Compact course notes

Combinatorics and Optimization 442/642, Fall 2012

Graph Theory

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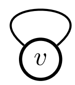
Contents

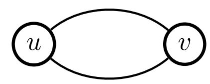
0.1 Foundations

All graphs in this course will be finite. Graphs may have multiple edges and loops:

a simple loop



multiple / parallel edges



Definition 0.1.1. A graph is a triple (V, E, φ) where V, E are finite sets and $\varphi : V \times E \to \{0, 1, 2\}$ is a function such that

$$\sum_{v \in V} \varphi(v, e) = 2 \;\; \forall \, e \in E$$

The function φ will be omitted when its action is clear. When the vertex set of G is not clear, it is given by $V(G)$, and similarly the edge set of G is given by $E(G)$.

A simple graph is a graph without loops and multiple edges. The simplification of a graph G is the graph that results in deleting the least number of edges from G such that the resulting graph has no loops or parallel edges.

Definition 0.1.2. Let $G = (V, E)$ be a graph. For $X_1 \subset V$ and $X_2 \subset E$, let $G[X_1]$ be the vertex set X_1 and the set of edges in E with both ends in X_1. Similarly, let $G[X_2]$ be the edge set X_2 and the set of vertices in V that are an end of an edge in X_2.

1 Graph minors

1.1 Contraction

Definition 1.1.1. Let $e = (u, v)$ be an edge of a graph $G = (V, E)$. The deletion of e is the removal of e from the graph, the resulting graph denoted by $G \setminus e$, with

$$G \setminus e = (V, E \setminus \{e\}, \varphi')$$

$$\varphi'(w, f) = \begin{cases} \varphi(w, f) & \text{if } w \notin \{u, v\} \\ \varphi(w, f) - 1 & \text{if } u \neq v, w \in \{u, v\} \\ \varphi(w, f) - 2 & \text{if } u = v, w = u \end{cases}$$

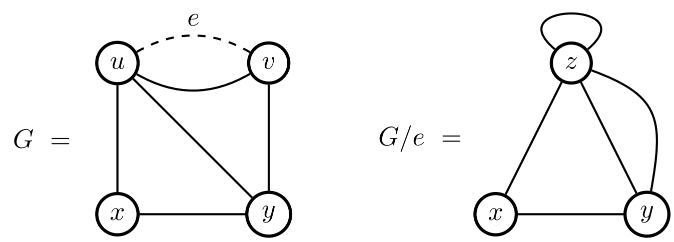
Definition 1.1.2. Let $e = (u, v)$ be an edge of (G, E, φ). The contraction of e is the identification of u and v and the deletion of e, the resulting graph denoted by G/e. If $u = v$, then $G/e = G \setminus e$. If $u \neq v$, then the resulting graph is defined as

$$G/e \;\; = \;\; (V \setminus \{u, v\} \cup \{z\}, E \setminus \{e\}, \varphi') \quad \text{for } z \notin V$$

$$\varphi'(w, f) \;\; = \;\; \begin{cases} \varphi(w, f) & \text{if } w \neq z \\ \varphi(u, f) + \varphi(v, f) & \text{if } w = z \end{cases}$$

For $X = \{e_1, \ldots, e_n\} \subset E$, let $G \setminus X = (\cdots ((G \setminus e_1) \setminus e_2) \cdots \setminus e_n)$.

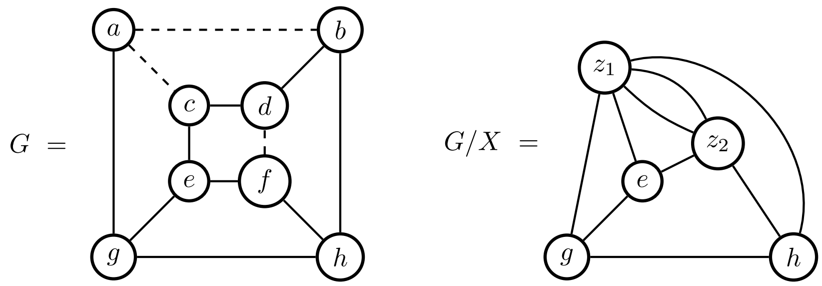
Example 1.1.3. This is a simple contraction:



Definition 1.1.4. A graph H is termed a minor of a graph G if H is obtained from a subgraph of G by contracting some (possibly none) edges.

A graph G has an H-minor K if K is a minor of G that is isomorphic to H.

Remark 1.1.5.
1. The number of components of G is equal to $|V(G/E)|$.
2. For $X \subseteq E(G)$, there is a bijection between the components of the subgraph of $G[V, X]$ and the vertices of G/X. For example, consider as below $X = \{(a, c), (a, b), (d, f)\}$:



Lemma 1.1.6. If $H = (V', E', \varphi')$ is a minor of $G = (V, E, \varphi)$, then $E' \subseteq E$, and there exist vertex-disjoint trees $T_v : v \in V'$ in G such that for each $e \in E'$ and $v \in V'$, we have

$$\varphi'(v, e) = \sum_{u \in V(T_v)} \varphi(u, e)$$

The converse holds up to relabeling of the vertices of H. This gives a useful way to gain intuition for minors.

Definition 1.1.7. A class of graphs $\mathcal{G}$ is termed minor closed if for any $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$.

Example 1.1.8. Some examples of minor closed classes are:
· Planar graphs
· Forests
· Apex graphs
· Graphs that embed in a closed connected 3-dimensional manifold without boundary without crossings
· For $k \in \mathbb{Z}_+$, the graphs with no path of length k
· For $k \in \mathbb{Z}_+$, the graphs with no cycle of length $\geqslant k$
· For $k \in \mathbb{Z}_+$, the graphs that do not have k vertex-disjoint cycles
· Knotless graphs (graphs that embed in $\mathbb{R}^3$ such that each cycle is embedded as the unknot)

Definition 1.1.9. A graph (V, E) is termed an apex graph if there is $v \in V$ such that $(V \setminus \{v\}, E \setminus E(v))$ is a planar graph.

Definition 1.1.10. Given a graph G with a cycle C, a chord of the cycle C is an edge $e \in E(G)$ such that $e \notin C$, but the ends of e are vertices of C.

Definition 1.1.11. Given a graph $G = (V, E)$ with $X \subset V$, the subgraph of G induced by X is the graph $H = (X, F)$ where $F \subset E$ and $e = (x, y) \in F$ iff $x, y \in X$. Such a graph is denoted by $H = G[X]$.

1.2 Excluded minors

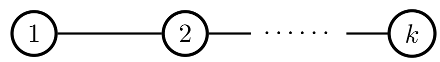
Definition 1.2.1. A graph G is termed an excluded minor for a minor-closed class $\mathcal{G}$ of graphs if $G \notin \mathcal{G}$, but each proper minor of G is contained in $\mathcal{G}$.

Kuratowski's theorem states that the excluded minors of the class of planar graphs are K_5 and $K_{3,3}$.

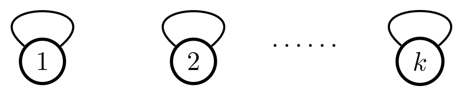
Remark 1.2.2. Kuratowski's theorem is equivalent to: A graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.

Remark 1.2.3.

1. The only excluded minor for the class of graphs with no path of length $\geqslant k$ is



2. The only excluded minor for the class of graphs without k vertex-disjoint cycles is



3. There are 32 excluded minors for the closed projective planar graphs (graphs that can be emebedded in planar form in the projective plane, or equivalently, the Möbis band).

4. The set of excluded minors for the set of apex graphs has at least 100 elements.

Theorem 1.2.4. [Graph minor theorem - Robertson, Seymour 1985-2012]
Each minor closed class of graphs has finitely many excluded minors.

Remark 1.2.5. The above theorem may be equivalently stated as:

1. Each infinite set of graphs has two graphs, one of which is isomorphic to a minor of the other.
2. There are countably many minor closed classes.
3. For each minor closed class, the membership testing problem is decidable.

Although there is no decidable algorithm known for checking whether or not a graph is knotless, the last statement guarantees the existence of such an algorithm.

Theorem 1.2.6. [Robertson, Seymour]
There is an $\mathcal{O}(n^3)$ running-time algorithm that, given an n-vertex graph H, tests an input graph G for an H-minor.

Corollary 1.2.7. For each minor closed class of graphs, there is an $\mathcal{O}(n^3)$ running-time algorithm for testing membership.

1.3 Edge density in minor closed classes

Observe that

1. $|E(K_n)| = \binom{n}{2}$ *(quadratic)*
2. If $G = (V, E)$ is a simple planar graph with $|V| \geqslant 3$, then $|E| \leqslant 3|V| - 6$. *(linear)*

Theorem 1.3.1. [Mader 1967]
If $G = (V, E)$ is a simple graph with no K_m-minor, then $|E| \leqslant (2^m - 1)|V|$.

Proof: Consider a counterexample G with $|V(G)|$ minimal. Then G is simple, has no K_m minor, but $|E(G)| > (2^m - 1)|V(G)|$. Let v be a vertex with degree at least 1, and let H be the graph induced by the neighbors of v (i.e. all w such that $(v,w) \in E$). Note that H is simple, has no K_{m-1} minor, and $|V(H)| < |V(G)|$. By our choice of G, $|E(H)| \leqslant (2^{m-1}-1)|V(H)|$. Recall that

$$2|E(H)| = \sum_{u \in V(H)} \mathsf{deg}_H(u)$$

So H has a vertex w of degree

$$\mathsf{deg}_H(w) \leqslant \frac{2|E(H)|}{|V(H)|} \leqslant 2^m - 2$$

Let $e = (v,w)$ and let G' be the simplification of G/e. Then $|V(G')| = |V(G)| - 1$, and

$$\begin{aligned} |E(G')| &= |E(G)| - 1 - \mathsf{deg}_H(w) \\ &> (2^m-1)|V(G)| - 1 - (2^m-2) \\ &= (2^m-1)|V(G)| - (2^m-1) \\ &= (2^m-1)(|V(G)|-1) \\ &= (2^m-1)|V(G')| \end{aligned}$$

However, G' is simple, $|V(G')| < |V(G)|$ and G' has no K_m minor. This contradicts our choice of G, as we have found a smaller graph. ■

Corollary 1.3.2. For any proper minor closed class $\mathcal{G}$ of graphs, there exists $c_{\mathcal{G}} \in \mathbb{R}$ such that each simple graph $G \in \mathcal{G}$ satisfies $|E(G)| \leqslant c_{\mathcal{G}}|V(G)|$.

Remark 1.3.3. Let $h_t(n)$ be the maximum number of edges in a simple n-vertex graph with no K_t-minor. Then:

1. $h_t(n) \geqslant (t-2)n - \binom{t-1}{2}$ for all $n \geqslant t \geqslant 2$
2. Equality holds for $t \in \{2, \ldots, 7\}$, *(proved by Mader)*
3. $h_t(n) = (\alpha + \mathcal{O}(1))t\sqrt{t}n$ for $\alpha \approx 0.319\ldots$, *(proved by Thomassen)*

Remark 1.3.4. A simple graph with no K_3-minor is equivalent to a forest. Also, then $|E| \leqslant |V| - 1$.

Theorem 1.3.5. Let $G = (V,E)$ be a simple graph with $|V| \geqslant 2$ and no K_4-minor. Then $|E| \leqslant 2|V| - 3$.

Proof: Consider a counterexample G with $|V(G)|$ minimal. We may assume that $|V| \geqslant 3$ (as for 2 the cases are simple). Let v be a vertex with degree $\geqslant 1$, and let H be the subgraph induced by the neighbors of v, so H is simple and has no K_3-minor. Thus H is a forest and has a vertex w with $\mathsf{deg}_H(w) = 1$. Let $e = (v,w)$ and let G' be the simplification of G/e. Note that $|V(G')| = |V(G)| - 1$ and $|E(G')| \leqslant |E(G)| - 1$, giving a smaller counterexample. ■

Theorem 1.3.6. Let $G = (V,E)$ be a simple graph with no K_5 minor and with $|V| \geqslant 3$. Then $|E| \leqslant 3|V| - 6$.

