Compact course notes

Riemann surfaces

Fall 2011

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December 23, 2011

Contents

<u>*Note:*</u> For a more complete exposition, see the following textbooks, which were used in accompaniment with this course:
· *Algebraic Curves and Riemann Surfaces,* Rick Miranda

1 Structures

Definition 1.0.1. A subset $U \subset X$ is termed open if $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}$ is open $\forall\ \alpha$.

Definition 1.0.2. A set $F \subset X$ is termed closed if $X \setminus F$ is open.

Remark 1.0.3. The empty set is defined to be open.

Definition 1.0.4. A topological space is a set X together with a set S of subspaces of X such that
- **a.** $\emptyset \in S$
- **b.** $X \in S$
- **c.** If $U_0, U_1, \dots \in S$ then $\bigcup_j U_j \in S$
- **d.** If $U, V \in S$ then $U \cap V \in S$

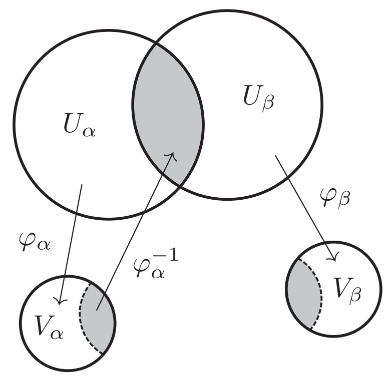
Definition 1.0.5. A homeomorphism is an injective function between topological spaces that conserves all the topological properties of the given space.

Definition 1.0.6.A complex chart on a set X is a homeomorphism $\varphi : U \to V$ for open sets $U \subset X$, $V \subset \mathbb{C}$.

Definition 1.0.7. A conformal mapping is a transformation that preserves local angles. A function is conformal wherever it has nonzero derivative.

Definition 1.0.8. Given a topological space X and a point p (or a set S), a neighborhood of p (or S) is any open set $T \subset X$ containing p (or S).

Definition 1.0.9. For U open and $\varphi : U \to X$ a homeomorphism (a chart), the set (U, φ), abbreviated to just U, is termed a coordinate neighborhood.



Definition 1.0.10. A Riemann surface is a set X with the properties:
1. $X = \bigcup_\alpha U_\alpha$ where all U_α are coordinate neighborhoods
2. For each U_α there exists a bijection $\varphi_\alpha : U_\alpha \to V_\alpha$ for $V_\alpha \subset \mathbb{C}$ open
3. $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a bijection from $\varphi_\alpha(U_\alpha \cap U_\beta)$ to $\varphi_\beta(U_\alpha \cap U_\beta)$ and a conformal mapping for each (α, β)

The function $\varphi_\beta \circ \varphi_\alpha^{-1}$ above is termed a transition function.

Theorem 1.0.11. [Implicit function theorem]
Let $f : U \to \mathbb{C}$ for $U \subset \mathbb{C}$ be a function on two variables z_1, z_2. Suppose that $\frac{\partial f}{\partial z_1}\Big|_p \neq 0$ at $p = (p_1, p_2) \in U$ and $f(p) = 0$. Then there exist open neighbourhoods $U_1, U_2 \subset \mathbb{C}$ and a holomorphic map $\varphi : U_1 \to U_2$ such that $\{(z, \varphi(z)) \mid z \in U_1\} = \{(z_1, z_2) \mid f(z_1, z_2) = 0\} \cap (U_1 \cup U_2)$.

2 Framework of Riemann surfaces

2.1 Basic configuration

Definition 2.1.1. A topological space X is termed compact if $X = \bigcup_j U_j \implies$ there exist $j_1, \dots j_k$ such that $X = U_{j_1} \cup \dots \cup U_{j_k}$ for U_j open sets.

Definition 2.1.2. The Riemann sphere is $\bar{\mathbb{C}} := \mathbb{C}^2 \cup \{\infty\}$. It is a compact Riemann surface.

Definition 2.1.3. A lattice is a set $\Gamma = \{n_1 w_1 + \dots + n_k w_k \mid n_1, \dots, n_k \in \mathbb{N} \text{ and } w_1, \dots, w_k \in \mathbb{C} \setminus \{0\}\}$.

Theorem 2.1.4. [Inverse Function Theorem]
If a function $f : X \to X$ has a non-zero derivative at 0, then there exists a neighborhood $U \subset X$ such that $f^{-1} : f(U) \to U$ is also smooth.

Definition 2.1.5. A Riemann surface X is <u>not connected</u> if $X = U \cup V$ and $U \cup V \neq \emptyset$ for U, V open nonempty sets.

Definition 2.1.6. Define the set $\Gamma = \{n_1 w + n_2 z \mid n_1, n_2 \in \mathbb{Z}\}$ for some $w, z \in \mathbb{C} \setminus \{0\}$ to be a <u>lattice</u>. Note that Γ is a subgroup of $\mathbb{C}$. Define $X = \mathbb{C} \,/\, \Gamma = \{\text{equivalence classes of } x \in X \mid z \sim w \iff z - w \in \Gamma\}$ to be an <u>elliptic curve</u>.

