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Compact course notes

Applied Math 391, Winter 2013

From Fourier to Wavelets

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1 The Fourier transform

· The `.jpeg` extension, named after the Joint Photographic Experts Group, uses the discrete cosine transform, which reduces the size of an image by a factor of 10-20.

Definition 1.1. A Hilbert space is a complete metric space with an inner product.

Definition 1.2. The dispersion of a function f about a in its domain is defined as

$$\Delta_a f = \frac{1}{\|f\|^2}\int_{-\infty}^{\infty}(t-a)^1|f(t)|^2dt$$

Theorem 1.3. [The uncertainty principle]
Let f be a function and F its Fourier transform. Then

$$(\Delta_a f)(\Delta_a F) \geqslant \frac{1}{4}$$

This inequality relates the dispersion of f in time to the dispersion of F in frequency, also implying that f and F can not both have finite support.

Theorem 1.4. [Plancherel]
For any $f \in L^2(\mathbb{R})$, with $\mathcal{F}[f]$ the Fourier transform of f,

$$\|f\|^2 = \|\mathcal{F}[f]\|^2$$

2 Wavelets and multiresolution analysis

$$\underline{\text{Haar scaling function}} \; : \; \phi(t) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$\underline{\text{Haar wavelet}} \; : \; \psi(t) = \begin{cases} 1 & x \in [0,1/2) \\ -1 & x \in [1/2,1) \\ 0 & \text{else} \end{cases}$$

The scaling and wavelet functions always must satisfy the following conditions, for some h_k and g_k:

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k\sqrt{2}\phi(2x-k) \qquad \psi(x) = \sum_{k\in\mathbb{Z}} g_k\sqrt{2}\psi(2x-k) \qquad \psi(x) = 2\phi(2x) - \phi(x)$$

The coefficients h_k must satisfy the relation $\sum_{k\in\mathbb{Z}} h_k = \sqrt{2}$, and the relationship between h_k and g_k may be given by $g_k = (-1)^k h_{1-k}$. By taking the Fourier transform of the first equation above, we find that the Fourier transform $\Phi(\omega)$ of $\phi(x)$ satisfies

$$\Phi(\omega) = \frac{\Phi(\omega/2)}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k e^{-ik\omega/2}$$

Definition 2.1. A function $f \in L^2(\mathbb{R})$ is termed Ω-bandlimted iff its Fourier transform $F(\omega)$ vanishes outside the interval $[-\Omega, \Omega]$. The spaces V_j that together constitute a multiresolution analysis are given by

$$V_j = \{f \in L^2(\mathbb{R}) \ : \ f \text{ has bandlimit } \Omega_j = 2^j \pi\}$$

This, and the sampling theorem, implies that $V_0 = \text{span}\{\text{sinc}(x-k) \ : \ k \in \mathbb{Z}\}$, for $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. Further, the space V_j may be decomposed as

$$V_j = V_{j-1} \oplus W_{j-1}$$

where W_j is orthogonal to V_j.

Theorem 2.2. [Sampling theorem]
Let $f(t)$ be an Ω-bandlimited function with piecewise-continuous Fourer transform $F(\omega)$. Then

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \text{sinc}\left(\frac{\Omega t}{\pi} - k\right) = \sum_{k=-\infty}^{\infty} \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}$$