Compact course notes

Topology I

Fall 2011

Professor: A. Penskoi
transcribed by: J. Lazovskis
Independent University of Moscow
December 23, 2011

Contents

1 Spaces and operations

1.1 Continuity and metric spaces

Definition 1.1.1. Let $f : \mathbb{R} \to \mathbb{R}$. Then f is termed continuous at $x \in \mathbb{R}$ if for set $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|y - x| < \delta \implies |f(y) - f(x)| < \epsilon$.

The same analog can be made with $f : \mathbb{R}^n \to \mathbb{R}^k$, with

$$\sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2} < \delta \implies \sqrt{(f(y)_1 - f(x)_1)^2 + \cdots + (f(y)_k - f(x)_k)^2} < \epsilon$$

Definition 1.1.2. A metric space (M, d) is a pair consisting of a set M and a function $d : M \times M \to \mathbb{R}$, termed a metric, such that d has the following properties, for $x, y, z \in M$:

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geqslant 0$
3. $d(x, y) = 0 \iff x = y$
4. $d(x, y) \leqslant d(x, z) + d(z, y)$

Example 1.1.3. Here are some examples of metrics:

· the Euclidean metric:
$(\mathbb{R}^n, d_e)$ for $d_e(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$

· the Manhattan metric:
$(\mathbb{R}^2, d_m)$ for $d_m(x, y) = |x_1 - y_1| + |x_2 - y_2|$

· the discrete metric:
(M, d) for $d(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

Note that any function is continuous on the discrete metric.

Definition 1.1.4. The limit $\lim\limits_{n\to\infty} [x_n] = x$ exists if and only if given $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for all $n > N$, $|x_n - x| < \epsilon$.

Remark 1.1.5. Let $\rightrightarrows$ denote uniform convergence. Then

$$f_n \to f \text{ in } C[a, b] \iff f_n \rightrightarrows f \text{ in } [a, b]$$

Definition 1.1.6. Let $f : (M, d_M) \to (N, d_N)$. Then f is continuous at $x \in M$ if for given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $d_M(x, y) < \delta \implies d_N(f(x), f(y))$ for $y \in M$.

Definition 1.1.7. The function f is continuous on a set U if f is continuous at every $x \in U$.

Remark 1.1.8.

$$(f \text{ cont. on } U) \bigwedge \left(\lim_{n\to\infty} [x_n] = x\right) \bigwedge (x_n, x \in U) \implies \left(\lim_{n\to\infty} [f(x_n)] = f(x)\right)$$

Definition 1.1.9. Define the open ball centred at p with radius $r > 0$ as $O_r(p) = \{q \mid d(p, q) < r\}$.

Definition 1.1.10. Let (M, d) be a metric space. A subset $U \subset M$ is termed open if for all $p \in U$ there exists $r > 0$ such that $O_r(p) \subset U$. The set U is closed if $M \setminus U$ is open.

Definition 1.1.11. Let (M, d) be a metric space with $N \subset M$. Then $(N, d|_N)$ is also a metric space, and $d|_N$ is termed an induced metric.

Theorem 1.1.12. A function $f : (M, d_M) \to (N, d_N)$ is continuous $\iff$ $f^{-1}(U) \subset M$ is open, for all $U \subset N$ open.

Theorem 1.1.13. Let (M, d_M) be a metric space with $\{x_n\} \in M$. Then $\lim\limits_{n\to\infty} [x_n] = x \in M$ if and only if for every open set $U \in M$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geqslant N$.

1.2 Topological spaces

Definition 1.2.1. A topological space is a pair (X, τ) consisting of a set X and a set τ of subsets of X with

i. $X, \emptyset \in \tau$

ii. $U, V \in \tau \implies U \cap V \in \tau$

iii. $U, V \in \tau \implies U \cup V \in \tau$

X is termed a set of points, τ is termed a topology on X, and $U \in \tau$ is termed an open subset of X.
A set $U \subset X$ is termed a closed subset if $X \setminus U$ is open.

Definition 1.2.2. Let (M, d_M) be a metric space. Then a topology τ may be defined by stating $U \in \tau \iff$ for all $x \in U$, there exists $r > 0$ such that $O_r(x) \in U$. Then (M, τ) is a topological space, and τ is termed an induced topology, specifically a topology on M induced by a metric d.

Definition 1.2.3. Consider $(\mathbb{R}, d_{Euclidean})$ a metric space and (X, τ)a topological space. Given some $B \subset \tau$, if any $U \in \tau$ is such that $U = \bigcup U_\alpha$ for all $U_\alpha \in B$, then B is termed a base of topology.