2.2 Holomorphic functions

Definition 2.2.1. A function $f : X \to \mathbb{C}$ is termed <u>holomorphic</u> if for each U_α the following composition is well-defined:

$$\begin{tikzcd}
U_\alpha \subset X \arrow[dr, "f|_{U_\alpha}"'] \arrow[dr, bend left, "f"] & \\
V_\alpha \arrow[u, dashed, "\varphi^{-1}"] \arrow[r, "(f|_{U_\alpha}) \circ \varphi_\alpha^{-1}"'] & \mathbb{C}
\end{tikzcd}$$

Proposition 2.2.2. If X is a compact and connected Riemann surface, then any holomorphic function $f : X \to \mathbb{C}$ is constant.

Definition 2.2.3. A function f has a <u>pole</u> at p if there exists a coordinate neighborhood $U_\alpha \ni p$ with $U\alpha \xrightarrow{\varphi_\alpha} V_\alpha$ such that $f \circ \varphi_\alpha^{-1}$ has a pole at $\varphi_\alpha(p)$.

Definition 2.2.4. A <u>meromorphic function</u> on X is a function $f : X \setminus S \to \mathbb{C}$ where $S \subset X$ is a nonempty set without cluster points and f has, at worst, poles at points of S.

Remark 2.2.5. Any meromorphic function on $\bar{\mathbb{C}}$ is a rational function of z.

Theorem 2.2.6. If f is meromorphic on $\bar{\mathbb{C}}$, then $\big(\#\text{ of zeros of } f\big) = \big(\#\text{ of poles of } f \text{ counting multiplicities}\big)$. The same holds if f is meromorphic on X an elliptic curve.

3 Complex projective planes

Definition 3.0.7. The <u>complex projective plane</u> is a 2-dimensional complex projective (and topological) space described by 3 complex coordinates: $\mathbb{C}P^2 = \{(z_0 : z_1 : z_2) \mid z_0 \neq 0 \text{ or } z_1 \neq 0 \text{ or } z_2 \neq 0\}$

The coordinates $(z_0 : z_1 : z_2)$ are termed <u>homogeneous coordinates</u>. They are uniquely defined up to scalar multiplication, i.e. $(1 : 2 : 5) = (4 : 8 : 20)$.

Remark 3.0.8. The complex plane may be embedded in $\mathbb{C}P^2$ three separate ways:

$$\left.\begin{array}{lcl} \mathbb{C}_0^2 & = & \{(1 : z : w)\} \\ \mathbb{C}_1^2 & = & \{(z : 1 : w)\} \\ \mathbb{C}_2^2 & = & \{(z : w : 1)\} \end{array}\right\} \subset \mathbb{C}P^2$$

Moreover, we have that $\mathbb{C}^2 \simeq \mathbb{C}P^2$.

Also, we may described this space as $\mathbb{C}P^2 = \{\text{lines in } \mathbb{C}^3 \text{ passing through the origin}\}$.

Definition 3.0.9. The set $\{(z_0, z_1, z_2) \mid p_0 z_0 + p_1 z_1 + p_2 z_2 = 0\}$ with at least one $p_i \neq 0$ is a <u>line</u> in $\mathbb{C}P^2$.

3.1 Differential forms

Definition 3.1.1. Let $F(z_0, z_1, z_2)$ be a function in three variables, Then Euler's identity is given as

$$dF = z_0 \frac{\partial F}{\partial z_0} + z_1 \frac{\partial F}{\partial z_1} + z_2 \frac{\partial F}{\partial z_2}$$

Definition 3.1.2. Let $U \subset \mathbb{C}$ be open. A holomorphic 1-form on U is an expression $\omega = f(z)dz$ for f a holomorphic function on U.Then ω is a holomorphic 1-form in the coordinate z.

Definition 3.1.3. Let X be a Riemann surface with $X = \bigcup U_\alpha$ and coordinate neighborhoods $z_\alpha : U_\alpha \to \mathbb{C}$. Then a holomorphic 1- form on X is a collection of holomorphic 1-forms $\{\omega_\alpha\}$, one for each z_α. If $U_\alpha \cap U_\beta \neq \emptyset$, then $f_\alpha(z_\alpha)dz_\alpha = f_\alpha(z_\alpha)\frac{dz_\alpha}{dz_\beta}dz_\beta = f_\beta(z_\beta)dz_\beta$.

Remark 3.1.4. There are no non-zero holomorphic forms on $\bar{\mathbb{C}}$.

Definition 3.1.5. Let $f : Y \to X$ be a holomorphic function of Riemann surfaces. Let ω be a meromorphic form on X. Then the inverse image of ω is $\varphi^*\omega$. In local coordinates, $z = \varphi(\omega)$ with $\varphi^*(f(z)dz) = f(z(\omega))\varphi'(\omega)$.

