Compact course notes

Topology II

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Contents

Note: These notes are neither coherent nor orderly; they have not been properly systematized to at least somewhat accurately reflect the content of this course. Refer to the following AMS GSM books for a complete exposition:

· *Elements of Combinatorial and Differential Geometry,* V.V. Prasolov

· *Elements of Homology Theory,* V.V. Prasolov

1 Definitions

1.1 Foundational

Definition 1.1.1. Let R be a ring with an identity element. A left module M over R is a set with two binary operations, $+ : M \times M \to M$ and $\cdot : R \times M \to M$, such that

1. $(u + v) + w = u + (v + w)$ for all $u, v, w \in M$
2. $u + v = v + u$ for all $u, v \in M$
3. There exists an element $0 \in M$ such that $u + 0 = u$ for all $u \in M$
4. For any $u \in M$, there exists an element $v \in M$ such that $u + v = 0$
5. $a \cdot (b \cdot u) = (a \cdot b) \cdot u$ for all $a, b \in R$ and $u \in M$
6. $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ for all $a \in R$ and $u, v \in M$
7. $(a + b) \cdot u = (a \cdot u) + (b \cdot u)$ for all $a, b \in R$ and $u \in M$

A right module is defined analogously, except that the function $\cdot$ goes from $M \times R$ to M and the scalar multiplication operations act on the right.

Definition 1.1.2. Let M_1, M_2 be R-modules. The tensor product of M_1 and M_2 is a set of elements $M_1 \bigotimes_R M_2 = \{\sum m_i \otimes m_j \mid m_i \in M_1, m_j \in M_2\}$ so every element in $M_1 \bigotimes_R M_2$ can be expressed as a sum.

If R is not commutative, M_1 must be a right R-module and M_2 must be a left R-module.

Definition 1.1.3. An Abelian group G is free if there exists $\{f_\alpha\}$ such that for all $g \in G$, $g = n_{\alpha_1} f_{\alpha_1} + \cdots + n_{\alpha_k} f_{\alpha_k}$ is a unique representation. Then $\{f_\alpha\}$ is a basis of G. If all C_k in a chain complex are fee then the chain complex is free.

Definition 1.1.4. The fundamental group of a topological space X is the group representing topological objects homotopic to X, and is denoted $\pi_1(X)$.

Definition 1.1.5. In category theory, a variety of algebras is a class of algebraic structures satisfying a given set of identities.

A subvariety is a variety and a subclass of a variety sharing the same properties as its parent variety.

Theorem 1.1.6. [Fixed Point Theorem - Brouwer]
Every continuous function $f : X \to X$ from a closed ball X of a topological space to itself has a fixed point $p \in X$, such that $f(p) = p$.

1.2 Homological

Definition 1.2.1. [Homotopy]
Two *maps* $f, g : X \to Y$ for topological spaces X, Y are termed homotopic if there exists a continuous map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Two *spaces* X, Y are termed homotopic if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$. This relation is expressed $X \sim Y$.

Definition 1.2.2. A loop is a continuous map $f : [0, 1] \to X$ with $f(0) = f(1) = p \in X$. The trivial loop is the continuous map $f : [0, 1] \to X$ with $f(x) = p$ for all $x \in [0, 1]$.

2 Basic homology

2.1 Homology groups

Definition 2.1.1. A simplex is a generalization of a tetrahedral region of space to n dimensions. A k-simplex has $k + 1$ vertices, $k(k + 1)/2$ edges and $\binom{k+1}{i+1}$ i-faces. A simplex with coordinates $a_0, \ldots, a_n$ in that order is denoted by $[a_0, \ldots, a_n]$.

Definition 2.1.2. The boundary of a simplex is defined to be

$$\partial[0,1,\ldots,n] = \sum_i (-1)^i[0,\ldots,\hat{i},\ldots,n] = \sum_i (-1)^i[0,\ldots,i-1,i+1,\ldots,n]$$

with $\partial[a] = 1$ for a 1-cycle.

Theorem 2.1.3. For any simplex $\triangle$, $\partial\partial\triangle = 0$.

Definition 2.1.4. A simplical complex K is a set of simplices in $\mathbb{R}^n$ satisfying:
1. All faces of simplices from K belong to K
2. The intersection of any two simplices from K is a face for each of them
3. For any $p \in K$ belonging to a simplex of K has a neighborhood that intersects only finitely many simplices from K

The dimension of a simplical complex K is the maximum dimension of all the simplices in K.

Definition 2.1.5. Let G be an Abelian group with $a \in G$ and $\triangle^k$ be a simplex of dimension k. Define a k-chain to be a finite sum

$$\sum a_i \triangle_i^k$$

The group of k-chains is denoted $C_k(K;G)$ or $C_k(K)$ or C_k.

