Compact course notes

Combinatorics and Optimization 351, Fall 2012

Network Flow Theory

Professor: B. Richter
transcribed by: J. Lazovskis
University of Waterloo
October 21, 2012

Contents

1 Paths and walks

1.1 Definitions

Definition 1.1.1. A directed graph (or digraph) is a pair (N, A) of finite sets such that for each $a \in A$ there exist distinct $t(a), h(a) \in N$.

The elements of N are termed nodes and the elements of A are termed arcs.
The element $t(a)$ for $a \in A$ is the tail of a, and $h(a)$ is the head of a.

Definition 1.1.2. A directed walk (or diwalk) in a digraph (N, A) is a sequence $\{a_1, \ldots, a_k\}$ of arcs of (N, A) such that for all $i = 1, \ldots, k-1$ we have $h(i) = t_{i+1}$. The length of a diwalk is the number of elements in the sequence. A directed walk $\{a_1, \ldots, a_k\}$ is termed closed if $t(a_1) = h(a_k)$.

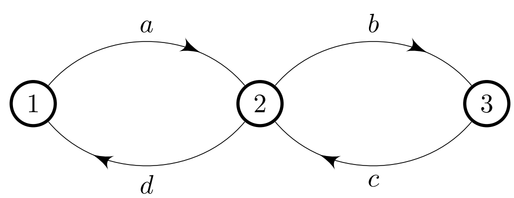
Definition 1.1.3. A directed walk $\{a_1, \ldots, a_k\}$ is su-directed if $s = t(a_1)$ and $u = h(a_k)$.

Definition 1.1.4. A directed walk $\{a_1, \ldots, a_k\}$ is termed a directed path (or dipath) if $h(a_1), \ldots, h(a_k)$ are all unique.

Theorem 1.1.5. If there is an su-directed walk, then there is an su-directed path.

Definition 1.1.6. A closed directed walk $\{a_1, \ldots, a_k\}$ is termed a directed cycle (or dicycle) if $h(a_1), \ldots, h(a_k)$ are all unique.

Example 1.1.7. Consider the following digraph:



Here we have that $\{a, b, c, d\}$ is a diwalk (a closed one), and both $\{a, d\}$ and $\{b, c\}$ are dicycles.

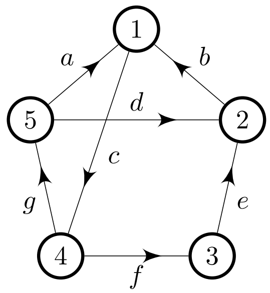
1.2 Deconstruction of walks

Theorem 1.2.1. If $\{a_1, \ldots, a_k\}$ for $k > 1$ is a closed diwalk, then there exist indexes $1 \leqslant i < j \leqslant k$ such that $\{a_i, a_{i+1}, \ldots, a_{j-1}, a_j\}$ is a dicycle.

Theorem 1.2.2. Let W be a closed diwalk in a digraph D. Then there exists a collection C of directed cycles of D such that each edge of D occurs in precisely the same number of cycles in C as it has occurrences in W.

Proof: Let $W = a_1, \ldots, a_k$ be the closed diwalk.
By above theorem, there exist indices i, j such that $a_i, \ldots, a_j$ is a dicycle.
Now we have that $t(a_i) = h(a_j)$.
Let $W' = a_1, \ldots, a_{i-1}, a_{j+1}, a_k$.
It is clear that $W1$ is still a closed diwalk, as from construction $t(a_i) = h(a_{i-1})$ and $h(a_j) = t(a_{j+1})$ and these expressions are equal as well.
Now we add $\{a_i, \ldots, a_j\}$ to C and by induction, every dicycle of W will be in C eventually. ■

Example 1.2.3. Consider the following closed diwalk:



$$\begin{aligned} W &= acfeb\underline{cga}cfebcg \\ W' &= a\underline{cfeb}cfebcg \\ W'' &= ac\underline{febc}g \\ W''' &= \underline{acg} \\ \\ C &= \{cga, cfeb, febc, acg\} \end{aligned}$$

Theorem 1.2.4. If W is an su-diwalk in a digraph D, then there is an su-dipath P and a collection C of dicycles so that the number of occurrences of each arc a in W is equal to the number of elements of $P \cup C$ containing a.