Definition 3.1.6. Given two 1-forms $\omega = f_1 dz_1 + \cdots + f_n dz_n$ and $\eta = g_1 dz_1 + \cdots + g_n dz_n$ on an n-dimensional manifold, define their distributive product by

$$\omega \wedge \eta = (f_1 dz_1 + \cdots + f_n dz_n) \wedge (g_1 dz_1 + \cdots + g_n dz_n)$$

with the following properties:

i. $dz_k \wedge dz_k = 0$
ii. $dz_k \wedge dz_\ell = -dz_\ell \wedge dz_k$

3.2 Poincaré residue

Definition 3.2.1. Let $X = \{(z_1, \ldots, z_n) \mid f(z_1, \ldots, z_n) = 0\}$ be a Riemann surface with $f : \mathbb{C}^n \to \mathbb{C}$ holomorphic. Given a differential form $\omega = g\ dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$, the residue of ω on X is the second expression in the product

$$\omega = \frac{df}{f} \wedge (g_1 dz_1 + \cdots + g_n dz_n)$$

and is denoted $\mathrm{res}_X(\omega)$.

Definition 3.2.2. Given a piecewise smooth path $\gamma([a : b]) \subset X$ for $X = \bigcup U_\alpha$ a Riemann surface and ω a differential 1-form, the integral of ω is

$$\int_\gamma \omega = \int_{[a_0:a_1]} \omega \ + \cdots + \int_{[a_{n-1}:a_n]} \omega$$

for a division $a = a_0 < a_1 < \cdots < a_n = b$ of $[a : b]$ such that $[a_j, a_{j+1}] \subset U_\beta$ for some β and for all j.

Theorem 3.2.3. [Cauchy]
If γ_1 is homotopic to γ_2, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ for any differential 1-form ω.

Remark 3.2.4. Let X be a Riemann surface, ω a meromorphic differential form on X, $p \in X$ a pole of ω and $\gamma \in X$ a closed path in X that only encircles one pole of ω, namely p. Then

$$\mathrm{res}_p(\omega) = \frac{1}{2\pi i} \int_\gamma \omega$$

Proposition 3.2.5. Let X be a Riemann surface, ω a meromorphic differential form on X, $p \in X$ a pole of ω and z be a local coordinate at p such that $z(p) = 0$. Then

$$\omega = \left(\sum_{k=-N}^{\infty} c_k z^k \right) dz \quad \Longrightarrow \quad \operatorname{res}_p(\omega) = c_{-1}$$

Proposition 3.2.6. If ω is a meromorphic form on a compact Riemann surface X, then $\sum_{p \in X} \operatorname{res}_p(\omega) = 0$.

3.3 Triangulation

Theorem 3.3.1. Suppose that X is a compact Riemann surface. Then X may be presented as a disjoint union of a finite number of sets of vertices, edges and faces, such that

1. Each vertex is a point
2. Each edge is homeomorphic to an open line segment
3. Each face is homeomorphic to the interior of a triangle
4. The closure of an edge includes the edge and both vertex endpoints
5. The closure of a face includes that face, all three bordering edges and all three bordering vertices
6. Each edge is a piecewise smooth curve

This is termed a triangulation of X.

Theorem 3.3.2. Each compact Riemann surface is homeomorphic to a "sphere with handles."

Definition 3.3.3. The sphere is a sphere with 0 handles. To attach a handle, remove the interior of two disjoint disks on a surface and attach ends of a cylinder to the disks. The number of handles g of a surface is termed the genus of the surface.

Theorem 3.3.4. If a surface X has genus g and is triangulated with e edges, v vertices and f faces, then

$$v - e + f = 2g - 2$$

Proposition 3.3.5. Let f be a meromorphic function on $\mathbb{C}P^n$ in n variables. Then the zeros and poles of f are $(n-1)$-dimensional surfaces.

4 Ramification

4.1 Holomorphic functions

Definition 4.1.1. Let $U \subset \mathbb{C}$ be open and connected, and $f : U \to \mathbb{C}$ a holomorphic and non-constant function with $a \in U$ such that $f'(a) = 0$. Then f is said to ramified at a.

Further, if f around a is given by $f(a) = b + c_k(z-a)^k + c_{k+1}(z-a)^{k+1} + \cdots$ for k the smallest index such that $c_k \neq 0$, then k is termed the ramification index of f at a.

Proposition 4.1.2. With respect to the above conditions, the ramification index of f at a is k if and only if there exist punctured neighborhoods of a where f is k-to-1.

Definition 4.1.3. Let $f : X \to Y$ be a non-constant, holomorphic map of Riemann surfaces with $a \in X$. Then f is ramified at a with index n if and only if it is ramified in some local coordinates at a with index n.

Denote the set of ramification points by $R = \{a \in X \mid f \text{ is ramified on } X \text{ at } a\}$.

Proposition 4.1.4. Suppose X, Y are compact, connected Riemann surfaces with $f : X \to Y$ holomorphic and non-constant. Then exactly one of the following hold:

- **i.** $f(X) = Y$
- **ii.** there exist a finite number of ramification points of f

Proposition 4.1.5. Let $B = f(R) \subset Y$. Then $f|_{X \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \to Y \setminus B$ is a covering.