A chain $c \in C_k$ is termed a boundary if $c = \partial_{k+1}c'$ for some chain $c' \in C_{k+1}$.

Remark 2.1.6. The map ∂ works by extension as $\partial_k : C_k \to C_{k-1}$. This is termed a boundary homomorphism.

The group of k-dimensional boundaries is denoted B_k.

Definition 2.1.7. A chain $c \in C_k$ is termed a cycle if $\partial_k(c) = 0$.

The group of k-dimensional cycles is denoted Z_k.

Definition 2.1.8. Since $B_k \subset Z_k$, define the k-dimensional simplicial homology group to be the quotient group $H_k(K) = Z_k/B_k$. Its elements are equivalence classes of cycles; cycles are equivalent (homologous) if their difference is a boundary.

2.2 Homology of simplices

Theorem 2.2.1. If K is a connected simplicial complex, then $H_0(K;G) = G$.

In general, $H_0(K;G)$ denotes the number of connected components.

Theorem 2.2.2. If $k \geqslant 1$, then $H_k(\triangle^n) = 0$. However, $H_{n-1}(\partial\triangle^n) = \partial\triangle^n$.

Corollary 2.2.3. Let $\partial\triangle^n$ be the simplicial complex consisting of all simplices in $\triangle^n$ except $\triangle^n$ itself. Then $H_k(\partial\triangle^n) = \begin{cases} 0 & 0 < k < n-1 \\ G & k = n-1 \quad \text{(and } k \geqslant 2\text{)} \end{cases}$

Definition 2.2.4. A chain complex is a family of Abelian groups C_k and homomorphisms ∂_k satisfying $\partial_k\partial_{k+1} = 0$.

2.3 Chain homotopy

Definition 2.3.1. A chain map is a map between chains that commutes with ∂.

Definition 2.3.2. Suppose there is a simplicial map $f : K \to L$. Then there are two maps related to f:

$$f_* : H_k(L) \to H_k(K)$$
$$f_\# : C_k(K) \to C_k(L)$$

Definition 2.3.3. Given objects $A_1, \ldots, A_n$, an exact sequence is a sequence of the objects

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

such that $\mathrm{im}(f_i) = \ker(f_{i+1})$ for all $1 \leqslant i < n-2$.

Definition 2.3.4. Given objects K, L, M, a <u>short exact sequence</u> is an exact sequence of the form

$$0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$$

such that f is injective and g is surjective.

Definition 2.3.5. Suppose simplicial maps $f, g : K \to L$ and their related homology maps. <u>Chain homotopy</u> is the exact sequence of homomorphic maps $D_k : C_k(K) \to C_{k+1}(L)$ such that

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{k+2}"] & C_{k+1}(K) \arrow[r, "\partial_{k+1}"] \arrow[d, shift right, "f_{\#(k+1)}"'] \arrow[d, shift left, "g_{\#(k+1)}"] & C_k(K) \arrow[r, "\partial_k"] \arrow[d, shift right, "f_{\#k}"'] \arrow[d, shift left, "g_{\#k}"] \arrow[dl, "D_k"'] & C_{k-1}(K) \arrow[r, "\partial_{k-1}"] \arrow[d, shift right, "f_{\#(k-1)}"'] \arrow[d, shift left, "g_{\#(k-1)}"] \arrow[dl, "D_{k-1}"'] & C_{k-2}(K) \arrow[r, "\partial_{k-2}"] \arrow[d, shift right, "f_{\#(k-2)}"'] \arrow[d, shift left, "g_{\#(k-2)}"] \arrow[dl, "D_{k-2}"'] & \cdots \\
\cdots \arrow[r, "\partial_{k+2}"] & C_{k+1}(L) \arrow[r, "\partial_{k+1}"] & C_k(L) \arrow[r, "\partial_k"] & C_{k-1}(L) \arrow[r, "\partial_{k-1}"] & C_{k-2}(L) \arrow[r, "\partial_{k-2}"] & \cdots
\end{tikzcd}$$

Thus $g_{\#k} - f_{\#k} = \partial_{k+1} D_k + D_{k-1}\partial_k$, and for any $[z] \in H_k(K)$, $(g_{\#k} - f_{\#k})[z] = 0$, so they are chain maps.

Definition 2.3.6. A simplicial complex is <u>acyclic</u> if all its homology groups evaluate to zero.

Definition 2.3.7. Given a chain $c_k = \sum a_k \triangle_i^k \in C_k(K)$, a subcomplex $K' \subset K$ containing all the $\triangle_i^k$ is termed a <u>support</u> of c_k.