1.4 Coloring

Definition 1.4.1. A graph G is termed k-colorable iff its vertices can be colored with k colors so that no edge has both of its ends in the same color class.

Theorem 1.4.2. [Four color theorem]
Loopless planar graphs are 4-colorable.

Theorem 1.4.3. [Wagner]
Loopless planar graphs with no K_5 minor are 4-colorable.

Conjecture 1.4.4. [Hadwiger]
Loopless graphs with no K_{n+1} minor are n-colorable.

For all $n \geqslant 4$, this implies the four color theorem.

Lemma 1.4.5. If $G = (V, E)$ is a simple graph with no K_4-minor and $V \neq \emptyset$, G has a vertex of degree $\leqslant 2$.

Theorem 1.4.6. If G is a loopless graph with no K_4-minor, then G is 3-colorable.

Proof: Consider a counterexample with $|G| + |V|$ minimal. Then G is simple, has no K_4-minor, and has $|V| \geqslant 4$. By the lemma, G has a vertex v of degree $\leqslant 2$, and by the choice of example G, $G - v$ has a 3-coloring. Since $\mathsf{deg}(V) \leqslant 2$, this extends to a 3-coloring of G. ■

Theorem 1.4.7. [Wagner]
If G is a loopless graph with no K_n-minor, then G is $(2^{n+1} - 1)$-colorable.

Proof: Consider a counterexample with G with $|V(G)|$ minimal, so G has no K_n-minor but is not $(2^{n-1}+1)$-colorable. We may assume that the graph is simple. By Mader, $|E(G)| \leqslant (2^n - 1)|V(G)|$, so G has a vertex v with degree

$$\mathsf{deg}(v) \leqslant \frac{2|E(G)|}{|V(G)|} \leqslant 2(2^n - 1) < 2^{n+1} - 1$$

Since $|V(G - v)| < |V(G)|$, $G - v$ has a $(2^{n+1} - 1)$-coloring. Since $\mathsf{deg}(v) < 2^{n+1} - 1$, this extends to a $(2^{n+1} - 1)$-coloring of G, a contradiction. ■

1.5 Constructing minor closed classes

Given a minor-closed class of graphs $\mathcal{G}$, is it possible to give a constructive characterization of the graphs in $\mathcal{G}$?

Definition 1.5.1. A graph is termed series-parallel if it may be obtained from the empty graph by :
· adding isolated vertices
· adding any edge incident with an isolated vertex
· subdividing an edge
· adding loops or parallel edges

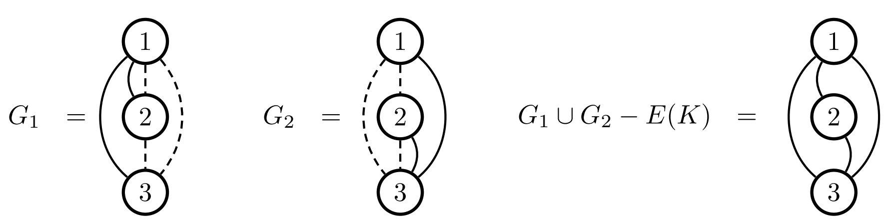
Remark 1.5.2.
1. The class of series-parallel graphs is minor closed.
2. K_4 is not series-parallel.
3. Any excluded minor for the class of series-parallel graphs is simple and has minimum degree $\geqslant 3$.
4. Any excluded minor for the class of series-parallel graphs contains a K_4-minor.

Theorem 1.5.3. A graph is series-parallel iff it does not contain a K_4-minor.

Definition 1.5.4. Given a graph G, a clique in G is a complete subgraph of G.

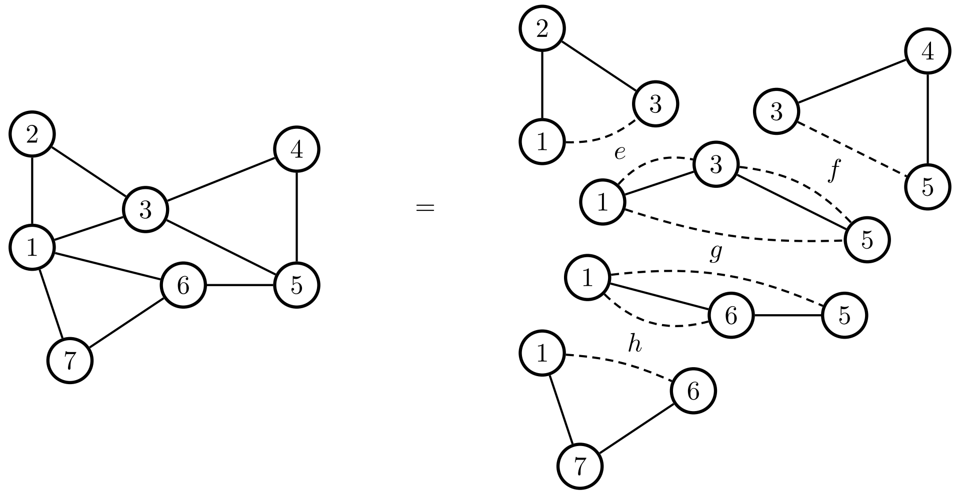
Suppose that K is a clique in G_1 and in G_2 with $|V(K)| = k$. If $V(G_1) \cap V(G_2) = V(K)$ and $E(G_1) \cap E(G_1) = E(K)$, then $(G_1 \cup G_2) - E(K)$ is termed the clique-sum, or k-sum, of G_1 and G_2.

Example 1.5.5. Consider the following graphs and their clique-sum for $K = K_3$.



Proposition 1.5.6. Every graph can be obtained from isomorphic copies of its 3-connected minors, by 0-, 1-, and 2-sums.

Example 1.5.7. Consider the following example of the previous proposition, only 2-sums being applied here:



Remark 1.5.8. Every 3-connected graph with at least 4 vertices has a K_4-minor, as its simplification has minimum degree $\geqslant 3$.

Theorem 1.5.9. A graph has no K_4-minor if and only if it can be obtained from graphs with $\leqslant 3$ vertices by ($\leqslant 2$)-sums.

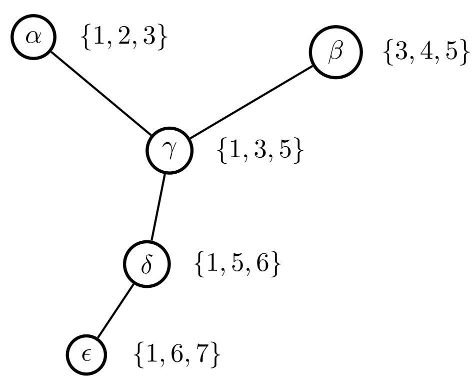
1.6 Tree decomposition

Definition 1.6.1. A tree decomposition of a graph G consists of a tree T and a collection B_v for $v \in V(T)$ called bags such that

1. for each $e = (u, v) \in E(G)$, there exists $w \in V(T)$ with $u, v \in B_w$
2. for each $v \in V(G)$, the set $\{w \in V(T) \mid v \in B_w\}$ induces a tree in T

Equivalently, we may say that G is the clique-sum of some graphs H_w with $w \in V(T)$ where $V(H_w) = B_w$.

Example 1.6.2. The tree decomposition for the graph given in the previous example would be as below, with vertices that correspond to the original graph indicated next to the vertices of the tree.



Definition 1.6.3. The width of a tree is defined by

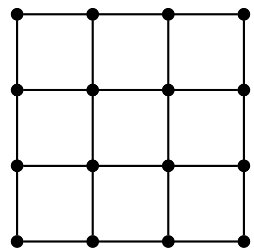
$$\text{width}(T, (B_w \mid w \in V(T))) = \max_{w \in V(T)} \{|B_w| - 1\}$$

The tree-width of a graph G is the minimum $k \in \mathbb{N}$ such that G has a tree decomposition of width k.

Remark 1.6.4.
1. Trees have tree-width $\leqslant 1$
2. A graph has no K_4-minor iff it has tree width $\leqslant 2$
3. If H is a minor of G, then tree-width$(H) \leqslant$ tree-width(G)

Theorem 1.6.5. [Grid theorem - Robertson, Seymour]
Given H planar, there exists $k \in \mathbb{Z}$ such that if G is a graph with no H-minor, then tree-width$(G) \leqslant k$.

Example 1.6.6. This is a 4×4 grid.



Remark 1.6.7.
1. The $k \times k$ grid is planar
2. For any planar graph G, there exists $k \in \mathbb{N}$ such that G is a minor of a $k \times k$ grid
3. tree-width$(k \times k$ grid$) = k$

Note that by the second remark, it suffices to prove the grid theorem for grids.

Theorem 1.6.8. [Qualitative structure theorem]
For $n \gg k$, the following inclusions of sets holds:

$$\{\text{graphs of tree-width} \leqslant k-1\} \subset \{\text{graphs with no } k \times k \text{ grid minor}\} \subset \{\text{graphs of tree-width} \leqslant n\}$$

Remark 1.6.9. Let $\mathcal{C}_k$ be the set of graphs without k vertex-disjoint cycles (recall that $\mathcal{C}_k$ is minor closed). By the grid theorem, for every $k \in \mathbb{N}$, there is some $t_k \in \mathbb{N}$ such that every graph in $\mathcal{C}_k$ has tree-width $\leqslant t_k$.

Definition 1.6.10. A hitting set for a graph G is a set $X \subset V(G)$ such that $G - X$ is a forest. That is, X meets every cycle in G.

Note that $\{$graphs with a hitting set of size $k-1\} \subset \mathcal{C}_k$.

Lemma 1.6.11. [Helly property of trees]
Let $\mathcal{F}$ be a collection of subtrees of a tree T and let $k \in \mathbb{N}$. Then either
1. there are k vertex-disjoint trees in $\mathcal{F}$, or
2. there is a set $X \subset V(T)$ with $|X| \leqslant k-1$ such that each tree in $\mathcal{F}$ contains a vertex of X

Theorem 1.6.12. [Erdos, Posa]
For each $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for each graph G, exactly one of the following hold:
1. G has k vertex-disjoint cycles
2. G has a hitting set of size $\leqslant \ell$

Proof: Consider $G \in \mathcal{C}_k$. The graph G has tree decomposition $(T, (B_w \mid w \in V(T)))$ of width $\leqslant t_k$. For each cycle C of G, define

$$S_C = \{w \in V(T) \mid B_w \cap V(C) \neq \emptyset\}$$

Note that S_C induces a subtree of T, so let $\mathcal{F}$ be the collection of all such subtrees. Since $G \in \mathcal{C}_k$ and by the definition of $\mathcal{F}$, there do not exist k vertex-disjoint subtrees in $\mathcal{F}$. By the lemma, there is a set X of at most $k-1$ vertices in T that meet each tree in $\mathcal{F}$. Now $\bigcup_{w \in X} B_w$ is a hitting set in G and has size $\leqslant (k-1)(t_k+1)$. Therefore the result holds with $\ell = (k-1)(t_k+1)$. ■

Remark 1.6.13. If $\mathcal{H}$ is the set of excluded minors for a minor-closed class $\mathcal{G}$ of graphs, then

1. By the grid theorem, if $\mathcal{H}$ contains a planar graph, then there exists $t \in \mathbb{N}$ such that each graph in $\mathcal{G}$ has tree-width $\leqslant t$.

2. If $\mathcal{H}$ contains no planar graph, then $\mathcal{G}$ contains all planar graphs, and hence the class $\mathcal{G}$ has unbounded tree-width.

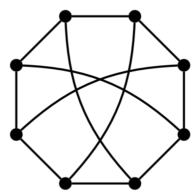
The second remark may be generalized: a minor-closed class $\mathcal{G}$ has unbounded tree-width iff $\mathcal{G}$ contains all planar graphs.