That is, for each $y \in Y \setminus B$, there exists a neighborhood $U \ni y$ with $U \cap B = \emptyset$ such that $f^{-1}(U) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_d$ with $V_i \subset X \setminus f^{-1}(B)$ open and $f|_{V_i} : V_i \to U$ an isomorphism for each i.

Theorem 4.1.6. Let X, Y be manifolds with $f : X \to Y$ a continuous function such that:

1. For each $x \in X$, there exist open sets $V \ni x$ with $f|_V : V \to f(V)$ homomorphisms and $f(V) \subset Y$

2. If $K \subset Y$ is compact, then $f^{-1}(K) \subset X$ is also compact

Then R is a covering.

Corollary 4.1.7. If $y \in Y \setminus B$, then $\#(f^{-1}(y))$ does not depend on y and is termed the degree of f.

Proposition 4.1.8. Let $f : X \to Y$ be a holomorphic, non-constant map of compact Riemann surfaces with $\deg(f) = d$. For $y \in Y$, we have $f^{-1}(y) = \{x_1, \ldots, x_m\}$ for $1 \leqslant m \leqslant d$. Then

$$\sum_{i=1}^{m} \begin{pmatrix} \text{ramification} \\ \text{index of } f \text{ at } x_i \end{pmatrix} = d = \deg(f)$$

Note that if f is not ramified at some x_i, then its ramification index is 1.

Theorem 4.1.9. [Riemann-Hurwitz]
Let $f : X \to Y$ be a non-constant holomorphic mapping of compact Riemann surfaces X, Y. Suppose that $R = \{x_1, \ldots, x_n\}$ with $e_i =$ (ramification index of f at x_i) and $\deg(f) = d$. Then

$$2 - 2g(X) = d(2 - 2g(Y)) - \sum_{i=1}^{r}(e_i - 1)$$

4.2 Divisors

Definition 4.2.1. Given a compact Riemann surface X, a divisor $\mathcal{D}$ is a finite formal linear combination of points of X with integer coefficients

$$\mathcal{D} = \sum_j n_j p_j \quad \text{for } n_j \in \mathbb{Z}, p_j \in X$$

Definition 4.2.2. Let $f : X \to Y$ be a meromorphic function. Then a principal divisor is denoted by

$$(f) = \sum_{p \in X} \operatorname{ord}_p(f) \cdot p$$

Definition 4.2.3. The degree of a divisor $\mathcal{D} = \sum_j n_j p_j$ is $\deg(\mathcal{D}) = \sum_j n_j \in \mathbb{Z}$.

Proposition 4.2.4. If $\mathcal{D}$ is a principal divisor, then $\deg(\mathcal{D}) = 0$.

Proposition 4.2.5. For functions f, g over identical spaces, $(fg) = (f) + (g)$. Moreover, the sum of principal divisors is a principal divisor.

Definition 4.2.6. Two divisors $\mathcal{D}_1, \mathcal{D}_2$ are termed equivalent if $\mathcal{D}_1 - D_2$ is principal, and is denoted $\mathcal{D}_1 \sim \mathcal{D}_2$. This is an equivalence relation.

Remark 4.2.7. If $\mathcal{D}_1 \sim \mathcal{D}_2$, then $\deg(\mathcal{D}_1) = \deg(\mathcal{D}_2)$.

Proposition 4.2.8. On the Riemann sphere, every divisor of degree 0 is principal.

Proposition 4.2.9. On the Riemann sphere, given a finite set $\{p_1, \ldots, p_n\}$ with principal parts at each p_i fixed, there exists a meromorphic function with poles at each p_i and principal parts as given, and no other poles.

4.3 Effective and canonical divisors

Definition 4.3.1. Let $\mathcal{D}$ be a divisor. Then $L(\mathcal{D}) = \{f \mid f \text{ is meromorphic on } X \text{ and } (f) + \mathcal{D} \geqslant 0\}$.

We write $\mathcal{D} \geqslant 0 \iff n_j \geqslant 0$ for all j where $\mathcal{D} = \sum_j n_j p_j$. Such a divisor $\mathcal{D}$ is termed effective.

Proposition 4.3.2. If $\mathcal{D} \geqslant 0$, then $\dim(L(\mathcal{D})) \leqslant \deg(\mathcal{D}) + 1$.

Definition 4.3.3. If ω is a meromorphic form on X, then the following divisor is termed canonical.

$$(\omega) = \sum_{p \in X} \mathrm{ord}_p(\omega) \cdot p$$

Proposition 4.3.4. Given a compact Riemann surface X, any two canonical divisors are equivalent. They then belong to a common canonical class.

Proposition 4.3.5. If X is a compact Riemann surface of genus g, then deg(canonical class) $= 2g - 2$.

Proposition 4.3.6. If $\mathcal{D}_1 \sim \mathcal{D}_2$, then $L(\mathcal{D}_1) = L(\mathcal{D}_2)$.