Theorem 2.3.8. Suppose $\varphi_k, \psi_k : C_k(K) \to C_k(L)$ are chain maps that preserve augmentation and whose coefficient groups are rings. Suppose that for all $\triangle \in K$ there exists $L(\triangle) \subset L$ such that
 1. If $\triangle' \subset \triangle$, then $L(\triangle') \subset L(\triangle)$
 2. $L(\triangle)$ is acyclic, or $H_i(L(\triangle)) = 0$ if $i \neq 0$
 3. $L(\triangle^k)$ is in the support of both chains $\varphi_k(\triangle^k)$ and $\psi_k(\triangle^k)$
Then φ_k and ψ_k are chain homotopic, moreover, $\varphi_* = \psi_*$.

The above shows that homology is a homotopic invariant, or that an object does not change homologies under a homotopy.

Theorem 2.3.9. [Mayer-Vietoris]
Given a simplicial complex K with subcomplexes K_0, K_1 with $K_0 \cup K_1 = K$ and $K_0 \cap K_1 = L$,

$$\cdots \xrightarrow{\partial} H_k(L) \xrightarrow{\partial} H_k(K_0) \oplus H_k(K_1) \xrightarrow{\partial} H_k(K) \xrightarrow{\partial} H_{k-1}(L) \xrightarrow{\partial} \cdots$$

is an exact sequence.

Theorem 2.3.10.

$$\begin{array}{ccc} K_1 & \sim & K_2 \\ \cup & & \cup \\ L_1 & \sim & L_2 \end{array} \quad \Longrightarrow \quad H_*(K_1; L_1) = H_*(K_2; L_2)$$

3 Cohomology

3.1 Structures

Definition 3.1.1. Given a simplicial complex K and an Abelian group G, a homomorphism $c^k : C_k(K;\mathbb{Z}) \to G$ is termed a k-dimensional <u>cochain</u> with coefficients in G.

The group of k-dimensional cochains is denoted $C^k(K;G) = \mathrm{Hom}(C_k(K;\mathbb{Z}), G)$.

Remark 3.1.2. As above, we have more homological objects:

i. The group of k-dimensional cocyles is $Z^k = \{z \in C^k(K) \;\big|\; \delta z = 0\}$

ii. The group of k-dimensional boundaries is $B^k = \{b \in C^k(K) \;\big|\;$ there exists $c \in c^{k-1}(K)$ with $\delta c = b\}$

iii. The cohomology group is $H^k = Z^k / B^k$

Theorem 3.1.3. Let G be an additive group of a field $\mathbb{F}$. Then $H^i(K)$ is dual to $H_i(K)$.

Remark 3.1.4. The operator $\delta : C^k(K;G) \to C^{k+1}(K;G)$ is used as a dual to $\partial : C_k(K;G) \to C_{k-1}(K;G)$, expressed by the relation

$$\langle \delta c^k, c_{k+1} \rangle = (-1)^{k+1} \langle c^k, \partial c_{k+1} \rangle$$

Theorem 3.1.5. An exact sequence on objects U, V, W induces a dual exact sequence on the dual objects.

$$U \xrightarrow{A=\partial} V \xrightarrow{B=\partial} W$$

$$U^* \xleftarrow{A^*=\delta} V^* \xleftarrow{B^*=\delta} W^*$$

with homology $H_i = \ker(B)/\mathrm{Im}(A)$ and cohomology $H^i = \ker(A^*)/\mathrm{Im}(B^*)$.

3.2 Universal coefficient formula

Definition 3.2.1. Let A, B be Abelian groups defined by generators and relations, and F, R free Abelian groups with F defined by generators and R by relations. Then a free resolution of the group A is an exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

with induced exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow R \otimes B \xrightarrow{\varphi} F \otimes B \longrightarrow A \otimes B \longrightarrow 0$$

and induced exact homology sequence

$$0 \longleftarrow \mathrm{Coker}(\varphi) \longleftarrow \mathrm{Hom}(R,B) \xleftarrow{\varphi} \mathrm{Hom}(F,B) \longleftarrow \mathrm{Hom}(A,B) \longleftarrow 0$$

Then we also define the torsion group $\mathrm{Tor}(A,B) = \ker(\varphi)$ and the extension group $\mathrm{Ext}(A,B) = \mathrm{Coker}(\varphi)$.

Definition 3.2.2. Given an exact sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$, the sequence is split if any one of the following equivalent conditions is satisfied.

i. the sequence is of the form $0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$ for i/p the natural embedding/projection

ii. there exists a homomorphism $\alpha : B \to A$ with $\alpha \circ \varphi = \mathrm{Id}_A$

iii. there exists a homomorphism $\beta : C \to B$ with $\psi \circ \beta = \mathrm{Id}_C$

Theorem 3.2.3. Homologies are related to coefficients in G by the following exact sequence:

$$0 \longrightarrow H_k(K;\mathbb{Z}) \longrightarrow H_k(K;G) \longrightarrow \mathrm{Tor}(H_{k-1}(K;\mathbb{Z});G) \longrightarrow 0$$

$$0 \longleftarrow \mathrm{Hom}(H_k(K;\mathbb{Z});G)) \longleftarrow H^k(K;G) \longleftarrow \mathrm{Ext}(H_{k-1}(K;\mathbb{Z});G) \longleftarrow 0$$

Moreover, both exact sequences are split.