Theorem 1.6.14. [WAGNER]
Every graph with no K_5-minor may be obtained by ($\leqslant 3$)-sums from:

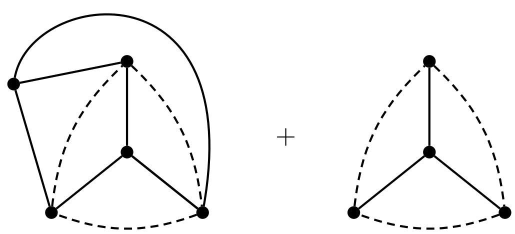
1. planar graphs, and
2. copies of V_8

where V_8 is the non-planar graph given by

$V_8 =$



For example, below is $K_{3,3}$ constructed by a 3-sum of planar graphs:



$= K_{3,3}$

Definition 1.6.15. Let $\mathcal{G}_k$ be the class of graphs that do not contain k vertex-disjoint non-planar subgraphs. Then $\mathcal{G}_k$ is minor-closed.

Let Φ_k be the orientable surface of genus k.
Let Ψ_k be the non-orientable surface of genus k.

Then $\mathcal{G}_k$ contains all graphs that embed in Φ_{k-1} or Ψ_{k-1}.

Theorem 1.6.16. [ROBERTSON, SEYMOUR]
For all $z \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that for each graph G, either

1. G contains k vertex-disjoint non-planar subgraphs, or
2. there exists $X \subset V(G)$ with $|X| \leqslant t_k$ such that $G - X$ embeds in Φ_{k-1} or Ψ_{k-1}

Remark 1.6.17. The graph minor structure theorem gives a structural description of graphs with no K_t-minor.

2 Coloring

In this section, all discussed graphs will be simple.

Definition 2.0.1. Given a graph G, the chromatic number of G is the least $k \in N$ such that G is k-colorable. This number is denoted by $\chi(G)$.

2.1 Critical graphs

Definition 2.1.1. A graph G is termed k-critical iff $\chi(H) < \chi(G) = k$ for all proper subgraphs H of G.

Remark 2.1.2.

1. A graph is $(k-1)$-colorable iff it has no k-critical subgraphs
2. A graph is 3-critical iff it is an odd cycle

Proposition 2.1.3. If a graph G is k-critical, then each vertex of G has degree $\geqslant k-1$.

Proof: Consider $v \in V(G)$. Since G is k-critical, $G - v$ has a $(k-1)$-coloring. If $\mathsf{deg}(v) \leqslant k-2$, then a $(k-1)$-coloring of $G - v$ extends to a $(k-1)$-coloring of G. This contradicts the fact that G is k-critical. ■

Definition 2.1.4. A graph G is termed d-degenerate iff every non-empty subgraph of G has a vertex of degree $\leqslant d$.

Example 2.1.5. These are some examples of d-degenerate classes.

· forests are 1-degenerate
· simple planar graphs are 5-degenerate
· simple planar graphs with no K_4-minor are 2-degenerate

Proposition 2.1.6. Every d-degenerate graph is $(d+1)$-colorable.

Proof: By the above proposition, no d-degenerate graph has a $(d+2)$-critical subgraph (else every vertex would have degree $\geqslant d+1$). Then by the previous remark, such graphs are $(d+2-1) = (d+1)$-colorable. ■

2.2 Graphs on orientable surfaces

Lemma 2.2.1. Let G be a simple graph that embeds in Φ_{k-1}. Then $|E(G)| \leqslant 3|V(G)| + 6(k-1)$.

Lemma 2.2.2. Let G be an 8-critical graph that embeds in Φ_k. Then $|V(G)| \leqslant 12(k-1)$.

Proof: Each vertex of G has degree $\geqslant 7$, so

$$\tfrac{7}{2}|V(G)| \leqslant |E(G)| \leqslant 3|V(G)| + 6(k-1)$$

This directly implies that $|V(G)| \leqslant 12(k-1)$, completing the proof. ■

Remark 2.2.3.

1. There are finitely many 8-critical graphs that embed in Φ_k
2. There are finitely many 6-critical graphs that embed in Φ_k, *(Thomassen)*

Theorem 2.2.4. For each $k \in \mathbb{N}$, there is a polynomial-time algorithm for testing whether a given Φ_k-embeddable graph is 7-colorable.

Proof: Let G be a graph that embeds in Φ_k, so it has at most $12(k-1)$ vertices. Then G is 7-colorable iff every one of its at most $12(k-1)$ vertex-neighborhood subgraphs is 6-colorable. There is a finite number of subgraphs to check. ■

Remark 2.2.5.

1. Thomassen's idea allows us to replace in the above theorem 7 with 5.
2. 3-coloring planar (or on any other surface) graphs is an NP-hard problem.
3. 4-coloring graphs on arbitrary surfaces is an open problem.

Theorem 2.2.6. Let G be a simple graph that embeds in Φ_k. Then either:

1. G is 7-colorable, or
2. G has a non-contractible cycle of length $\leqslant 12(k-1)$

Proof: If G is not 7-colorable, then it has an 8-critical subgraph H. Note that H is not planar with $|V(H)| \leqslant 12(k-1)$, and so H has a non-contractible cycle of length $\leqslant 12(k-1)$. ■

Remark 2.2.7.

1. Thomassen's idea gives a similar result with 5 instead of 7.
2. A similar result holds for Ψ_k.

2.3 Clique cutsets

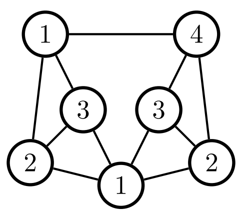
Definition 2.3.1. Let G be a graph, and a clique a set of pairwise adjacent vertices. A set $X \subset V(G)$ is termed a clique cutset if X is a clique and $G - X$ is not connected.

The pair (G_1, G_2) is termed a separation of G if G_1, G_2 are subgraphs of G and $G = G_1 \cup G_2$. The order of the separation is $|V(G_1) \cap V(G_2)|$. The separation is proper if $V(G_1) - V(G_2) \neq \emptyset$ and $V(G_2) - V(G_1) \neq \emptyset$.

Lemma 2.3.2. If G is a k-critical graph, then G has no clique cutset.

Proof: Suppose that (G_1, G_2) is a proper separation in a graph G and $G_1 \cap G_2$ is complete. Note that G_1 has a $\chi(G_1)$-coloring and G_2 has a $\chi(G_2)$-coloring. Since $G_1 \cap G_2$ is complete, we may assume that these colorings agree on $V(G_1) \cap V(G_2)$. Hence $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$, so G is not k-critical. ■

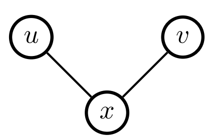
The above implies that k-critical graphs are 2-connected. Recall that a graph is k-connected iff there exists no $X \subset V(G)$ of size $k-1$ such that $G - X$ is disconnected. Consider, for example, the following graph that is 4-critical and 2-connected, whose vertex labels denote colors.



Definition 2.3.3. A chord in a cycle is an edge not in the cycle, but with both ends in the cycle. A hole in a graph is a chordless cycle of length $\geqslant 4$.

Lemma 2.3.4. If G is a graph with no clique cutset, then either G is complete or has a hole.

Proof: Suppose that G is not complete and has no clique cutset, so G is connected. Since G is connected and not complete, there exist distinct vertices $x, u, v \in V(G)$ such that $(x, u), (x, v) \in E(G)$, but $(u, v) \notin E(G)$.



Choose distinct vertices $u, v \in G$ and a non-empty clique X in G such that

1. u and v are complete to X (i.e. $X \subset N(u) \cap N(v)$)
2. $(u, v) \notin E(G)$
3. subject to the first two conditions, take $|X|$ maximal

Now X is not a clique cutset, so there is a shortest chordless uv-path P in $G - X$.
We may assume that each vertex on P is complete to X, as otherwise we can find a hole.
For $P = (v_0, v_1, \ldots, v_t)$ the choice of (u, v, X) is contradicted by $(v_0, v_2, X \cup \{v_1\})$. ■

Remark 2.3.5.

1. Holes do not have clique cutsets.

2. The proof above was constructive.

Definition 2.3.6. Let H be an induced subgraph of G. Let u, v be distinct non-adjacent vertices in H. Let P be an induced uv-path in G with $V(P) \cap V(H) = \{u, v\}$. Then $G[V(H) \cup V(P)]$ is obtained from H by adding an induced ear.

Remark 2.3.7.

1. If H has no clique cutset and H' is obtained from H by adding an induced ear, then H' has no clique cutset.

2. If G is not complete and G has no clique cutset, then G may be obtained from a hole by successively adding induced ears.

Algorithm 2.3.8. [Greedy algorithm]
Input: A graph $G = (V, E)$ and a set of colors $(c_1, \ldots, c_k)$
Output: A coloring $((v_1, c_{v_1}), (v_2, c_{v_2}), \ldots, (v_n, c_{v_n}))$ of the vertices of G

· Order the vertices of G as $(v_1, \ldots, v_n)$.
· Color $v_1, \ldots, v_n$ in that order, assigning each vertex the first available color.

Note that for $\Delta(G)$ the maximum degree of a vertex in G, the greedy algorithm uses at most $\Delta(G) + 1$ colors. Hence $\chi(G) \leqslant \Delta(G) + 1$. Moreover, the only graphs with $\chi(G) = \Delta(G) + 1$ are complete graphs and odd cycles (this is Brooks' theorem, discussed below).

Lemma 2.3.9. If u, v, w are distinct vertices in a simple graph G with $(u, w), (v, w) \in E(G)$ but $(u, v) \notin E(G)$, and $G - u - v$ is connected, then $\chi(G) \leqslant \Delta(G)$.

Proof: Take an ordering $(v_1, v_2, \ldots, v_n)$ of $V(G)$ with $v_1 = u, v_2 = v, v_n = w$, and so that for each $1 \leqslant i \leqslant n - 1$, v_i has a neighbor v_j with $j > i$. Note that such an ordering exists, as $G - u - v$ is connected. Color the vertices using the greedy algorithm. Now u, v get the same color, so the algorithm only needs $\leqslant \Delta(G)$ colors. ∎

Lemma 2.3.10. If G is a simple graph with no clique cutset and G is neither a cycle nor complete, then G contains distinct vertices u, v, w such that $(u, w), (v, w) \in E(G)$ but $(u, v) \notin E(G)$ and $G - u - v$ is connected.

Proof: Since G is not complete and has no clique cutset, G has a hole C. Since G is not a cycle, $G \neq C$.

Case 1: There is a component H of $G - N(u)$ such that $N(u) \cap V(H) = \emptyset$.
By (possibly) changing our choice of u, it will be that:

1. $(u, w) \in E(C)$
2. $(w, v) \in E(G)$
3. $N(u) \cap V(H) = \emptyset$

Since G has no clique cutset, G is 2-connected and hence $G - v$ is connected. Now each component of $(G - C) - v$ has a neighbor in $N(u) - \{u\}$. Hence $G - u - v$ is connected.

Case 2: Not case 1.
Take u, v, w as 3 consecutive vertices of C. Then $G - u - v$ is connected. ∎

Theorem 2.3.11. [Brooks]
If G is a simple connected graph with $\chi(G) = \Delta(G) + 1$, then G is either an odd cycle or is complete.

Proof: Let G be a simple connected graph with $\chi(G) = \Delta(G) + 1$. We may assume that G is not a cycle and that G is not complete. Note that G has a $(\Delta(G)+1)$-critical subgraph H. Therefore H is not $\Delta(G)$-regular. Since G is connected, $G = H$. So then G is $(\Delta(G) + 1)$-critical, and hence G has no clique cutset. Then by lemmas 1 and 2 we have that $\chi(G) \leqslant \Delta(G)$, which is a contradiction. ∎

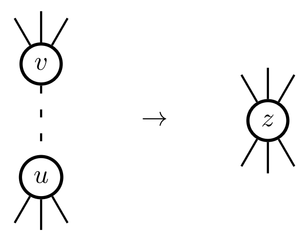
Definition 2.3.12. A graph G is termed k-regular if every vertex in $V(G)$ has degree k.