Proposition 4.3.7. If K is a canonical divisor, then $L(K) \simeq \begin{pmatrix} \text{space of holomorphic} \\ \text{forms on } X \end{pmatrix}$.

Corollary 4.3.8. The space of holomorphic forms on X is always finite-dimensional.

Further on, for any divisor $\mathcal{D}$, denote $\dim(L(\mathcal{D})) = \ell(\mathcal{D})$.

Theorem 4.3.9. For K a canonical divisor, $\ell(K) = g$.

Theorem 4.3.10. [Riemann, Roch]
For K a canonical divisor,

$$\ell(\mathcal{D}) = \deg(\mathcal{D}) + 1 - g + \ell(K - \mathcal{D})$$

Proposition 4.3.11. Suppose that f is a meromorphic function on a Riemann surface X. If $a \in X$ is a zero or pole of f, then df/f has a simple pole at a and $\mathrm{res}_a(df/f) = \mathrm{ord}_a(f)$.

Proposition 4.3.12. Let $\Omega(X) = \{\omega \mid \omega \text{ is a holomorphic form on } X\}$. Then $\dim(\Omega(X)) = g(X)$.

Proposition 4.3.13. If $g(X) = 0$, then $X \simeq \bar{\mathbb{C}}$.

Definition 4.3.14. Let X be a Riemann surface defined by $X = \{(z, w) \mid w^2 = (z-a_1)(z-a_2)\cdots(z-a_n)\}$. For $n \geqslant 4$, X is termed a hyperelliptic curve.

Proposition 4.3.15. For X a Riemann surface, if $g(X) > 0$ and $p \in X$, then $\ell(p) = 1$.

Moreover, $\ell((n+1) \cdot p) \geqslant \ell(n \cdot p)$ for $n \in \mathbb{Z}_{\geqslant 0}$.

Proposition 4.3.16. If X is a compact Riemann surface of genus g with $\deg(\mathcal{D}) > 2g - 2$, then $\ell(\mathcal{D}) = \deg(\mathcal{D}) + 1 - g$.

5 Linear systems

5.1 Separation of points

Definition 5.1.1. For $\mathcal{D}$ a divisor on X with $\ell(\mathcal{D}) > 0$, the complete linear system defined by $\mathcal{D}$ is

$$|\mathcal{D}| = \{\mathcal{D}' \mid \mathcal{D}' \geqslant 0, \mathcal{D}' \sim \mathcal{D}\}$$

Definition 5.1.2. $|\mathcal{D}| = P(L(\mathcal{D})) = \Big(L(\mathcal{D}) \setminus \{0\}\Big) \Big/ \mathbb{C}^*$

Remark 5.1.3.

1. $\mathcal{D}_1 \sim \mathcal{D}_2 \implies |\mathcal{D}_1| = |\mathcal{D}_2|$
2. If $\mathcal{D}$ is effective, then $\mathcal{D} \in |\mathcal{D}|$

Corollary 5.1.4. The space of ordered n-tuples of points of $\mathbb{S}^2$ is homeomorphic to $\mathbb{C}P^n$.

Definition 5.1.5. Let $|\mathcal{D}|$ be a complete linear system. A point $p \in X$ is termed a basepoint of $|\mathcal{D}|$ if $\mathcal{D}' \geqslant p$ for all $\mathcal{D}' \in |\mathcal{D}|$.

Definition 5.1.6. If X is a Riemann Surface with genus > 1 and K_X is its canonical class, then $|K_X|$ is termed a canonical linear system.

Proposition 5.1.7. If $g \geqslant 1$ for a Riemann surface X, then $|K_X|$ has no basepoints.

Theorem 5.1.8. A divisor $\mathcal{D}$ has no basepoints $\iff \ell(\mathcal{D} - p) = \ell(\mathcal{D}) - 1$ for all $p \in X$.

Definition 5.1.9. Define the following mapping for a Riemann surface X and a divisor $\mathcal{D}$:

$$\varphi_{|\mathcal{D}|} : X \to \mathbb{C}P^n = |\mathcal{D}|^* \qquad \text{by} \qquad x \mapsto \{\mathcal{D}' \bigm| \mathcal{D}' \in |\mathcal{D}|, \mathcal{D} \geqslant x\}$$

We note that if $\mathcal{D}$ has no basepoints, then this mapping is well-defined. Moreover, if $\mathcal{D}$ has no basepoints, then it defines a holomorphic mapping into projective space.

Proposition 5.1.10. If $L(\mathcal{D}) = \langle f_0, \ldots, f_n \rangle$, then $\varphi_{|\mathcal{D}|} : x \mapsto (f_0(x) : \cdots : f_n(x))$

Proposition 5.1.11. If $\mathcal{D}$ has no basepoints, then $\varphi_{|\mathcal{D}|}$ is injective $\iff$ for all $p \neq q$, $\ell(\mathcal{D}-p-q) = \ell(\mathcal{D})-2$.

