Equations of Mathematical Physics

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1. Mains

1.1. **Defintions.**

Def 1.1. Let $f : X \to Y$ be a function. The support of f is the smallest closed set A such that $f|_{X\setminus A} = 0$.

Def 1.2. A 1-jet of a function $u(x_1, \dots, x_n)$ at a point $(x_1^0, \dots, x_n^0)$ in the domain of u is an $(n+1)$-tuple of the form

$$\left(\frac{\partial u}{\partial x_1}(x_1^0, \dots, x_n^0), \dots, \frac{\partial u}{\partial x_n}(x_1^0, \dots, x_n^0), u(x_1^0, \dots, x_n^0)\right)$$

Def 1.3. A functional is a function $f : X \to Y$ such that X is a space of functions, where

$$\varphi \mapsto \langle f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx$$

Functionals satisfy the following relations:
1. $\langle f(x)g(y), \varphi(x)\psi(y)\rangle = \langle f(x), \varphi(x)\rangle\langle g(y), \psi(y)\rangle$
2. $\langle f(x - x_0), \varphi(x)\rangle = \langle f(x), \varphi(x + x_0)\rangle$

Def 1.4. A distribution is a continuous linear functional.

Def 1.5. Let a differential operator be given, for $\alpha \in \mathbb{R}^n$, by

$$\mathcal{D} = \sum_{|\alpha| \leqslant m} a_\alpha \partial^\alpha$$

Then a fundamental solution of $\mathcal{D}$ is a distribution u such that $\mathcal{D}u = \delta$.

Def 1.6. Given two functions f, g on $\mathbb{R}^n$, the convolution of the two functions is given by

$$(f * g)(x) = \int^{\mathbb{R}^n} f(x - y)g(y)dy$$

1.2. **Theorems.**

Thm 1.1. Let L be a differential operator with constant coefficients, and $\mathcal{E}(x)$ its fundamental solution (i.e. $L\mathcal{E} = \delta$). Then $\mathcal{E} * f$ is a particular solution to the equation $Lu = f$.

Thm 1.2. The general solution of a non-homogeneous equation is the sum of the general solution of the homogeneous equation and a particular solution of the non-homogeneous equation.

2. Fundamentals

$$a = \sqrt{\frac{T}{\sigma}}$$
$$u_{tt} = a^2 u_{xx}$$

a : Acceleration, ms^{-2}
T : Tension, $kg \cdot ms^{-2}$
σ : Density, $kg \cdot m^{-3}$
u_{ij} : Derivative wrt i of derivative wrt j of u

3. Strings

$\varphi(x)$: Initial shape
$\psi(x)$: Initial velocity

· D'Alembert equation, half-bounded

$$u(x,t) = \frac{1}{2}\left(\varphi(x+at) + \varphi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)dy$$

· String equation, bounded

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\left(\frac{\pi ant}{\ell}\right) + B_n \cos\left(\frac{\pi ant}{\ell}\right)\right)\sin\left(\frac{\pi nx}{\ell}\right)$$
$$A_n = \frac{2}{\pi an}\int_0^{\ell} \psi(x)\sin\left(\frac{\pi nx}{\ell}\right)dx$$
$$B_n = \frac{2}{\ell}\int_0^{\ell} \varphi(x)\sin\left(\frac{\pi nx}{\ell}\right)$$

· Total energy

$$E(t) = \frac{1}{2}\int_a^b \rho u_t^2 dx + \frac{1}{2}\int_a^b T u_x^2 dx$$

4. Operators and special functions

· Laplacian

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

· Gamma function of Poisson integral

$$\Gamma(t,x) = (2a\sqrt{\pi t})^n exp\left\{\frac{-x^2}{4a^2t}\right\}$$

5. Fundamental solutions

σ_n : Volume of the sphere $\mathbb{S}^n$

· Particular for Laplacian

$$\mathcal{E}_n(x) = \begin{cases} \frac{r^{2-n}}{(2-n)\sigma_{n-1}} & n \geqslant 3 \\ \frac{\ln(r)}{2\pi} & n = 2 \end{cases}$$