2.4 Building k-chromatic graphs

Remark 2.4.1.

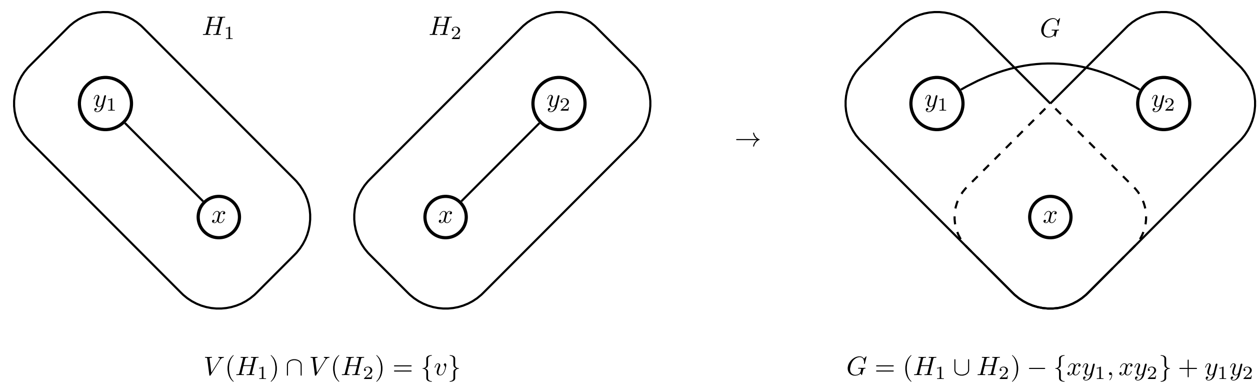
1. Testing $\chi(G) = 3$ is NP-complete.
2. It is conjectured that $NP \neq coNP$
3. Applying the first two remarks, it follows that there is no succinct proof that $\chi(G) \geqslant 4$.

Definition 2.4.2. Let $G = (V, E)$ be a graph, and $(u, v) = uv \notin E$. Then the vertex identification of u and v is the simplification of the graph $(G + uv)/uv = G \circ uv$.



Then coloring the graph on the left with u, v having the same color is equivalent to coloring the graph on the right, so $\chi(G \circ uv) \geqslant \chi(G)$.

Definition 2.4.3. Given two graphs H_1, H_2 with $V(H_1) \cap V(H_2) = \{v\}$, the Hajos construction of these two graphs results in the new graph $G = (H_1 \cup H_2) - \{xy_1, xy_2\} + y_1y_2$, for $xy_1 \in H_1$ and $xy_2 \in H_2$. Below is an illustration of such a construction.



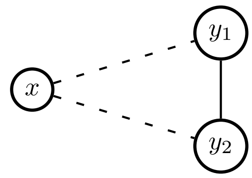
Proposition 2.4.4. If $\chi(H_1) \geqslant k$ and $\chi(H_2) \geqslant k$, then $\chi(G) \geqslant k$.

Proof: In any $(k-1)$-coloring of $H_1 - xy_1$, x and y_1 get the same color, WLOG. Similarly for $H_2 - xy_2$. Hence in any $(k-1)$-coloring of $(H_1 - xy_1) \cup (H_2 - xy_2)$, y_1 and y_2 get the same color. Hence G is not $(k-1)$-colorable. ■

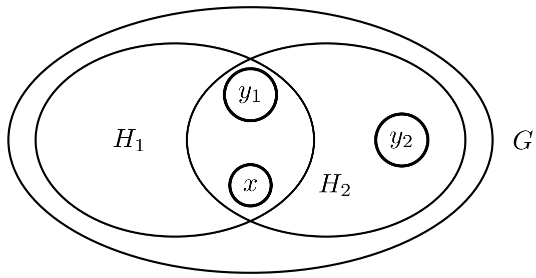
Definition 2.4.5. A graph G is termed k-constructable if it may be obtained from copies of K_k by a sequence of vertex identifications and Hajos constructions. Note that if G is k-constructable, then $\chi(G) \geqslant k$.

Theorem 2.4.6. [Hajos' amazing theorem]
A graph G has chromatic number $\geqslant k$ iff G has a k-constructable subgraph.

Proof: Let G be a simple counterexample with $|V(G)|$ minimal and $|E(G)|$ maximal subject to $|V|$ minimal. Thus $\chi(G) \geqslant k$ and G does not have a a K-constructible subgraph, so G is k-critical. Since G is not complete, it contains adjacent vertices x, y_1. Observe that the neighbor set of y_1 is not contained in the neighbor set of x, as G is k-critical. So there exists $y_2 \in N(y_1) - N(x)$, and we have found the following subgraph in G:



By our choice of G, $G + xy_1$ and $G + xy_2$ contain K-constructible subgraphs, say H_1, H_2. Note that $xy_1 \in E(H)$ and $xy_2 \in E(H_2)$.

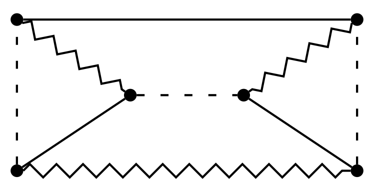


Now G contains a k-constructable subgraph. To find it, split the vertices and identify the two copies. ■

2.5 Edge colorings

Definition 2.5.1. An edge coloring of a graph G is a coloring of the edges of G so that no vertex has two incident edges with the same color.

Below is an edge coloring of a graph. Instead of colors we will use different graphical styles to represent different edge colors.

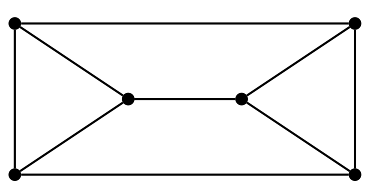


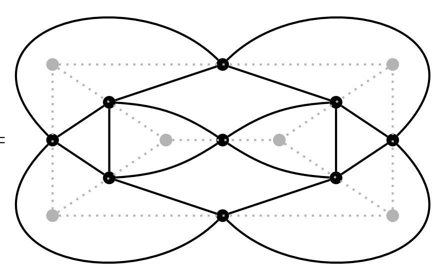
By convention, graphs with loops are not edge-colorable.

Definition 2.5.2. The edge chromatic number (or chromatic index) of G, denoted by $\chi'(G)$ is the minimum number of colors required for an edge coloring.

Definition 2.5.3. Given a graph G, the line graph of G, denoted by $L(G)$, is the graph induced by replacing edges of G with vertices and connecting two vertices with an edge for every vertex the edges they represent are both incident on.

Below is a graph G with its line graph $L(G)$ overlaid on G.

G = 

$L(G)$ = 

Remark 2.5.4.

1. $\chi'(G) \leqslant k$ iff $E(G)$ can be partitioned into k matchings.
2. $\chi'(G) = \chi(L(G))$
3. $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(L(G)) + 1 \leqslant 2(\Delta(G) - 1) + 1 = 2\Delta(G) - 1$

Theorem 2.5.5. If G is a bipartite graph, then $\chi'(G) = \Delta(G)$. In this case G need not be simple.

Lemma 2.5.6. Let v be a vertex of a simple graph G and let $k \in \mathbb{Z}$. If the following conditions hold:

i. v and its neighbors have degree $\leqslant k$
ii. at most 1 neighbor of v has degree $= k$
iii. $\chi'(G - v) \leqslant k$

Then $\chi'(G) = k$.

Proof: Let (v, G, k) be a counterexample with k minimal (note that $k > 0$). By possibly adding pendent (dangling) edges, we may assume that V has a neighbor w with degree k and all other neighbors have degree $k - 1$. Consider an edge coloring of $G - v$ with colors $1, 2, \ldots, k$. For each $i \in \{1, 2, \ldots, k\}$, let X_i be the set of neighbors of v that are not incident with an edge of color i. Note that $\sum_{i=1}^{k} |X_i| = 2\mathsf{deg}(v) - 1$, as w contributes 1 to the sum and all other neighbors contribute 2. We may assume that our coloring is chosen so as to minimize $\min\{|X_i| \mid |X_i| \text{ is odd}\}$. Let X_i be a set of odd cardinality attaining this minimum.

Case 1: $|X_i| = 1$

Suppose that $X_i = \{u\}$. Let M_i be the edges of $G - v$ with color i and let $G' = G - (M_i \cup \{uv\})$. Then $(v, G', k - 1)$ satisfies points i, ii, iii. By our choice of (v, G, k), $\chi'(G') = k - 1$ and $\chi'(G) = k$, contradicting the minimality of $|V(G)|$.

Case 2: $|X_i| \geqslant 3$

Since the average degree of the sets X_i is < 2 (by above), there is $j \in \{1, 2, \ldots, k\}$ with $|X_j| < 2$. By our choice of X_i, $|X_j| = 0$. Let M_i, M_j be the sets of edges with colors i, j respectively. Note that $G[M_i \cup M_j]$ consists of paths and cycles, with $z \in X_i$ at the end of one such path P. Define a new edge coloring of $G - v$ by swapping the colors i and j on P. Let X'_ℓ denote the set of neighbors of v that do not see an edge of color ℓ in the new coloring. If P has one end in X_i, then the other end of P is not a neighbor of v, and

$$|X'_\ell| = \begin{cases} 1 & \text{if } \ell = j \\ |X_\ell| - 1 & \text{if } \ell = i \\ |X_\ell| & \text{else} \end{cases}$$

In this case $|X'_j| = 1 < |X_i|$, which contradicts our choice of X_i. If P has both ends in X_i, then

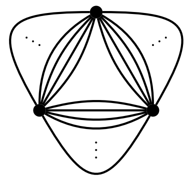
$$|X'_\ell| = \begin{cases} 2 & \text{if } \ell = j \\ |X_\ell| - 2 & \text{if } \ell = i \\ |X_\ell| & \text{else} \end{cases}$$

In this case $|X'_i| < |X_i|$ and $|X'_i|$ is odd, again contradicting our choice of X_i. Since the only two possible cases result in contradictions, there is no counterexample. ■

Theorem 2.5.7. [Vizing]
If G is a simple graph, then $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$.

Proof: Consider a counterexample G with $|V(G)|$ minimal. Let $v \in V(G)$ and $k = \Delta(G) + 1$. Then (v, G, k) satisfy points i, ii, iii of the lemma above, so $\chi'(G) \leqslant k$. This contradicts the minimality of $|V(G)|$. ■

Remark 2.5.8. In the case of Vizing's theorem, it is important for G to be simple.



This graph G of 3 vertices and k edges between each pair of vertices has:

· $\chi'(G) = 3k$
· $\Delta(G) = 2k$

2.6 Cut spaces and cycle spaces

Definition 2.6.1. Let G be a graph. For $X \subset V(G)$, let $\delta(X)$ be the set of edges (x, y) with $x \in X$ and $y \in V(G) - X$. Then $\delta(X)$ is termed a cut of G. If both $G[X]$ and $G[V(G) - X]$ are connected, then $\delta(X)$ is termed a bond of G.

Proposition 2.6.2. Let G be a graph. For $X, Y \subset V(G)$, $\delta(X) \triangle \delta(Y) = \delta(X \triangle Y)$, where $\triangle$ is the symmetric difference operator on sets.

Definition 2.6.3. Let $\mathcal{C}^*$ be the set of all cuts of a graph G, or

$$\mathcal{C}^* = \{\delta(X) \mid X \subset V(G)\}$$

This is termed the cut space of the graph G. Note that:

1. If $A, B \in \mathcal{C}^*$, then $A \triangle B \in \mathcal{C}^*$, so $\mathcal{C}^*$ is a vector space over $GF(2)$.

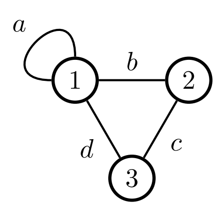
2. For each $A \in \mathcal{C}^*$, there exists $X \subset V(G)$ with $A = \delta(X) = \bigtriangleup_{x \in X} \delta(\{x\})$

So the cut space is generated by elementary cuts (cuts with $|X| = 1$).