Definition 5.1.12. Let $X = \bar{\mathbb{C}}$ and $\mathcal{D}$ be a divisor of X with $\deg(\mathcal{D}) = n > 1$. Note that $\mathcal{D}$ does not have basepoints, and $\varphi_{|\mathcal{D}|} : X \to \mathbb{C}P^2$ is injective. Then $\varphi_{|\mathcal{D}|}(X) = X_n$ is termed a rational normal curve, or Veronese curve.

5.2 Separation of tangent vectors

Definition 5.2.1. Let $\varphi : X \to \mathbb{C}^n$ be a map of a Riemann surface $X \subset \mathbb{C}$ with $\varphi : z \mapsto (\varphi_1(z), \ldots, \varphi_n(z))$. Then $\varphi'(z) = (\varphi_1'(z), \ldots, \varphi_n'(z))$ is termed degenerate at $p \in X$ if $\varphi_1'(p) = \cdots = \varphi_n'(p) = 0$.

Proposition 5.2.2. The derivative of $\varphi_{|\mathcal{D}|}$ is non-degenerate at $p \in X \iff \ell(\mathcal{D} - 2p) = \ell(\mathcal{D}) - 2$.

Theorem 5.2.3. If $|\mathcal{D}|$ is a complete linear system without basepoints and $\ell(\mathcal{D} - p - q) = \ell(\mathcal{D}) - 2$ for all $p, q \in X$, then $\varphi_{|\mathcal{D}|} : X \to P^{\dim(|\mathcal{D}|)}$ is an embedding, and $\varphi_{|\mathcal{D}|}(X)$ is a smooth curve.

Proposition 5.2.4. Suppose that X is a compact Riemann surface, $\mathcal{D}$ is a divisor of X with $\deg(\mathcal{D}) \gg 0$. Then $\mathcal{D}$ has no basepoints and $\varphi_{|\mathcal{D}|}$ embeds X as a smooth curve.

Proposition 5.2.5. $\varphi_{|\mathcal{D}|}$ is not an embedding $\iff$ there exists a $\mathcal{D}$ such that $\deg(\mathcal{D}) = 2$ and $\ell(\mathcal{D}) = 2$. In this case X is a hyperelliptic curve.

Corollary 5.2.6. $|K_X|$ does not define an embedding $\iff$ X is a hyperelliptic curve.

Remark 5.2.7. If $g > 2$, then a Riemann surface of genus g is not hyperelliptic. Moreover, if X is not hyperelliptic, then $\varphi_{|K_X|} : X \hookrightarrow \mathbb{C}P^{g-1}$ is an embedding, and $\deg(\varphi_{|K_X|}(X)) = 2g-2$. Here $\varphi_{|K_x|}$ is termed a canonical curve.

Proposition 5.2.8. A Riemann surface of genus 1 is isomorphic to an elliptic curve.

Definition 5.2.9. Define the Weierstrass p-function to be $\wp : X \to \mathbb{R}$, given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right)$$

Note that this function converges on $X = \mathbb{C}/\Gamma$ an elliptic curve.

Theorem 5.2.10. [PROPERTIES OF THE $\wp$-FUNCTION]

1. $\wp(z) = \wp(-z)$
2. $\wp'(z) = -\wp'(-z)$
3. $\wp'(z) = -2\sum_{\gamma\in\Gamma} \frac{1}{(z-\gamma)^3}$
4. $\wp'(z+\alpha) = \wp'(\alpha)$ for all $\alpha \in \Gamma$
5. $\wp(z+\alpha) = \wp(\alpha)$ for all $\alpha \in \Gamma$

Theorem 5.2.11. Let $X = \mathbb{C}/\Gamma$ for $\Gamma = \langle \omega_1, \omega_2 \rangle$ a lattice. Then $\wp'(z)$ has three distinct zeros on X.

The value of $\wp$ at these points is denoted:

$$e_1 = \wp\left(\frac{\omega_1}{2}\right) \qquad e_2 = \wp\left(\frac{\omega_2}{2}\right) \qquad e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$$

Moreover,

$$(\wp'(z))^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

5.3 Embeddings

Remark 5.3.1. Let X be the space of homogeneous polynomials of degree d in m variables. Then $\dim(X) = \binom{d+m-1}{d}$.

Proposition 5.3.2. Let X be a compact Riemann surface. For $n > 3$, X may be embedded in $\mathbb{C}P^{n-1}$.

Proposition 5.3.3. Let $X = \mathbb{C}/\Gamma$ be an elliptic curve for Γ a lattice. Then X is isomorphic to the Riemann surface $Y = \{(x,y) \mid y^2 = 4(x-e_1)(x-e_2)(x-e_3)\}$ for e_i as above.

Remark 5.3.4. A Riemann surface of genus 1 is isomorphic to a cubic curve. Any Riemann surface of genus 2 is a hyperelliptic curve.

Proposition 5.3.5. If $\mathcal{D} \geqslant 0$, then $\ell(\mathcal{D}) = \deg(\mathcal{D})$.

Note the following new notation on divisors of a Riemann surface X. The sets below are groups with the group operation of addition.