Proof: Let $W' = W \cup \{a\}$ such that $t(a) = u$ and $h(a) = s$, which then reduces to the previous theorem. ■

2 Shortest path algorithms

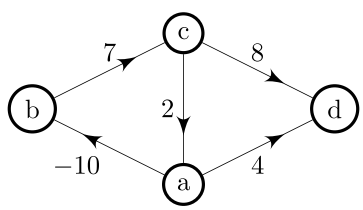
2.1 Dantzig's algorithm

Remark 2.1.1. If a digraph D has an su-directed walk that has length less than the shortest su-directed path, then d contains a negative cycle. We already showed that every su-directed walk W decomposes into an su-directed path P and a collection C of dicycles with the property that

$$\ell(W) = \ell(P) + \sum_{c \in C} \ell(c)$$

So if we have that $\ell(W) < \ell(P)$, then there must be some $c \in C$ with $\ell(c) < 0$.

Definition 2.1.2. Let $D = (N, A)$ be a digraph with $ab, bc \in A$. Then a is termed an in-neighbor of b and c is an out-neighbor of b. This is illustrated with an example below:



Proposition 2.1.3. Let u, v be nodes of a digraph D, and let vt be an arc of D. Then a shortest uv-dipath that contains t contracts to a shortest ut-dipath iff the tv-dipath does not contain a negative cycle.

Theorem 2.1.4. Dantzig's algorithm terminates either with a negative cycle (this will always happen if such a cyclele exists), or a tree with all paths from u being shortest to the nodes on them in the digraph.

Proof: Let T be an out-tree from u containing all the nodes of D.
For each node a of T, let y_a denote the length of the ua-dipath.
Let $ab \in A$.

Case 1: $ab \in T$.
Then $y_b = y_a + \ell(ab)$.

Case 2: $ab \notin T$.
■

Definition 2.1.5. Given a digraph $D = (N, A)$, for all $ab \in A$ define values y_a and y_b such that

$$y_b - y_a \leqslant \ell(ab)$$

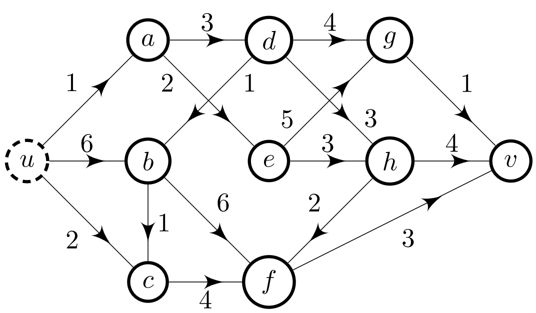
If such values for all arcs $ab \in A$ may be found, then they are termed feasible potentials.

Lemma 2.1.6. A digraph D has a set of feasible potentials iff it has no negative cycle.

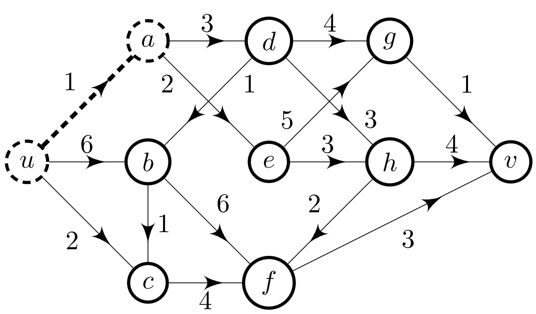
2.2 Dijkstra's algorithm

Algorithm 2.2.1. [Dijkstra]
Let A be the set of arcs ab for which $y_a - y_b = \ell(ab)$.
Let S be the set of nodes reachable from u using only arcs in A ($S = \{u\}$ at inception).
Set all $y_a = 0$.
While $v \notin S$:
    For each node $a \in S$, $b \notin S$, $ab \in A$, let $\epsilon_{ab} = y_a - y_b + \ell(ab) \neq 0$.
    Let $\epsilon = \min\{\epsilon_{ab}\}$.
    Add to A all arcs ab with $\epsilon_{ab} = \epsilon$.
    Add to S all nodes in the digraph formed by A.
    Add ϵ to all y_a with $a \notin S$.

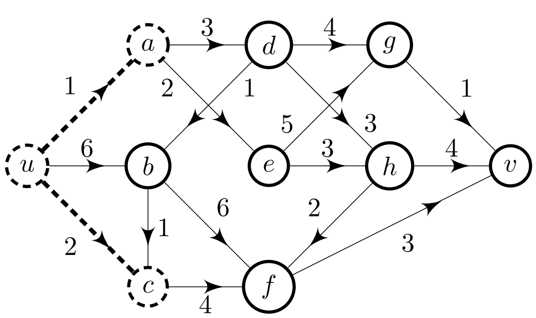
Here we will do a complete application of Djikstra's algorithm to a digraph.