Remark 3.2.4. For the Tor group as above, note that $\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A)$.

4 Duality

4.1 Cellular homology

Definition 4.1.1. A topological space X is termed a <u>CW-complex</u> if $X = \bigcup X^i$ for X^0 a discrete space and X^{i+1} generated by attaching a disjoint union $\bigsqcup D_j^{i+1}$ of $(i+1)$-disks to X^i via a continuous map $\varphi : \bigsqcup \partial D_j^{i+1} \to X^i$.

Definition 4.1.2. For X a CW-complex as above, the space X^i is termed the <u>i-dimensional skeleton</u> of X.

Definition 4.1.3. Let $\triangle$ be an n-dimensional simplex. Then $\triangle^*$ is termed the <u>dual</u> simplex of $\triangle$ generated by barycentric division. Then $\triangle$ and $\triangle^*$ are termed <u>transversal</u>. Moreover, we have that

$$\langle\langle \triangle_i, \triangle_j^* \rangle\rangle = \delta_{ij}$$

Theorem 4.1.4. [Poincare duality]
Classifying surfaces with barycentric triangulation and other methods, we find that, for p prime

$$\begin{aligned} H_k(K;\mathbb{Z}) &\simeq H^{n-k}(K;\mathbb{Z}) \\ H_k(K,\mathbb{Z}_p) &\simeq H^{n-k}(K;\mathbb{Z}_p) \\ H_k &\simeq (H^k)^* \end{aligned}$$

Proposition 4.1.5. Let $\triangle_i, \triangle_j$ be triangles of dimension $k, k-1$. Then

$$\langle\langle \partial\triangle_i, \triangle_j^* \rangle\rangle = (-1)^k \langle\langle \triangle_i, \partial(\triangle_j^*) \rangle\rangle$$

Definition 4.1.6. Let M be manifold then the <u>Euler characteristic</u> of M is defined as

$$\begin{aligned} \chi(M) &= \sum_k (-1)^k |C_k(K)| \\ &= \sum_k (-1)^k \dim(H_k) \end{aligned}$$

Note that χ is homotopy invariant, or $X \simeq Y \implies \chi(X) = \chi(Y)$.

Proposition 4.1.7. Given a short exact sequence $0 \to A \to B \to C \to 0$ for A, B, C simplicial complexes,

$$\chi(B) = \chi(A) + \chi(C)$$

Definition 4.1.8. Let K be a finite simplicial complex and $f : K \to K$ a continuous map. Consider the induced map $f_* : H_k(K) \to H_k(K)$. As $H_k(K)$ is a finite-dimensional vector space, consider the trace $\mathrm{tr}(f_*)_k$. Define the <u>Lefschetz number</u> to be

$$\Lambda(f) = \sum (-1)^k \mathrm{tr}(f_*)_k$$

Theorem 4.1.9. [Fixed Point Theorem - Lefschetz]
Let K be a simplicial complex. If $\Lambda(f) \neq 0$, then the map $f : K \to K$ has a fixed point.

Proposition 4.1.10. Let $f_\# : C_k(K;R) \to C_k(K;R)$, and then

$$\sum (-1)^k \mathrm{tr}(f_\#)_k = \sum (-1)^k \mathrm{tr}(f_*)_k$$

4.2 Homotopy groups

Every loop on the sphere S^2 is contractible to a point, so its fundamental group, $\pi_1(S^2)$, is trivial.

Let $H_n(S^2, \mathbb{Z})$ denote the n-th homology group of S^2.We can compute all of these groups using the basic results from algebraic topology:

- S^2 is a compact orientable smooth manifold, so $H_2(S^2, \mathbb{Z}) = \mathbb{Z}$;
- S^2 is connected, so $H_0(S^2, \mathbb{Z}) = \mathbb{Z}$;
- $H_1(S^2, \mathbb{Z})$ is the abelianization of $\pi_1(S^2)$, so it is also trivial;
- S^2 is two-dimensional, so for $k > 2$, we have $H_k(S^2, \mathbb{Z}) = 0$

In fact, this pattern generalizes nicely to higher-dimensional spheres:

$$H_k(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{else} \end{cases}$$

This also provides the proof that the hyperspheres S^n and S^m are non-homotopic for $n \neq m$, for this would imply an isomorphism between their homologies.