- $\mathrm{Div}(X) = \{\text{all divisors on } X\}$
- $\mathrm{Div}^0(X) = \{\mathcal{D} \in \mathrm{Div}(X) \mid \deg(\mathcal{D}) = 0\}$
- $\mathrm{Principal}(X) = \{\mathcal{D} \in \mathrm{Div}(X) \mid \mathcal{D} \text{ is principal}\}$
- $\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\mathrm{Principal}(X)$, the Picard group

5.4 Elliptic curves

Theorem 5.4.1. [ABEL, JACOBI]
Let $+, -$ denote operations for points of a divisor. Let $\oplus, \ominus$ denote operations on group elements in $\mathbb{C}/\Gamma$.

Suppose that $\mathcal{D} = m_1a_1 + \cdots + m_ka_k$ is a divisor on $X = \mathbb{C}/\Gamma$ with $m_j \in \mathbb{Z}, a_j \in X$, and $\deg(\mathcal{D}) = m_1 + \cdots + m_k = 0$. Then $\mathcal{D}$ is principal $\iff m_1a_1 \oplus \cdots \oplus m_ka_k = 0$.

Corollary 5.4.2. If $X = \mathbb{C}/\Gamma$, then $\mathrm{Pic}^0(X) \simeq X$.

Theorem 5.4.3. $\mathrm{Pic}^0(X) = \mathbb{C}^g/\Gamma$ for $\Gamma \subset \mathbb{C}^g$ a lattice of rank $2g$.

Definition 5.4.4. Let $k \in \mathbb{Z}_{\geqslant 2}$ and $\Gamma \subset \mathbb{C}$ a lattice. Then $G_k := \sum_{\gamma\in\Gamma\setminus\{0\}} \frac{1}{\gamma^{2k}}$ is termed an Eisenstein series.

Remark 5.4.5. By rewriting $\wp$ in terms of Eisenstein series, the following conclusions are reached:

$$\begin{aligned}
(\wp'(z))^2 &= 4\wp^3(z) - 60G_2\wp(z) + 140G_3 \\
e_1 + e_2 + e_3 &= 0 \\
e_1e_2 + e_1e_3 + e_2e_3 &= -15G_2 \\
e_1e_2e_3 &= 20G_3
\end{aligned}$$

Definition 5.4.6. Let $C \subset \mathbb{C}P^n$ be a smooth curve of degree d, and L a line in $\mathbb{C}P^n$. If $L \cap C = \{q\}$ and the divisor $L \cap C = d \cdot q$, then q is termed an <u>inflection point</u> of C.

Proposition 5.4.7. Let C be as above, and q_1, q_2 two inflection points of C. Then the line that passes through q_1 and q_2 also passes through a distinct third inflection point of C.

Proposition 5.4.8. Let Γ be a lattice on $\mathbb{C}$ and $f : X \to X \subset \mathbb{C}$ a holomorphic map with a fixed point $f(0) = 0$. Then there exists $\lambda \in \mathbb{C}$ such that $f(z) = \lambda z \pmod{\Gamma}$.

Proposition 5.4.9. Let $X_1 = \mathbb{C}/\Gamma_1$ and $X_2 = \mathbb{C}/\Gamma_2$ be two elliptic curves. Then $X_1 \simeq X_2 \iff$ there exists non-zero $\lambda \in \mathbb{C}$ such that $\lambda\Gamma_1 = \Gamma_2$.

6 Line bundles

6.1 Construction

Definition 6.1.1. Suppose that X is a complex manifold with $U \subset X$ open. A <u>line bundle</u> on X is both:

1. A complex manifold T, the "total space" of the bundle

2. A holomorphic mapping $\pi : T \to X$, the "projection"

that satisfies the following conditions:

a. Φ is an isomorphism such that $\pi' \circ \Phi = \tilde{\pi}$

b. For each $y \in U$, $\Phi(\pi^{-1}(y)) = (\pi')^{-1}(y)$

c. For each $x \in X$, $\pi^{-1}(x) \simeq \mathbb{C}$ and $0 \in \pi^{-1}(x)$

This may be envisioned as the following graph that commutes:

$$\begin{tikzcd}
T \arrow[dd, "\pi"'] & \pi^{-1}(U) \arrow[l, "i"'] \arrow[dd, "\tilde{\pi}"] \arrow[dr, dashed, "\Phi"] & \\
 & & U \times \mathbb{C} \arrow[dl, "\pi'"] \\
X & U \arrow[l, "i"] &
\end{tikzcd}$$

where: $\tilde{\pi} = \pi|_{\pi^{-1}(U)}$
$\pi' : (x, \lambda) \mapsto x$ is also a projection
$i : U \hookrightarrow X$ is an inclusion

Moreover, the function Φ is termed the <u>local trivialization</u>.

Definition 6.1.2. Consider two domains $U, V \subset X$ with local trivializations over them:

$$\begin{aligned}
\Phi_U &: \pi^{-1}(U) \to U \times \mathbb{C} \\
\Phi_V &: \pi^{-1}(V) \to V \times \mathbb{C}
\end{aligned}$$

Then for all $x \in U \cap V$, both functions are defined, and they differ by a linear automorphism given by

$$\Phi|_{\pi^{-1}(x)} = g_{UV}\Phi_V(x)$$

The functions of the type g_{UV} are termed <u>transition functions</u> of the line bundle.