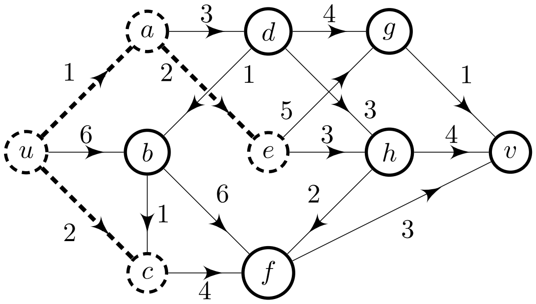
$y_u = 0$	$y_e = 0$	$A = \{\}$	$S = \{u\}$
$y_a = 0$	$y_f = 0$	$\epsilon_{ua} = 1$	
$y_b = 0$	$y_g = 0$	$\epsilon_{ub} = 6$	
$y_c = 0$	$y_h = 0$	$\epsilon_{uc} = 2$	
$y_d = 0$	$y_v = 0$		



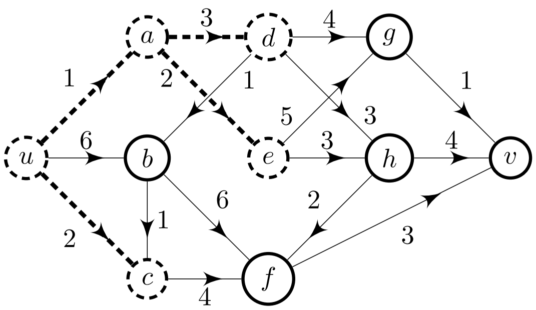
$y_u = 0$	$y_e = 1$	$A = \{ua\}$	$S = \{u, a\}$
$y_a = 1$	$y_f = 1$	$\epsilon_{ub} = 5$	$\epsilon_{ad} = 3$
$y_b = 1$	$y_g = 1$	$\epsilon_{uc} = 1$	$\epsilon_{ae} = 2$
$y_c = 1$	$y_h = 1$		
$y_d = 1$	$y_v = 1$		



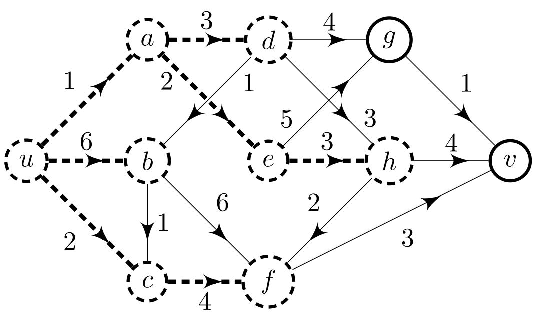
$y_u = 0$	$y_e = 2$	$A = \{ua, uc\}$	$S = \{u, a, c\}$
$y_a = 1$	$y_f = 2$	$\epsilon_{ub} = 4$	$\epsilon_{ad} = 2$
$y_b = 2$	$y_g = 2$	$\epsilon_{cf} = 4$	$\epsilon_{ae} = 1$
$y_c = 2$	$y_h = 2$		
$y_d = 2$	$y_v = 2$		



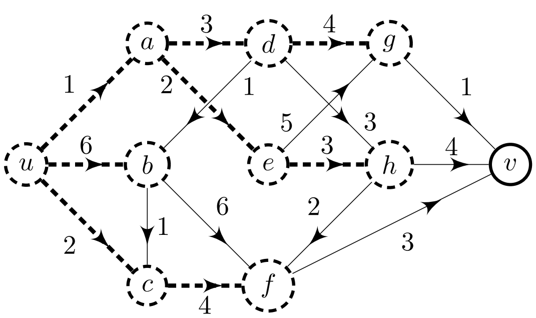
$y_u = 0$	$y_e = 3$	$A = \{ua, uc, ae\}$	$S = \{u, a, c, e\}$
$y_a = 1$	$y_f = 3$	$\epsilon_{ub} = 3$	$\epsilon_{ad} = 1$
$y_b = 3$	$y_g = 3$	$\epsilon_{cf} = 3$	$\epsilon_{eg} = 5$
$y_c = 2$	$y_h = 3$	$\epsilon_{eh} = 3$	
$y_d = 3$	$y_v = 3$		