Definition 2.6.4. Let $G = (V, E)$ be a graph and $A \in \{0, 1, 2\}^{|V \times E|}$ be the incidence matrix of G, with $[A]_{v,e} = \#$ of ends of e at v. Moreover, if $G = (V, E, \varphi)$ is our graph, then $A = \varphi$.

· The rows of A correspond to elementary cuts.
· The rowspace of A over $GF(2)$ is the cut space.

Example 2.6.5. Consider the following graph G and its incidence matrix.



$$A = \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Proposition 2.6.6. Let A be the incidence matrix of G with $X \subset E(G)$ and $x \in GF(2)^{|E(G)|}$ such that $x_e = \left\{\begin{smallmatrix} 1 & \text{if } e \in X \\ 0 & \text{if } e \notin X \end{smallmatrix}\right.$. Then $X \subset \mathcal{C}^*$ iff $x \in \text{rowspace}_{GF(2)}(A)$.

Theorem 2.6.7. If a graph G has c components, then $\dim(\mathcal{C}^*) = |V(G)| - c$.

Proof: Let A be the incidence matrix, and consider $x \in GF(2)^{|V(G)|}$. Note that $x^T A = 0$ iff $x_u = x_v$ whenever u, v are in the same component, and

$$\dim\left(\left\{x \in GF(2)^{|V(G)|} \mid x^T A = 0\right\}\right) = c$$

Hence $\text{rank}_{GF(2)}(A) = |V(G)| - c$. So by the above proposition, $\dim(\mathcal{C}^*) = |V(G)| - c$. ■

Definition 2.6.8. A graph G is termed even if every one of its vertices has even degree.

Definition 2.6.9. Let G be a graph and its cycle space $\mathcal{C}$ defined by

$$\mathcal{C} = \{X \subset E(G) \mid G[V, X] \text{ is even}\}$$

This space has the following properties:
1. If $A, B \in \mathcal{C}$, then $A \triangle B \in \mathcal{C}$, so $\mathcal{C}$ is a vector space over $GF(2)$.
2. If C is a cycle of G, then $C \in \mathcal{C}$.

Proposition 2.6.10. Let G be a graph with $E \subset E(G)$. Then $X \in \mathcal{C}$ iff there is a partition $(C_1, \dots, C_k)$ of X such that $G[C_1], \dots, G[C_k]$ are all cycles.

Definition 2.6.11. Let G be a graph and its orthogonal space space $\mathcal{C}^\perp$ defined by

$$\mathcal{C}^\perp = \{X \subset E(G) \mid |C \cap \delta(X)| \text{ is even for all } C \in \mathcal{C}\}$$

This space has the following properties:
1. $\mathcal{C}^\perp$ is a vector space
2. $\dim(\mathcal{C}) + \dim(\mathcal{C}^\perp) = |E(G)|$
3. $(\mathcal{C}^\perp)^\perp = \mathcal{C}$

Note that if $C \subset E(G)$ is a cycle and $X \subset V(G)$, then $|C \cap \delta(X)|$ is even. Since cycles generate $\mathcal{C}$, we have that $\mathcal{C}^* \subset \mathcal{C}^\perp$ and $\mathcal{C}^\perp \subset (\mathcal{C}^*)^\perp$.

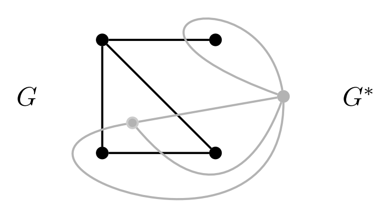
Theorem 2.6.12. Let G be a graph. Then $\mathcal{C}^* = \mathcal{C}^\perp$. Equivalently, if $X \subset E(G)$ intersects each cycle an even number of times, then X is a cut.

Proof: Note that $G/(E(G) - X)$ has no odd cycles. So $G/(E(G) - X)$ is bipartite. Therefore X is a cut of G. ■

2.7 Planar graphs

Definition 2.7.1. Given a planar graph G, a plane graph of G is a specific embedding of G in the plane. The dual G^* of a plane graph is the dual of a planar graph.

Example 2.7.2. This is a graph G overlaid with its dual G^*.



Definition 2.7.3. Let F be the set of faces on a plane graph $G = (V, E)$. For $e \in E$ and $f \in F$, define

$$\varphi^*(f, e) = \# \text{ of sides of } e \text{ in the embedding of } G \text{ that are in face } f$$

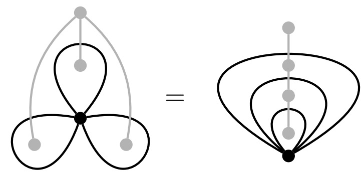
Remark 2.7.4. For a plane graph G, we have that
1. G^* is planar
2. G^* is connected
3. If G is connected and we take the natural embedding of G^*, then $(G^*)^* = G$.

Theorem 2.7.5. Let G be a plane graph. Then $\mathcal{C}(G) = \mathcal{C}^*(G^*)$ and $\mathcal{C}^*(G) = \mathcal{C}(G^*)$.

Proof: If G is not connected, identify pairs of vertices in different components but on the same face. Since G is now connected, $(G^*)^* = G$. It suffices now to prove that $\mathcal{C}(G) = \mathcal{C}^*(G^*)$, as the other statement is then equivalent.

Consider a cycle C of G. Let F be the set of faces "inside" C in the embedding of G. Then $E(G) = \delta_{G^*}(F)$. Hence $\mathcal{C}(G) \subset \mathcal{C}^*(G^*)$, and dually $\mathcal{C}(G^*) \subset \mathcal{C}^*(G)$. It suffices to show for $X \subset V(G)$ with $G[X]$ and $G[V(C) - X]$ connected that $\delta_G(X)$ is a cycle in G^*. In this case X is a bond. ■

Remark 2.7.6. For the above theorem to hold, it is necessary for G to be a plane graph, as planar graphs may induce plane graphs with varying duals and varying cut spaces, for example:

$G =$ 

Corollary 2.7.7. Euler's formula follows. That is, for G a connected graph with vertex set V, edge set E and face set F,

$$
\begin{aligned}
|E| &= \dim(\mathcal{C}^*(G)) + \dim(\mathcal{C}(G)) \\
&= \dim(\mathcal{C}^*(G)) + \dim(\mathcal{C}^*(G^*)) \\
&= (|V| - 1) + (|F| - 1) \\
\implies |V| + |F| &= |E| + 2
\end{aligned}
$$

Remark 2.7.8. Let e be an edge of a plane graph G. Then
1. $(G \setminus e)^* = G^*/e$
2. If e is not a loop, then $(G/e)^* = G^* \setminus e$

Definition 2.7.9. Given a connected graph G, an edge $e \in E(G)$ is termed a bridge iff $G \setminus e$ is disconnected.

Proposition 2.7.10. The following are equivalent:
1. The four-color theorem (every loopless planar graph is 4-colorable)
2. If G is a loopless planar graph, then there exist $C_1, C_2 \in \mathcal{C}^*(G)$ with $E(G) = C_1 \cup C_2$.
3. If G is a bridgeless planar graph, then there exist $C_1, C_2 \in \mathcal{C}(G)$ such that $E(G) = C_1 \cup C_2$. That is, G is the union of two even subgraphs.

Theorem 2.7.11. [Jaeger]
Every bridgeless graph is the union of 3 even subgraphs.

Theorem 2.7.12. [Jaeger]
For every graph G, there is a partition (C, C^*) of $E(G)$ such that $c \in \mathcal{C}(G)$ and $C^* \in \mathcal{C}^*(G)$. Equivalently, a cut may be deleted from G to get an even graph.

Definition 2.7.13. A simple connected plane graph is termed a plane triangulation, or triangulation, iff every face of the graph has degree 3. As a result of the Jordan curve theorem, a simple connected plane graph is a triangulation iff its dual is cubic.

Lemma 2.7.14. If G is a minimal counterexample to the four-color theorem, then G is a simple planar triangulation.

Proof: A minimal counterexample is a 5-critical plane graph G, which is simple and 2-connected. If G is not a triangulation, is has a face f of degree $\geqslant 4$. Let $x_1, \ldots, x_4$ be consecutive vertices on f. Note that both $(x_1, x_3), (x_2, x_4)$ are not in G. WLOG $(x_1, x_3) \notin G$. Identifying x_1 and x_3 gives a smaller counterexample, a contradiction. ■

Lemma 2.7.15. Every minimal counterexample to the four-color theorem is 4-connected.

Proposition 2.7.16. The following are equivalent:

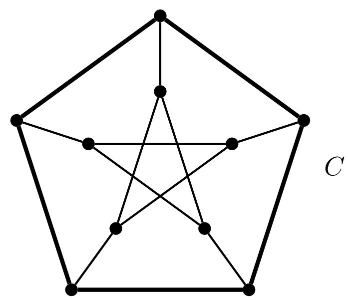
1. The four-color theorem.
2. Every loopless plane graph is a triangulation.
3. Every bridgeless cubic plane graph is the union of two even subgraphs.

Lemma 2.7.17. Let G be a cubic graph. Then G is the union of two even subgraphs iff G is 3-edge-colorable.

Remark 2.7.18. In 1879, Tait conjectured that the four color theorem is equivalent to the statement that every bridgeless subic plane graph is 3-edge-colorable. This turned out to be false, as Julius Petersen showed with a counterexample that bears his name.

Theorem 2.7.19. [Petersen 1891]
The Petersen graph P is not 3-edge-colorable.

Proof: No perfect matching of P uses exactly one edge of the outer cycle C.

$P =$ 

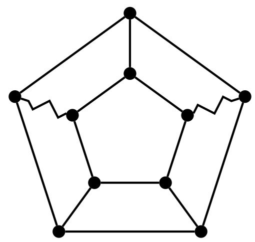
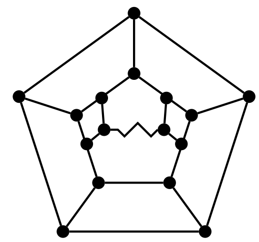
This follows from trial and error. So every perfect matching uses an even number of edges from C. But $|E(C)|$ is odd, so P is not 3-edge-colorable. ■

Proposition 2.7.20. Let G be a cubic graph with a Hamilton cycle C (a cycle that meets every vertex). Then G is 3-edge-colorable.

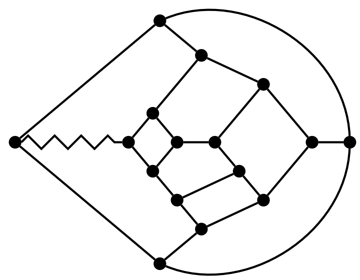
Proof: Color the complement of the cycle color 1. Since G is cubic, $|V(G)|$ is even. Hence the edges of C may be colored with colors 2,3. ■

Conjecture 2.7.21. [Tait 1879]
Every bridgeless (equivalently, 3-connected) cubic planar graph has a Hamilton cycle.

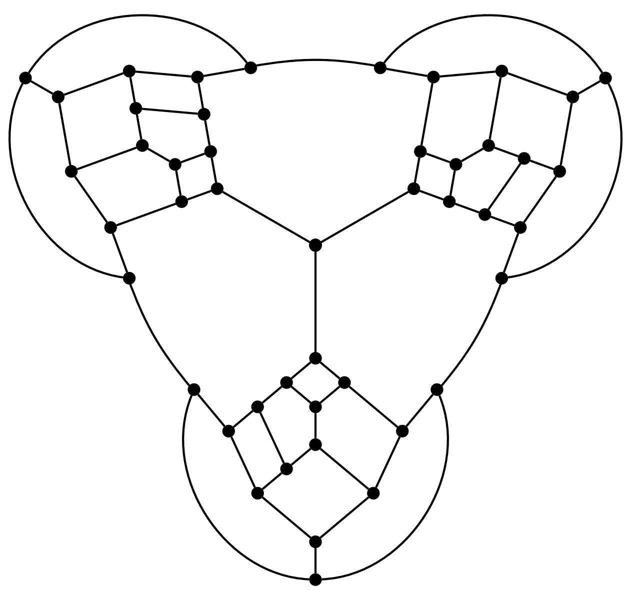
Tutte proved this theorem false in 1946 with a graph built as follows. First, note that no Hamilton cycle of the graph G_1 contains both accented edges.