Proposition 6.1.3. [THE COCYCLE CONDITION]
Let $U, V, W \subset X$ and $x \in U \cap V \cap W$. Then $g_{UW}(x) = g_{UV}(x) g_{VW}(x)$.

Proposition 6.1.4. Suppose that X is a complex manifold and $\bigcup U_j = X$ is an open covering with, for each $i \neq j$ a holomorphic function $g_{ij} : U_i \cap U_j \to \mathbb{C} \setminus \{0\}$ such that on $U_i \cap U_j \cap U_k$ with $k \notin \{i, j\}$

$$g_{ij} g_{jk} = g_{ik}$$
$$g_{ij} g_{ji} = 1$$

Then there exists a line bundle $T \xrightarrow{\pi} X$ such that T has trivializations over each U_j for which the g_{ij} are transition functions.

Remark 6.1.5. The line bundle as constructe.d above is usually denoted $\mathcal{O}_X(-1)$.

6.2 Sections

Definition 6.2.1. A <u>section</u> of a line bundle $\mathcal{O}_X(-1)$ is a holomorphic mapping $s : X \to T$ such that $\pi \circ s = \mathrm{id}_X$. Note that if $x \in X$, then $s(x) \in \pi^{-1}(x)$.

If s_1, s_2 are sections, then we may define addition of sections by $(s_1 + s_2) : x \mapsto s_1(x) + s_2(x)$.

Definition 6.2.2. Let $s : X \to T$ be a section with $x \in U \subset X$ and $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ the local trivialization. Then there exists a holomorphic function $s_U : U \to \mathbb{C}$ such that

$$\Phi_U(s(x)) = (x, s_U(x))$$

Moreover, if $x \in U \cap V$ for $V \subset X$, then

$$s_U(x) = g_{UV}(x) s_V(x)$$

Proposition 6.2.3. There is a 1-1 correspondence between sections of a given line bundle and collections of the holomorphic $s_U : U \to \mathbb{C}$ with $s_U = g_{UV} s_V$.

Remark 6.2.4. Every section of $\mathcal{O}_{\mathbb{C}P^n}(-1)$ is identically zero.

Definition 6.2.5. Suppose that L, M are line bundles on X with transition functions on $\bigcup U_j$

$$g_{ij} : U_i \cap U_j \to \mathbb{C} \setminus \{0\} \quad \text{for } L$$
$$h_{ij} : U_i \cap U_j \to \mathbb{C} \setminus \{0\} \quad \text{for } M$$

Then the tensor product $L \otimes M$ is the line bundle with transition functions $g_{ij} h_{ij}$ over $U_i \cap U_j$

Definition 6.2.6. Using the same notation as above, $(\mathcal{O}_{\mathbb{C}P^n}(-1))^* = \mathcal{O}_{\mathbb{C}P^n}(1)$.

6.3 On Riemann surfaces

Proposition 6.3.1. (Sections of $\mathcal{O}_X(\mathcal{D})) \simeq L(\mathcal{D})$.

Proposition 6.3.2. Any line bundle on a compact Riemann surface X is of the form $\mathcal{O}_X(\mathcal{D})$ for some divisor $\mathcal{D}$.

Definition 6.3.3. (The space of sections of a line bundle L on X) $= H^0(X, L)$.
Here H^0 denotes the 0th cohomology group.

Proposition 6.3.4. Suppose that $S_1, S_2 \in H^0(X, \mathcal{O}_X(\mathcal{D}))$. Then $(S_1) \sim (S_2)$.

Remark 6.3.5. Let $L_1 = \mathcal{O}_X(\mathcal{D}_1)$ and $L_2 = \mathcal{O}_X(\mathcal{D}_2)$. Then $L_1 \otimes L_2 = \mathcal{O}_X(\mathcal{D}_1 + \mathcal{D}_2)$.

Proposition 6.3.6. Let $\mathcal{D}$ be a divisor on a Riemann surface X. Then $\mathcal{O}_X(\mathcal{D})$ is trivial $\iff$ $\mathcal{D}$ is principal.

Proposition 6.3.7. Let $\mathcal{D}_1, \mathcal{D}_2$ be divisors on a Riemann surface X. Then the following are equivalent:

- **i.** $\mathcal{D}_1 \sim \mathcal{D}_2$
- **ii.** $\mathcal{O}_X(\mathcal{D}_1) \simeq \mathcal{O}_X(\mathcal{D}_2)$
- **iii.** $\mathcal{O}_X(\mathcal{D}_1) \otimes \mathcal{O}_X(\mathcal{D}_2)^{-1} = \mathcal{O}_X(\mathcal{D}_1 - \mathcal{D}_2)$ is trivial

Corollary 6.3.8. Line bundles are equivalence classes of divisors, with the canonical line bundle having sections of holomorphic forms.