Compact course notes

Combinatorics and Optimization 250, Winter 2011

Introduction to optimization

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April 5, 2011

Contents

1 Linear Programs

Problem 1.0.1. [Production problem]
A firm produces products $1, \dots, n$ for the production of which certain amounts of input $1, \dots, m$ are necessary. Find how many units of each product should be produced to maximize profit and not exceed available inputs.

1.1 Fundamentals

Definition 1.1.1. Optimization is the maximization of a function subject to given constraints.

Definition 1.1.2. Given $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ constant, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\beta \in \mathbb{R}$, the expression $\alpha^T x + \beta$ is a linear function.

Definition 1.1.3. A linear constraint is an equation in one of the forms: $\begin{array}{l} \alpha^T x \leqslant \beta \\ \alpha^T x \geqslant \beta \\ \alpha^T x = \beta \end{array}$

Definition 1.1.4. A linear program is a linear function coupled with a finite set of linear constraints.

Definition 1.1.5. An integer program is a linear program with the added condition that at least one variable must have an integer value.

Problem 1.1.6. [Facility location problem]
Given potential facility locations $1, \dots,$ and involved parties situated at locations $1, \dots, m$, find how many and where facilities should be built to minimize cost and satisfy reachability.

Problem 1.1.7. [Inventory problem]
Given time periods $1, \dots, n$ and demand and price for a certain product for the periods, find how much of the product should be bought each time period, with possibility of storage.

Problem 1.1.8. [Assignment problem]
Given assignments $1, \dots, n$ and interested parties $1, \dots, m$, find which party should get which assignment to maximize profits and not exceed a 1:1 ratio of assignments:parties for each party.

2 Graph theory

2.1 Matching problem

Definition 2.1.1. A set $G = (V, E)$ is a graph, where V is a finite set of points, termed vertices, and E is a finite set of unordered pairs of points of V, termed edges.

Definition 2.1.2. A graph $G = (V, E)$is termed bipartite if there exists a partition $V_1, V_2 \subset V$ with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$ and such that for all $e = \{x, y\} \in E$, $x \in V_1$ and $y \in V_2$.

Theorem 2.1.3. A graph $G = (V, E)$is bipartite if and only if it has no cycles with an odd number of vertices.

Definition 2.1.4. For $G = (V, E)$a graph, $M \subseteq E$ is a matching of G if every $v \in M$ is at most 1-regular.

Problem 2.1.5. [Matching problem]
Given a graph $G = (V, E)$and a weight function $f(e) = w_e$ for all $e \in E$, find a matching $M \subseteq E$ that maximizes weight.

Problem 2.1.6. [Scheduling problem]
Given a list of tasks $1, \dots n$ with associated profits and start/end times, find a collection of tasks to be done that maximize profits and are constrained by no multitasking.

2.2 Probabilistic solving

Problem 2.2.1. [Portfolio optimization]
Given an amount a and potential returns $1, \ldots, n$ with associated probabilities, find the expected rate of return and volatility.

Definition 2.2.2. A solution x is a feasible solution to an LP if x satisfies all the constraints of the program.

Definition 2.2.3. An LP is termed infeasible if it has no feasible solutions.

Definition 2.2.4. A solution x is an optimal solution to an LP given:

1. x is feasible
2. If y is also feasible, then $y \leqslant x$ (if max) or $y \geqslant x$ (if min)

Definition 2.2.5. An LP is termed unbounded if feasible solutions exist, but optimal solutions do not.