Definition 1.2.4. Suppose (X, τ) is a topological space with $Y \subset X$. Consider $\sigma = \{U \cap Y \;\big|\; U \in \tau\}$. Then (Y, σ) is a topological space, and σ is termed a topology induced by inclusion.

Definition 1.2.5. A topological space (X, τ) is termed a Hausdorff space iff it has the following property: For all $x, y \in X$ with $x \neq y$, there exist open sets $U \ni x$ and $V \ni y$ in X such that $U \cap V = \emptyset$.

Remark 1.2.6. Let $\mathbb{F}$ be a field and A^n an n-dimensional space over $\mathbb{F}$. Then $X \subset A^n$ is closed if and only if X is a zero set of polynomials, that is, the solution set to a set of polynomials such that the polynomials evaluate to zero. This is termed the Zavisky topology, and it is not Hausdorff.

1.3 Compactness

Definition 1.3.1. Let (X, τ) be a topological space with $X = \bigcup U_\alpha$. Then X is termed compact if for $U_{\alpha_i} \in \{U_\alpha\}$, $X = \bigcup U_{\alpha_i}$ is a finite union.

Theorem 1.3.2. [Completeness axiom]
Let $X, Y \subset \mathbb{R}$. Then for all $x \in X, y \in Y$ with $x \geqslant y$, there exists $c \in \mathbb{R}$ such that $x \leqslant c \leqslant y$. In this case, c is said to separate X and Y.

Theorem 1.3.3. Let $I = [a, b]$ and $I \supset I_1 \supset \cdots$ for $I_i = [a_i, b_i]$. Then $\bigcap\limits_{i=1}^{\infty} I_i \neq \emptyset$.

Definition 1.3.4. Any open $X \ni x$ is a neighborhood of x.

Definition 1.3.5. A topological space (X, τ) is connected if for all $x, y \in X$, there exists a path in X joining them.

2 Geometry

2.1 Construction

For all the definitions below, we assume that (X, τ) is a topological space.

Definition 2.1.1. The cone over X is denoted $C(X) = (X \times [0, 1])/\sim$ where the equivalence relation brings one end of the interval together to a point.

Definition 2.1.2. The suspension over X is denoted $\Sigma(X) = X \times [-1, 1]/\sim$ and consists of a cone on either side of X.

Definition 2.1.3. A map $f : X \to Y$ is termed a homeomorphism if it is continuous and has a continuous inverse map, $f^{-1} : Y \to X$ with $f^{-1} \circ f = \mathrm{id}$.

Definition 2.1.4. An <u>n-cell</u> in X is D^n, the n-dimensional disk.

Definition 2.1.5. There are several classical surfaces, with labeled planarization diagrams presented below.

Left to right, they are the Mobius strip, the torus, the Klein bottle, and the projective plane $\mathbb{R}P^2$.

Definition 2.1.6. Given a topological object X and its defined construction with cells, the construction only of k-cells is termed the <u>k-skeleton</u> of X, and denoted by $\mathrm{Sk}_k(X)$.

2.2 Triangulation

Definition 2.2.1. Let Σ be a surface and $\triangle \subset \mathbb{R}^2$. Then a <u>triangulation</u> T of Σ is the image of homeomorphic maps $\varphi_i : \triangle \to \Sigma$ such that for any vertex v of $\triangle$, $\varphi_i(v) = \varphi_j(v)$ for $i \neq j$ and for any edge e, $\varphi_i(e) = \varphi_j(e)$ for $i \neq j$.

Definition 2.2.2. Given a triangulation T of Σ, a <u>subtriangulation</u> T' is a triangulation of Σ such that T' has all the same (possibly more) vertices than T, and either the same edges as T, or subdivisions of those edges.

Remark 2.2.3. Some facts about triangulations:
1. Any compact surface has a triangulation
2. Any two triangulations have a common subtriangulation

Definition 2.2.4. Let Σ be a surface and T a triangulation of Σ. For T, let

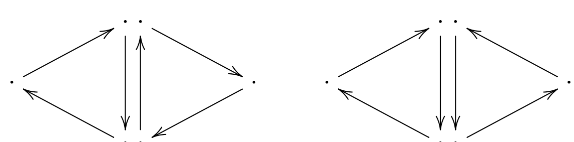
$$\left.\begin{array}{l} V - \text{ \# of vertices} \\ E - \text{ \# of edges} \\ F - \text{ \# of faces} \end{array}\right\} \text{Then the } \underline{\text{Euler characteristic}} \text{ of } \Sigma \text{ is } \chi(\Sigma) = V - E + F.$$

Proposition 2.2.5. For a surface Σ, $\chi(\Sigma)$ does not depend on the choice of triangulation.