G_1 =  G_2 = 

It follows that no Hamilton cycle of the graph G_2 avoids the accented edge. Otherwise if that edge were to be removed and the remaining 4-cycles contracted, there would be a cycle using both the accented edges of G_1. Redraw G_2 as follows, centered aroud the accented edge.

G_2 = 

Then G, known as Tutte's graph, has no Hamilton cycle. It is composed of three partial copies of G_2, as below.



Theorem 2.7.22. [Tutte]
Every 4-connected planar graph has a Hamilton cycle.

2.8 The proof of the four-color theorem

Remark 2.8.1.
· 1976: Appel and Haken prove the four-color theorem finding 1476 unavoidable configurations.
· 1997: Robertson, Sanders, Seymour, and Thomas prove it with 633 unavoidable configurations.
The proof has three main steps:

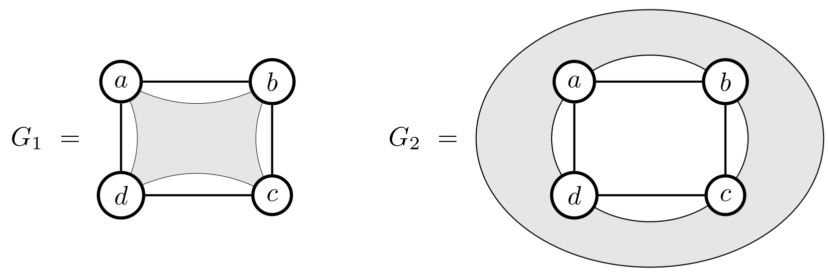
Step 1: Prove that every minimal counterexample to the theorem is a simple, internally 6-connected plane triangulation (i.e. 6-connected with vertices of degree 5 allowed).

Step 2: Show that every internally 6-connected plane triangulation contains one of 633 unavoidable configurations.

Step 3: Show that each of these configurations is reducible, i.e. they cannot occur in a minimal counterexample to the theorem, by contracting them for smaller counterexamples.

Lemma 2.8.2. Every minimal counterexample to the four-color theorem is a 5-connected triangulation.

Proof: Consider such a minimal counter example G. We have already seen that G is a simple 4-connected plane triangulation. Now suppose that G is not 5-connected, so then there is a proper separation (G_1, G_2) of G such that $G_1 \cap G_2$ is a 4-cycle. Label the cycle $C = (a, b, c, d, a)$.



Up to color symmetry, there are 4 colorings of C, listed here as colors for (a, b, c, d).

$$c_1 = (1, 2, 1, 2) \quad c_2 = (1, 2, 1, 3)$$
$$c_3 = (1, 3, 2, 3) \quad c_4 = (1, 3, 4, 2)$$

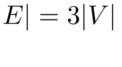
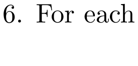
For $i \in \{1, 2\}$, let $\mathcal{C}_i$ be the set of 4-colorings of C that extend to G_i, and as G is not 4-colorable, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. By minimality, we have that $G_1 + ac$ and $G_2 + ac$ are 4-colorable. By possibly swapping G_1 and G_2, we may assume that $c_3 \in \mathcal{C}_1$ and $c_4 \in \mathcal{C}_2$. Similarly, $G_1 + bd$ is 4-colorable, so $c_2 \in \mathcal{C}_1$.

Now consider a coloring of G_2 that extends c_4. For colors i, j, let $G_2(i, j)$ be the subgraph of G_2 induced by vertices of color i, j. Then note that:

· If we swap the colors i, j on any component, of $G_2(i, j)$, we get another 4-coloring of G_2
· Since C bounds a face in G, there cannot exist both an ac-path in $G_2(1, 2)$ and a bd-path in $G_2(3, 4)$

By possibly relabelling (a, b, c, d) by (b, c, d, a), we may assume that there is no ac-path in $G_2(1, 2)$. Now swapping colors 1 and 2 on the component of $G_2(1, 2)$ that contains a, we see that $c_2 \in \mathcal{C}_2$. But $c_2 \in \mathcal{C}_1$, a contradiction. ■

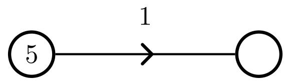
The observation on vertex swapping is due to Kempe.

Lemma 2.8.3. Every simple plane triangulation of minimum degree 5 contains either  or .

Proof: Let $G = (V, E)$ be a simple plane triangulation with minimum degree 5, so $|E| = 3|V| - 6$. For each $v \in V$, assign a charge φ by $\varphi(v) = 5(6 - \mathsf{deg}(v))$, so

$$\sum_{v \in V} \varphi(v) = 5 \sum_{v \in V} (6 - \mathsf{deg}(v)) = 5(6|V| - 2|E|) = 5(6|V| - 2(3|V| - 6)) = 60$$

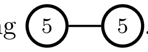
Define a discharging rule among the vertices:

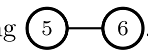


So each vertex of degree 5 sends one unit of charge to each of its neighbors. Now let φ' denote the resulting charge, and let α denote the number of degree 5 neighbors of a vertex, so

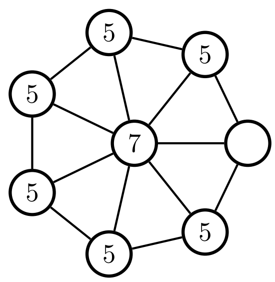
$$\varphi'(v) = \begin{cases} \varphi(v) + \alpha(v) & \text{if } \mathsf{deg}(v) > 5 \\ \alpha(v) & \text{if } \mathsf{deg}(v) = 5 \end{cases}$$

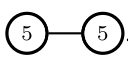
Note that $\sum_{v \in V} \varphi(v) = \sum_{v \in V} \varphi'(v) = 60$. For v a vertex of positive charge, we now consider all the possible cases.

Case 1: $\mathsf{deg}(v) = 5$
Then $0 < \varphi'(v) = \alpha(v)$, so v has at least one neghbor of degree 5, giving .

Case 2: $\mathsf{deg}(v) = 6$
Then $0 < \varphi'(v) = \varphi(v) + \alpha(v) = \alpha(v)$, so v has at least one neghbor of degree 5, giving .

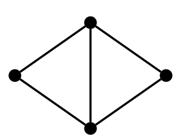
Case 3: $\mathsf{deg}(v) = 7$
Then $0 < \varphi'(v) = \varphi(v) + \alpha(v) = \alpha(v) - 5$, so v has at least 6 neighbors of degree 5, and the plane triangulation of v and its neighbors looks like



So we have found .

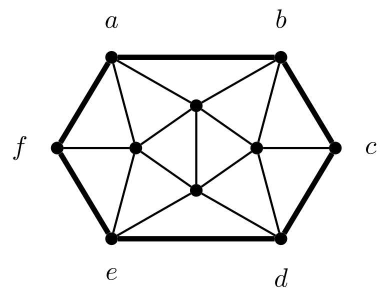
Case 4: $\mathsf{deg}(v) \geqslant 8$
Then $0 < \varphi'(v) = \varphi(v) + \alpha(v) \leqslant 5(6 - \mathsf{deg}(v)) + \mathsf{deg}(v) = 30 - 4\mathsf{deg}(v) \leqslant -2$, but this is a contradiction. Hence $\mathsf{deg}(v) < 8$ for all v in the triangulation, and all the cases have been checked. ■

Lemma 2.8.4. No minimum counterexample to the four-color theorem contains the Birkhoff diamond:



Proof: Suppose that the lemma does not hold. Consider the neighbor set of the Birkhoff diamond, along with their common edges. Then there is a minimal counterexample G with a proper planar separation (G_1, G_2), so

G_1 =



$G_1 \cap G_2 = C$

$C = (a, b, c, d, e, f, a)$

Let $\mathcal{C}_i$ for $i = 1, 2$ be the set of 4-colorings of C that extend to G_i, so $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. As G is a minimal counterexample, it is 5-connected, so the cycle C indicated above is an induced cycle, i.e. chordless. By the minimality of G, there is a 4-coloring of G s.t

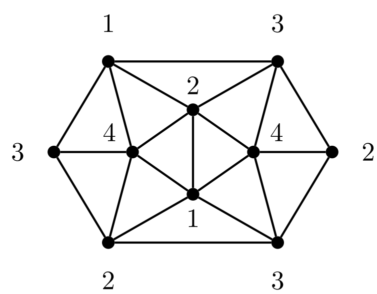
i. a, e have color 1
ii. c has color 2

This follows by identifying a and e, 4-coloring that graph, and then separating a and e, while keeping the same coloring. Up to color symmetry, there are then 6 colorings of C, namely

	c_1	c_2	c_3	c_4	c_5	c_6
top	1 3	1 3	1 3	1 3	1 4	1 4
middle	2 2	2 2	3 2	3 2	3 2	3 2
bottom	1 3	1 4	1 3	1 4	1 3	1 4

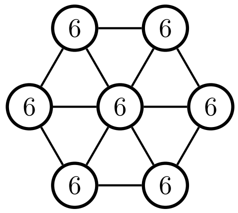
We know that at least one c_i must extend to G_2, i.e. be in $\mathcal{C}_2$, as G is minimal. To find it, we extend all colorings to G_1. By trial and error, we find that $c_1, c_2, c_3, c_5, c_6 \in \mathcal{C}_1$, so we must have that $c_4 \in \mathcal{C}_2$ and $c_4 \notin \mathcal{C}_1$.

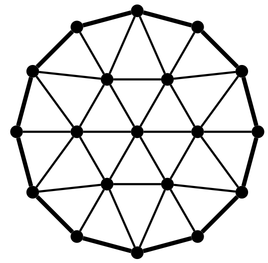
Now consider a 4-coloring of G_2 extending c_4. For distinct colors i, j, let $G_2(i, j)$ be the subgraph induced by the vertices of colors i, j. Note that then b, d, f are in the same component of $G_2(3, 4)$, as otherwise by swapping the colors 3, 4 in one of the components gives a restricted c_k-coloring. Then a, c, e are in distinct components of $G_2(1, 2)$. Switching the colors 1, 2 on the component of $G_2(1, 2)$ containing e, we have the coloring



Now we have a coloring c_7 of C with $c_7 \in \mathcal{C}_1 \cap \mathcal{C}_2$, a contradiction. Hence no minimal counterexample exists. ■

In a similar way to the proof above, it may be shown that the configuration on the left (another one of the 633 possible configurations) is reducible by considering the neighbors of the configuration, in the subgraph on the right.





The general method of this part of the four-color theorem is as follows:

· Get G_1, a known graph, in an induced cycle
· Compute $\mathcal{C}_1$ and set $\tilde{\mathcal{C}}_2 = \{$4-colorings of C not in $\mathcal{C}_1\}$ explicitly
· For every partition of pair of colors in a coloring in $\tilde{\mathcal{C}}_2$ find a coloring that extends to G_1

3 Extremal graph theory

In this section, all graphs will be simple.

3.1 Ramsey theory

Definition 3.1.1. Let G be a graph. Define the complement of G by the graph G^c with edge and vertex set given by

$$\begin{array}{rcl} V(G^c) & = & V(E) \\ E(G^c) & = & \{(x,y) \mid x,y \in V(G), (x,y) \notin E(G)\} \end{array}$$

If G^c has a clique set, then that clique is a stable set of G.

Theorem 3.1.2. [Ramsey]
For each $k \in \mathbb{N}$, there exists $R \in \mathbb{N}$ such that if G is a graph with $|V(G)| \geqslant R$, then G or G^c contains K_k.

Theorem 3.1.3. (Version 2 of (3.1.2))
There exists a function $R : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if G is a graph with $|V(G)| \geqslant R(k,\ell)$, then G has a clique of size k or a stable set of size ℓ.

