



· There are $n-1$ generators of B_n, namely



Remark 5.1.3. There exists a homeomorphism $\gamma : B_n \to S_n$ the permutation group, given by

$$\begin{array}{c}\text{braid with strand starting in } j\text{th position}\\ \text{and ending in } i_j\text{th position for all } j\end{array} \quad \mapsto \quad \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

Therefore B_n is not Abelian.

Theorem 5.1.4. [Artin]
Cosider $\hat{B}_n := \langle b_1, \dots, b_n \mid b_i b_{i+1} b_1 = b_{i+1} b_1 b_{i+1} \ \forall\ 1 \leqslant i \leqslant n-2 \text{ and } b_i b_j = b_j b_i \ \forall\ i, j \text{ with } |i-j| \geqslant 2\rangle$. Then $\hat{B}_n \simeq B_n$ for all $n \in \mathbb{N}$.

5.2 Link to knots

Definition 5.2.1. Given a braid $b \in B_n$, its closure is the link that results from connecting every ith open end at the top to its corresponding ith open end at the bottom, and is denoted $\mathrm{cl}(b)$.

Remark 5.2.2. For $b \in B_n$, if $\gamma(b)$ contains k cycles, then $\mathrm{cl}(b)$ is a k-component link.

Theorem 5.2.3. [Alexander]
For any $L \in \mathscr{L}$, there exists $b \in B_n$ such that $\mathrm{cl}(b) = L$.

Theorem 5.2.4. [Markov]
A braid $b \in B_{n-1}$ is invariant under the following equalities. Let $a \in B_{n-1}$ and $b_n \in B_n$ and $\notin B_{n-1}$. Then

$$b = aba^{-1}$$
$$b = bb_n$$

These are termed Markov moves.

Theorem 5.2.5. [Markov]
Let $b, b' \in B_n$, Then $\mathrm{cl}(b) \sim \mathrm{cl}(b') \iff b \xrightarrow{\text{Markov moves}} b'$.

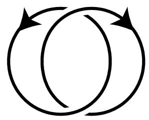
Proposition 5.2.6. [Kontsevich]
Let $K \in \mathscr{K}$ and $G_p \in \mathcal{G}_m$. Then there exists an invariant

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\tau < t_1 < \cdots < t_m < T} \left(\sum_{p = \{(z_j, z'_j)\}} (-1)^{\downarrow} G_p \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \right)$$

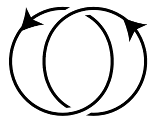
Note that the coefficient of every G_p is an element of $\mathbb{C}$, equivalently a Vassiliev invariant. Also, $\downarrow$ denotes the number of disjoint sections where the direction of flow along the knot is downward.

6 Some classification

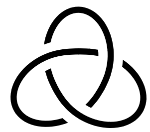
6.1 Knot and link tables



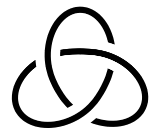
Right Hopf link



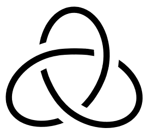
Left Hopf link



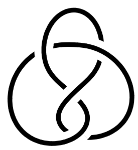
Right trefoil



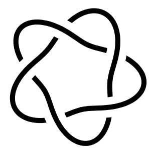
Left trefoil



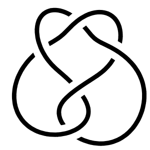
3_1



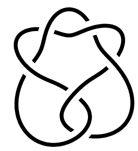
4_1



5_1



5_2



6_1

6.2 Knot and link summary

Link	**Conway polynomial**	**Kauffman bracket**	**Writhe number**	**Jones polynomial**
Right Hopf link	x	$-a^{-4} - a^4$	2	$-q^{1/2} - q^{5/2}$
Left Hopf link	$-x$	$-a^{-4} - a^4$	-2	$-q^{-5/2} - q^{-1/2}$
0	1	1	0	1
3_1 (right)	$1 + x^2$	$a^{-7} - a^{-3} - a^5$	3	$q^1 + q^3 - q^4$
3_1 (left)		$-a^{-5} - a^3 + a^7$	-3	$-q^{-4} + q^{-3} + q^{-1}$
4_1	$1 - x^2$	$a^{-8} - a^{-4} + 1 - a^4 + a^8$	0	$q^{-2} - q^{-1} + 1 - q^1 + q^2$
5_1	$1 + 3x^2 + x^4$		-5	$q^2 + q^4 - q^5 + q^6 - q^7$

Link	**Gauss linking number**
Right Hopf link	1
Left Hopf link	-1