Proposition 2.2.6. The formula for the Euler characteristic holds for any polygons.

Remark 2.2.7. A surface Σ with g holes has $\chi(\Sigma) = 2 - 2g$.

Definition 2.2.8. On a given triangulation T, assign a choice of orientation to every triangle of T as follows. The triangles on the left are <u>compatible</u>, whereas on the right they are <u>incompatible</u>.



Definition 2.2.9. A surface Σ is termed <u>orientable</u> if there exists a triangulation of Σ of only compatible triangles.

Definition 2.2.10. Given a surface X and its labeled planarization diagram, define the ordered sequence of letters $\omega_1\omega_2\cdots\omega_{2n}$ with $\omega_i \in \{a, a^{-1}, b, b^{-1}, \ldots, n, n^{-1}\}$ to be a <u>development</u> of X.

Theorem 2.2.11. Any compact surface without boundary has a development of one of the possible two forms:
1. $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$ a sphere with g handles
2. $a_1a_1b_1b_1\cdots a_na_n$ a sphere with n Mobius bands

Definition 2.2.12. A <u>connected sum</u> of topologicl objects X, Y is created by removing a disk from each of X, Y and identifying the resulting boundaries. This operation is denoted $X\#Y$.

Theorem 2.2.13. For topological objects X, Y, $\chi(X\#Y) = \chi(x) + \chi(Y) - 2$.

2.3 Algebraic topology

Definition 2.3.1. Consider two topological spaces X, Y and maps $f_0, f_1 : X \to Y$. Then equivalently:

1. f_0 is homotopy equivalent to f_1
2. there exists a homotopy between f_0 and f_1
3. there exists a continuous map $g : X \times [0,1] \to Y$ such that $g(x,0) = f_0(x)$ and $g(x,1) = f_1(x)$

Proposition 2.3.2. Homotopy equivalence is an equivalence relation and is denoted $\sim$.

Remark 2.3.3. Since $\sim$ is an equivalence relation, we may consider the quotient group

$$\{f \mid f : X \to Y \text{ is continuous}\}/\sim\, = [X, Y]$$

Then $[X, Y]$ is termed the set of homotopy equivalence classes of continuous maps from X to Y.

Definition 2.3.4. Define the homotopy groups by $\pi_k(Y) = [\mathbb{S}^k, Y]$.

Theorem 2.3.5. [Fixed Point Theorem - Brouwer]
Any continuous map $f : D^2 \to D^2$ has a fixed point. That is, there exists $x \in D^2$ such that $f(x) = x$.

Definition 2.3.6. A continuous map $r : X \to A \subset X$ is termed a retract if $r\big|_A = \mathrm{id}_A$.

3 Structure classification

3.1 Fundamental group

Definition 3.1.1. For X a topological space, a continuous map $\gamma : [0,1] \to X$ is termed a curve or path.

Definition 3.1.2. A path $\gamma : [0,1] \to X$ is termed a loop if $\gamma(0) = \gamma(1)$.

Definition 3.1.3. Let $\gamma_1, \gamma_2 : [0,1] \to X$ be paths for X a topological space, with $\gamma_1(1) = \gamma_2(0)$. Then define the product of γ_1 and γ_2 to be the curve

$$\gamma_1\gamma_2 : [0,1] \to X \qquad \text{given by} \qquad x \mapsto \begin{cases} \gamma_1(2x) & x \in [0, \frac{1}{2}] \\ \gamma_2(2x-1) & x \in (\frac{1}{2}, 1] \end{cases}$$

If γ_1 and γ_2 are such that $\gamma_1(1) = \gamma_2(0)$, then the two curves may be multiplied, or connected.

Remark 3.1.4. Any two loops may be connected.

Definition 3.1.5. Two curves γ_1 and γ_2 are termed equivalent if there exists a homotopy $F : [0,1] \times [0,1] \to X$ with $F(0,x) = \gamma_1(x)$ and $F(1,x) = \gamma_2(x)$. This relationship is denoted $\gamma_1 \sim \gamma_2$.

An extra condition is required, that $F(t,0) = F(t,1)$, or that the homotopy has fixed endpoints.

Definition 3.1.6. The set of equivalence classes generated by $\sim$ is termed a fundamental group, or $\pi_1(X, x_0)$. Moreover, $\pi_n(X, x_0)$ is the set of equivalence classes of maps $\gamma : \mathbb{S}^n \to X$ with basepoint x_0.