Proof: Define R, for which we will prove the result, recursively by

$$R(k,\ell) = \begin{cases} 1 & \text{if } k = 1 \text{ or } \ell = 1 \\ R(k-1,\ell) + R(k,\ell-1) & \text{else} \end{cases}$$

Suppose that R fails and there exists a counterexample (G,k,ℓ) with $k+\ell$ minimal. Note that (G^c,ℓ,k) is also a minimal counterexample in this case. Clearly we have $k \geqslant 2$ and $\ell \geqslant 2$.

Let $v \in V(G)$ and $X = N_G(v)$ with $X^c = N_{G^c}(v)$. As G has no clique of size k or stable set of size ℓ, $G[X]$ has no clique of size $k-1$ or stable set of size ℓ. By our choice of (G,k,ℓ), we have $|X| \leqslant R(k-1,\ell) - 1$ and $|X^c| \leqslant R(k,\ell-1)$. Hence

$$|V(G)| = 1 + |X| + |X^c| < R(k-1,\ell) + R(k,\ell-1) = R(k,\ell)$$

This contradicts the definition of R, so the original assumption is false. ∎

Note that the function R used in the above proof is given explicitly by $R(k,\ell) = \binom{k+\ell-2}{k-1}$.

Definition 3.1.4. Let $r(k,\ell)$ be the least integer so that each graph with $\geqslant r(k,\ell)$ vertices has a clique of size k or a stable set of size ℓ. Then $r(k,\ell) \leqslant R(k,\ell)$ and $r(k,\ell) \leqslant 4^{k-1}$.

Theorem 3.1.5. [Erdos]
For $k \geqslant 3$, $r(k,k) \geqslant \sqrt{2}^k$.

Proof: Let $\mathcal{G}_n$ denote the set of all simple graphs with vertex set $\{v_1, \ldots, v_n\}$. Let $\mathcal{G}_n^k$ denote the set of all graphs in $\mathcal{G}_n$ with a clique of size k. Let $X \subset \{v_1, \ldots, v_n\}$ be of size k. Then the number of graphs in $\mathcal{G}_n$ with a clique on X is

$$2^{\binom{n}{2}-\binom{k}{2}} \qquad \Longrightarrow \qquad |\mathcal{G}_n^k| \leqslant \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}$$

Then for $n \leqslant \sqrt{2}^k$,we have

$$\begin{aligned}
\frac{|\mathcal{G}_n^k|}{|\mathcal{G}_n|} &\leqslant \binom{n}{k} 2^{-\binom{k}{2}} \\
&< \frac{n^k}{k!} 2^{-\binom{k}{2}} \\
&\leqslant \frac{2^{k^2/2}}{k!} 2^{-(k(k-1))/2} \\
&= \frac{2^{k/2}}{k!} \\
&= \frac{\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{3} \cdots \frac{\sqrt{2}}{n} \\
&< \frac{1}{2}
\end{aligned}$$

Therefore less than $\frac{1}{2}$ of graphs in $\mathcal{G}_n$ have a clique of size k. So there exists $G \in \mathcal{G}_n$ such that $G \notin \mathcal{G}_n^k$ and $G^c \notin \mathcal{G}_n^k$. Hence G has nether a clique of size k nor a stable set of size k. ■

Example 3.1.6.

1. We proved that $\sqrt{2}^k \leqslant r(k,k) \leqslant 4^k$, which are bounds close to the bleeding edge of research
2. The above proof was not constructive
3. Known constructions give subexponential lower bounds
4. For fixed k, $r(k,\ell)$ is polynomial in ℓ:

$$r(k,\ell) \leqslant \binom{k+\ell-2}{k-1} \leqslant \left(\ell + \frac{k-1}{\text{(const.)}}\right)^{k-1}$$

Conjecture 3.1.7. [Erdos, Hajnal]
For any graph H, there is a polynomial $p_H(k)$ such that if $k \in \mathbb{N}$ and G is a graph with no induced subgraph isomorphic to H and with $|V(G)| \geqslant p_H(k)$, then G has a clique or a stable set of size k.

Proof: A proof to this conejcture is claimed by Gabor Sagi, at `http://arxiv.org/abs/1211.3876` ■

Theorem 3.1.8. Every infinite sequence of distinct numbers in $\mathbb{R}$ contains a monotonic subsequence.

Proof: Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of distinct real numbers. Then one of the following two sets is infinite:

$$\{\alpha_j \mid j \geqslant 2, \alpha_j > \alpha_i \;\forall\; i < j\} \qquad \{\alpha_j \mid j \geqslant 2, \alpha_j < \alpha_i \;\forall\; i < j\}$$

Therefore we can find an infinite subsequence $\{\beta_k\}$ of $\{\alpha_i\}$ by building it one element at a time, such that for all $i \in \mathbb{N}$, either

- **i.** $\beta_j > \beta_i$ for all $j > i$, or
- **ii.** $\beta_j < \beta_i$ for all $j > i$

Case 1: For infinitely many $i \in \mathbb{N}$, **i.** holds
This gives an infinite increasing subsequence of $\{\alpha_n\}$.

Case 2: For infinitely many $i \in \mathbb{N}$, **i.** does not hold.
So **ii.** holds , and this gives an infinite decreasing subsequence of $\{\alpha_n\}$. ■

Theorem 3.1.9. (Restatement of (3.1.2))
For any $k \in \mathbb{N}$, if $G = (V, E)$ is a graph with $|V| \geqslant 2^{2k-1}$, then G has a clique or a stable set of size k.

Proof: Choose vertex sets $(X_1, X_2, \ldots, X_{2k+2})$ and vertices $(v_1, v_2, \ldots, v_{2k+2})$ as follows:

· Set $X_1 = V$ and choose any $v_1 \in X$.
· For all $i \geqslant 1$, take X_{i+1} to be the larger of $(X_i - \{v_i\}) \cap N(v_i)$ and $(X_i - \{v_i\}) - N(v_i)$, and any $v_{i+1} \in X_{i+1}$.

Note that $|X_{i+1}| \geqslant 2^{2k-i}$. Now define four sets

$$K = \{v_i \mid i \in \{1, 2, \ldots, 2k-3\}, (v_i, v_{i+1}) \in E\} \qquad K' = K \cup \{v_{2k-2}\}$$
$$S = \{v_i \mid i \in \{1, 2, \ldots, 2k-1\}, (v_i, v_{i+1}) \notin E\} \qquad S' = S \cup \{v_{2k-2}\}$$

Then K' is a clique in G and S' is a stable set in G. Since $|K'| + |S'| = 2k - 1$, either $|K'| \geqslant k$ or $|S'| \geqslant k$, proving the theorem. ■

Theorem 3.1.10. (Version 3 of (3.1.2))
There is a function $\tilde{R} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any $k, t, n \in \mathbb{N}$ with $n \geqslant \tilde{R}(k, t)$, if we color the edges of a graph $G = K_n$ with t colors, then G contains a monochromatic (having a single color) copy of K_k.

Proof: This is done by induction on t.
$t = 1$: Trivial.
$t = 2$: Follows from the previous presentation of Ramsey's theorem.
$t \geqslant 3$: Assume that the result holds for all fewer colors. Define $\tilde{R}(k, t) = R(k, \tilde{R}(k, t-1))$ and let $n \geqslant \tilde{R}(k, t)$. Consider a coloring of K_n with colors $c_1, \ldots, c_t$. If G contains K_k as a monochromatic subgraph, then we are done. Else, assume that G does not have K_k as a monochromatic subgraph of color c_1. Then by the first restatement of Ramsey's theorem, G contains $K_{n'}$ as a subgraph with no edge of color c_1, with $n' = \tilde{R}(k, t-1)$. The result follows by induction. ■

Ramsey's theorem has applications, such as in the proof of the following theorem.

Theorem 3.1.11. [Schur]
For each $t \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that if $(A_1, \ldots, A_t)$ is a partition of $\{1, \ldots, s\}$, then there exists $i \in \{1, \ldots, t\}$ and $x, y, z \in A_i$ (not necessarily distinct) with $x + y = z$.

Proof: Let $s = \tilde{R}(3, t)$ and let $(A_1, \ldots, A_t)$ be any partition of $\{1, \ldots, s\}$. Color edge (i, j) of K_s by color ℓ iff $|i - j| \in A_\ell$. By the last restatement of Ramsey's theorem, there exist $0 < a < b < c \leqslant s$ and ℓ such that

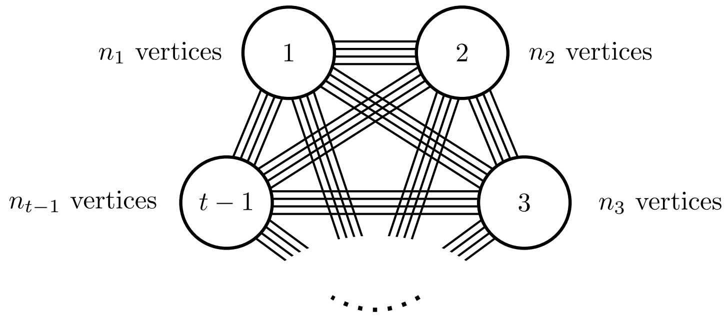
$$b - a, c - b, c - a \in A_\ell$$

Let $x = b - a$, $y = c - b$ and $z = c - a$, so $x + y = z$ and $x, y, z \in A_i$. ■

3.2 Forbidding subgraphs

Definition 3.2.1. For a graph H and $n \in \mathbb{N}$, let $\mathsf{ex}(H, n)$ be the maximum number of edges of a simple n-vertex graph with no subgraph isomorphic to H.

Remark 3.2.2.
1. If H is a subgraph of G, then $\chi(H) \leqslant \chi(G)$
2. The densest $(t-1)$-colorable graph with n vertices is the complete $(t-1)$-partite graph $K_{n_1, n_2, \ldots, n_{t-1}}$, denoted by $T(t, n)$ and termed the Turan graph:



Here, $\left\lfloor \frac{n}{t-1} \right\rfloor \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{t-1} \leqslant \left\lceil \frac{n}{t-1} \right\rceil$. Moreover, note that $e(t,n) = |E(T(t,n))| \approx \frac{t-2}{t-1}\binom{n}{2}$.

3. Hence $\mathsf{ex}(H,n) \geqslant e(\chi(H),n) \approx \frac{\chi(H)-2}{\chi(H)-1}\binom{n}{2}$, which is, in fact, asymptotically an equality.

Definition 3.2.3. Vertices u, v of a graph are termed clones iff $N(u) = N(v)$. By cloning a vertex v, we refer to the operation of adding a vertex v' to G with the same neighbors as v.

Note that of G has no K_t-subgraph, then no graph obtained from G by cloning has a K_t-subgraph.

Theorem 3.2.4. [Turan]
For $t, n \in \mathbb{N}$, $\mathsf{ex}(K_t, n) = e(t,n)$. Moreover, $T(t,n)$ is the only n-vertex graph with $e(t,n)$ edges that does not contain K_t as a subgraph.

Proof: Suppose that G is a simple n-vertex graph with no K_t-subgraph, and that $|E(G)|$ is maximal, subject to having no K_t-subgraph.

Claim 1: If vertices u, v of G are not adjacent, then $\mathsf{deg}(u) = \mathsf{deg}(v)$.
Suppose that $\mathsf{deg}(v) > \mathsf{deg}(u)$ for u, v not adjacent. Let G' be the graph obtained from $G - v$ by cloning u. Now G' has no K_t-subgraph, and $|E(G')| > |E(G)|$, contradicting the maximality of G.

Claim 2: If vertices u, v of G are not adjacent, then u, v are clones.
Suppose that they are not adjacent but are not clones. Then by possibly swapping u and v, we may assume that there exists $w \in N(v) - N(u)$. Let G' be obtained from $G - v - w$ by cloning u twice. Now G' has no K_t-subgraph and $|E(G')| > |E(G)|$, contradicting the maximality of G.