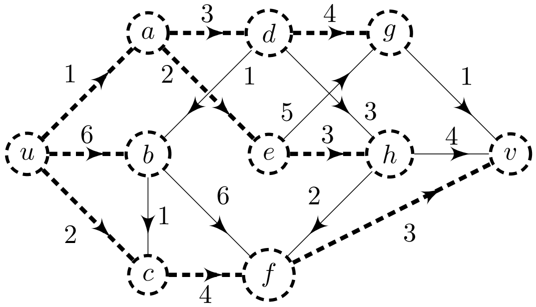
$y_u = 0$	$y_e = 3$	$A = \{ua, uc, ae, ad\}$	$S = \{u, a, c, e, d\}$
$y_a = 1$	$y_f = 4$		
$y_b = 4$	$y_g = 4$	$\epsilon_{ub} = 2$	$\epsilon_{dg} = 4$
$y_c = 2$	$y_h = 4$	$\epsilon_{cf} = 2$	$\epsilon_{dh} = 3$
$y_d = 4$	$y_v = 4$	$\epsilon_{eh} = 2$	$\epsilon_{eg} = 4$



$y_u = 0$	$y_e = 3$	$A = \{ua, uc, ae, ad, ub, cf, eh\}$	$S = \{u, a, c, e, d, b, f, h\}$
$y_a = 1$	$y_f = 6$		
$y_b = 6$	$y_g = 6$	$\epsilon_{fv} = 3$	$\epsilon_{eg} = 3$
$y_c = 2$	$y_h = 6$	$\epsilon_{hv} = 4$	$\epsilon_{dg} = 2$
$y_d = 4$	$y_v = 6$		



$y_u = 0$	$y_e = 3$	$A = \{ua, uc, ae, ad, ub, cf, eh, dg\}$	$S = \{u, a, c, e, d, b, f, h, g\}$
$y_a = 1$	$y_f = 6$		
$y_b = 6$	$y_g = 8$	$\epsilon_{fv} = 1$	$\epsilon_{gv} = 1$
$y_c = 2$	$y_h = 6$	$\epsilon_{hv} = 2$	
$y_d = 4$	$y_v = 8$		



$y_u = 0$	$y_e = 3$	$A = \{ua, uc, ae,$	$S = \{u, a, c, e,$
$y_a = 1$	$y_f = 6$	$ad, ub, cf, eh, dg,$	$d, b, f, h, g, v\}$
$y_b = 6$	$y_g = 8$	$fv\}$	
$y_c = 2$	$y_h = 6$		
$y_d = 4$	$y_v = 9$		

In the last iteration, we could have just as well added gv instead of fv. This means that there are two shortest uv-dipaths.

Theorem 2.2.2. At each iteration, the current potentials y_a are feasible. When an arc ab is added to A, the new potential at B is the length of the shortest ub-dipath.

Proof: Since $\ell(ab) \geqslant 0$ for all arcs ab, and $y_1 = y_b = 0$, we have that $0 + \ell(ab) \geqslant 0$.
Therefore the initial potentials are feasible.
At a given iteration, assume inductively that the current potentials are feasible.
We will prove that the updated potentials are also feasible.

Case 1: $a, b \in N \setminus S$
Then $y_a = y_b$.
Thus if ab is an arc with $a, b \in N \setminus S$, $y_a = y_b$ both before and after the update.
Then $y_a + \ell(ab) \geqslant y_b = y_a$ both before and after the update.

Case 2: $a, b \in S$
Then y_a, y_b are not changed by the update.
As the arc ab was feasible before the update, it is feasible after the update.

Case 3: $a \in S, b \in N \setminus S$

2.3 Bellman-Ford algorithm

Algorithm 2.3.1. [Bellman, Ford]
Let $y_u = 0$ and $y_a = \infty$ for all $a \in N \setminus \{u\}$. All p_a are undefined for $a \in N$.
Repeat $|N| - 1$ times:
Check each arc ab for feasibility.
If $y_b - y_a = \ell(ab)$, do nothing
If $y_b - y_a > \ell(ab)$, let $y_b = y_a + \ell(ab)$ and set p_b to be the length of the path to b

The outcome will be that if all the y_as are feasible, we have all shortest paths. If not all y_as are feasible, then we have a negative cycle.

3 Flows

3.1 Ford-Fullkerson algorithm

Algorithm 3.1.1. [Ford, Fullkerson]
Let x be an st-flow on a capacitated digraph D with arc ab having capacity x_{ab}.