Theorem 3.1.7. Multiplication extends to equivalence classes, i.e. $[\gamma_1][\gamma_2] = [\gamma_1\gamma_2]$.

Definition 3.1.8. Denote the curve $\gamma_e : [0,1] \to X$, given by $\gamma_e(x) = x_0$, to be the identity curve.

Definition 3.1.9. Given a curve $\gamma : [0,1] \to X$, denote the inverse of γ to be $\gamma^{-1} : [0,1] \to X$ with $\gamma^{-1}(x) = \gamma(1-x)$, so that $\gamma\gamma^{-1} = \gamma^{-1}\gamma = \gamma_e$.

Proposition 3.1.10. For X, Y topological spaces, $X \sim Y \implies \pi_1(X, x_0) \simeq \pi_1(Y, y_0)$.

Theorem 3.1.11. [Algorithm for computing fundamental group]

1. Find a cellular strucutre on X with one 0-cell, by combining path connected 0-cells.
2. In $\mathrm{Sk}_1(X)$, attach a letter with orientation to each 1-cell; these letters are generators of π_1.
3. Go along this path of 1-cells, get a sequence of letters which is the relation of π_1.

3.2 Covering spaces

Definition 3.2.1. A map $f : Y \to X$ is termed a covering if for all $x \in X$ there exists an open neighborhood $U \ni x$ such that $f^{-1}(U) = U \times D$ for D the discrete space.

Here, X is the base, Y is the covering space, and f is the projection.

If D consists of n points for any $x \in X$, then f is termed an n-fold covering.

Remark 3.2.2. A covering map $f : Y \to X$ induces a homomorphism $f_* : \pi_1(Y, \tilde{x}_0) \to \pi_1(X, x_0)$ such that $\tilde{x}_0 \in f^{-1}(x_0)$.

Definition 3.2.3. If γ is a path in X, then $\hat{\gamma} \in f^{-1}(\gamma) \subset Y$ is termed a lift of γ.

Proposition 3.2.4. As above, f_* is surjective $\iff$ for all loops γ with basepoint x_0, their lifts $\hat{\gamma}$ are loops. However, f_* is always injective.

Definition 3.2.5. A covering map $f : Y \to X$ is termed regular if for any loop $\hat{\gamma} \in Y$, all lifts of its projection $f(\hat{\gamma})$ are loops.

Theorem 3.2.6. For any subgroup $H \subset \pi_1(X, x_0)$, there exists a unique covering space Y such that $\pi_1(Y, y_0) \simeq H$.

Definition 3.2.7. Given a covering $p : \tilde{Y} \to Y$ and a continuous map $f : X \to Y$, a map $\tilde{f} : X \to \tilde{Y}$ is termed a lifting of f if $p \circ \tilde{f} = f$. This is described by:

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Theorem 3.2.8. Let $p : \tilde{Y} \to Y$ be a covering space. Let $f : X \to Y$ be a continuous map. Let X be path connected and locally path connected. If $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ where $p(\tilde{y}_0) = y_0 = f(x_0)$, then there exists a lift $\tilde{f} : X \to \tilde{Y}$ of f such that $\tilde{f}(x_0) = \tilde{y}_0$.

Definition 3.2.9. Suppose we have a covering as above such that $\pi_1(\tilde{Y}, \tilde{y}_0) = 0$. Then $\tilde{Y}$ is termed a universal covering.

Note that the universal cover covers any cover of the space.

3.3 Cellular approximation

Definition 3.3.1. Suppose we have a map $f : X \to Y$ of cellular space. Then, wrt to the maps $f, \tilde{f}$ as above, $f \sim \tilde{f}$ such that $\tilde{f}(\mathrm{Sk}_k(X)) \subset \mathrm{Sk}_k(Y)$.

Theorem 3.3.2. If we have a connected cell space X, then there exists a cell space $\tilde{X}$ such that $X \sim \tilde{X}$, but $\tilde{X}$ only has one 0-cell.

Corollary 3.3.3. Therefore for any connected X, $\mathrm{Sk}_1(X) = \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$.

4 Handy tables

Complex X	$\chi(X)$	$\pi_1(X)$
$\mathbb{S}^1$	2	0
T^2	0	$\langle a, b\rangle$
$\mathbb{R}P^1$	1	$\langle a \mid a^2 = 1\rangle$
Mb	0	$\langle a\rangle$
Kl	0	$\langle a, b \mid a^2 = 1\rangle$