By the second claim, G is a complete multipartite graph. Due to the assumptions, $\chi(G) = t - 1$. Let $X_1, \ldots, X_{t-1}$ be the color classes. Suppose that $G \neq T(t,n)$. Then there exist i, j such that $|X_i| \leqslant |X_j| - 2$. Let $u \in X_i$ and $v \in X_j$, so $\mathsf{deg}(v) \geqslant \mathsf{deg}(u) + 2$. Let G' be the graph obtained from $G - u$ by cloning v. Now G' has no K_t-subgraph and $|E(G')| > |E(G)|$, contradicting the maximality of G. ■

Definition 3.2.5. Given a graph H, define the Turan density of H to be

$$\pi(H) = \lim_{n\to\infty} \left[\frac{\mathsf{ex}(H,n)}{\binom{n}{2}} \right]$$

Note that $\lim_{n\to\infty} \left[\frac{e(t,n)}{\binom{n}{2}} \right] = \frac{t-2}{t-1}$, so Turan's theorem implies that $\pi(K_t) = \frac{t-2}{t-1}$.

Theorem 3.2.6. [Erdos, Stone 1946]
For any graph H with $E(H) \neq \emptyset$, $\pi(H) = \frac{\chi(H)-2}{\chi(H)-1}$.

Note that for $\chi(H) \geqslant 3$, the above theorem describes the asymptotic behavior of $\mathsf{ex}(H,n)$. As for $\chi(H) = 2$, we get $\pi(H) = 0$.

Lemma 3.2.7. If $G = (V, E)$ is an n-vertex graph with no $K_{t,t}$-subgraph, then

$$\sum_{v \in V} \binom{\mathsf{deg}(v)}{t} \leqslant (t-1)\binom{n}{t}$$

Proof: First note that every bipartite graph is contained in some complete bipartite graph. Therefore it suffices to prove the statement only for complete bipartite graphs.

Let X be a t-element subset of V. As G has no $K_{t,t}$-subgraph, the number of vertices v with $X \subset N(v)$ is $\leqslant t-1$. It follows directly that

$$\sum_{v \in V} \underbrace{\binom{\mathsf{deg}(v)}{t}}_{\substack{\text{\# of ways to choose} \\ \text{a } t\text{-element subset of } N(v)}} \leqslant (t-1) \underbrace{\binom{n}{t}}_{\substack{\text{\# of choices} \\ \text{for } X}}$$

■

Theorem 3.2.8. There exists $f : \mathbb{N} \times (0,1) \to \mathbb{N}$ such that if $n, t \in \mathbb{N}$ and $\alpha \in (0,1)$ with $n > f(t, \alpha)$, then $\mathsf{ex}(K_{t,t}, n) \leqslant \binom{\alpha}{2}$.

Proof: Note that $\binom{d}{t}$ is a degree t polynomial in d with leading coefficient $\frac{1}{t!}$. So there exists $c_t \in \mathbb{R}$ with $\binom{d}{t} \geqslant \frac{d^t}{t!} - c_t d^{t-1}$ for all $d \geqslant 1$. Define

$$f(t, \alpha) = \frac{t - 1 + c_t \cdot t!}{\alpha^{2t+2}} + 1$$

Let $G = (V, E)$ be a graph with $|V| = n \geqslant f(t, \alpha)$ and $|E| \geqslant \alpha\binom{n}{2}$. Let X be the set of vertices of G with degree $\geqslant \alpha^2(n-1)$.

Claim: $|X| \geqslant \alpha^2 n$
If not, then

$$\begin{aligned}
2\alpha n\binom{n}{2} &\leqslant 2|E| \\
&= \sum_{v \in V} \mathsf{deg}(v) \\
&< \alpha^2 n(n-1) + (1-\alpha^2)n\alpha^2(n-1) \\
&= \alpha^2 n(n-1)(1-\alpha^2) \\
&< \alpha n(n-1) && (\text{as } \alpha < \tfrac{1}{2}) \\
&= 2\alpha\binom{n}{2}
\end{aligned}$$

As this is a contradiction, the claim is proven.

Now note that

$$\begin{aligned}\sum_{v\in V}\binom{\mathsf{deg}(v)}{t} &\geqslant \sum_{v\in V}\left(\frac{\mathsf{deg}(v)^t}{t!} - c_t\mathsf{deg}(v)^{t-1}\right)\\ &\geqslant \sum_{v\in V}\left(\frac{\mathsf{deg}(v)^t}{t!}\right) - c_t n(n-1)\\ &\geqslant \alpha^n\frac{(\alpha^2(n-1))^t}{t!} - c_t n(n-1)^{t-1}\\ &= \frac{n(n-1)^{t-1}}{t!}\left(\alpha^{2t+2}(n-1) - c_t t!\right)\\ &\geqslant \binom{n}{t}(t-1)\end{aligned}$$

Then by the previous lemma, G has a $K_{t,t}$-subgraph. ■

Remark 3.2.9.

1. The above theorem implies the base case of the Erdos-Stone theorem (the case where H is bipartite)

2. Also, it implies that if $n \geqslant f(t, \frac{1}{k})$, then in any coloring of the edges of K_n with k colors, we can find a monochromatic $K_{t,t}$-subgraph. Hence it is termed a density Ramsey theorem for bipartite graphs.

4 The probabilistic method

Definition 4.0.1. Let $\mathcal{G}_n$ denote the set of all simple graphs with vertex set $\{1, 2, \ldots, n\}$. For $0 < p < 1$, let $\mathcal{G}_{n,p}$ denote the probability distribution of $\mathcal{G}_n$, so that

$$P(G) = p^{|E(G)|}(1-p)^{|E(G^c)|}$$

4.1 Applications

Existence theorems

Recall the lower bounds for Ramsey numbers. For $\alpha(G)$ the size of the largest stable set in G, if $n < \sqrt{2}^k$ and $G \in \mathcal{G}_{n,\frac{1}{2}}$, then

$$P(\alpha(G) \geqslant k) < \frac{1}{2}$$

Thus there exists $G \in \mathcal{G}_n$ with $\alpha(G) < k$ and $\alpha(G^c) < k$. This showed us that $r(k, k) \geqslant \sqrt{2}^k$.

Properties of random graphs

Theorem 4.1.1. For $0 < p < 1$, $G \in \mathcal{G}_{n,p}$ has asymptotically almost surely (a.a.s.) the property that every pair of vertices has a common neighbor. That is, if $\lambda_{n,p}$ is the probability that G has this property, then $\lim_{n\to\infty}[\lambda_{n,p}] = 1$.

Proof: Let $i, j \in V(G)$ be distinct. The probability that i, j have no common neighbor is equal to $(1-p^2)^{n-2}$. So the probability that there is no pair of vertices wih a common neighbor is $\leqslant \binom{n}{2}(1-p^2)^{n-2}$, and by l'Hopital's rule,

$$\lim_{n\to\infty}\left[\binom{n}{2}(1-p^2)^{n-2}\right] = \lim_{n\to\infty}\left[\frac{n(n-1)}{2}(1-p^2)^{n-2}\right] = 0$$

■

Corollary 4.1.2. A graph in $\mathcal{G}_{n,p}$ is a.a.s. connected.

Random processes of graphs

Proposition 4.1.3. Every loopless graph with m edges has a cut size of size at least $\frac{m}{2}$.

Proof: Choose $X \subset V(G)$ uniformly at random. For $e \in E(G)$, we have that $P(e \in \delta(X)) = \frac{1}{2}$. Hence $\mathbb{E}(|\delta(X)|) = \frac{m}{2}$. Therefore there exists $X \subset V(G)$ with $|\delta(X)| \geqslant \frac{m}{2}$.

This follows by observing that for the random variable $c_e = |\delta(X) \cap \{e\}|$ and the linearity of $\mathbb{E}$,

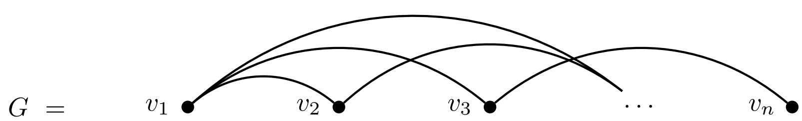
$$\mathbb{E}[|\delta(X)|] = \mathbb{E}\left[\sum_{e \in E(G)} c_e\right] = \sum_{e \in E(G)} \mathbb{E}[c_e] = \sum_{e \in E(G)} P(e \in \delta(X)) = \frac{m}{2}$$

■

Remark 4.1.4. Finding a maximum cut in a graph is NP-hard. The above shows that we can always find a cut of size $\geqslant \frac{m}{2}$, which is at least half the maximal size.

Theorem 4.1.5. For any simple graph G, $\alpha(G) \geqslant \sum_{v \in V(G)} \frac{1}{\mathsf{deg}(v)+1}$.

Proof: Choose an ordering $(v_1, \ldots, v_n)$ of $V(G)$ at random uniformly, and express G as



Let X be the set of all $v_i \in V(G)$ such that for each $v_j \in N(v_i)$, $j > i$. Then X is a stable set. For any $v \in V(G)$, $P(v \in X) = \frac{1}{\mathsf{deg}(v)+1}$, so $\mathbb{E}[|X|] = \sum_{v \in V(G)} \frac{1}{\mathsf{deg}(v)+1}$. Then there is an ordering giving

$$\alpha(G) \geqslant |X| \geqslant \sum_{v \in V(G)} \frac{1}{\mathsf{deg}(v)+1}$$

■

Remark 4.1.6. Equality is attained in the above theorem iff G is a disjoint union of cliques. Moreover, Turan's theorem is a consequence of it.

Conjecture 4.1.7. [Hajos]
If G does not contain K_t as a topological minor, then $\chi(G) \leqslant t - 1$.

Theorem 4.1.8. [Catlin]
Almost all graphs are counterexamples to Hajos' conjecture. That is, $G \in \mathcal{G}_{n,\frac{1}{2}}$ is a.a.s. a counterexample.

Remark 4.1.9.
- Hajos' conjecture remains open for K_5 and K_6.
- Hadwiger's conjecture is known to be a.a.s. true for $G \in \mathcal{G}_{n,\frac{1}{2}}$.

Lemma 4.1.10. For $G \in \mathcal{G}_{n,\frac{1}{2}}$, $\alpha(G) < 2\log_2(n)$ a.a.s.

Proof: Let $k = \lceil 2\log_2(n) \rceil$. Then $n \leqslant \sqrt{2}^k$, and

$$\begin{aligned}
P(\alpha(G) \geqslant k) &\leqslant \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&< \frac{n^k}{k!} 2^{\frac{-k(k-1)}{2}} \\
&= \frac{2^{k/2}}{k!} \\
&= \frac{\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{3} \cdots \frac{\sqrt{2}}{k} \\
&\xrightarrow{k,n\to\infty} 0
\end{aligned}$$

■

With the result of this lemma, we return to the proof of (4.1.8).

Proof: Let $G \in \mathcal{G}_{n,\frac{1}{2}}$. By the lemma, a.a.s.

$$\chi(G) \geqslant \frac{|V(G)|}{\alpha(G)} \geqslant \frac{n}{2\log_2(n)}$$

Let $\Bbbk = \lceil n/(2\log_2(n)) \rceil$. We may assume that N is large enough, so $\binom{\Bbbk}{2} \geqslant n^{3/2}$. It remains to show that G a.a.s. does not contain $K_{\Bbbk}$ as a topological minor.

Now suppose that G has a subgraph H that is a subdivision of $K_{\Bbbk}$. Note that H must have fewer than n degree 2 vertices. Hence G has a subgraph H' with $|V(H')| = \Bbbk$, and $|E(H')| \geqslant \binom{\Bbbk}{2} - n$. Then the probability of $\tilde{G} \in \mathcal{G}_{n\frac{1}{2}}$ containing such a subgraph is less than or equal to

$$\begin{aligned}
\binom{n}{\Bbbk}\binom{\binom{\Bbbk}{2}}{n}\left(\frac{1}{2}\right)^{\binom{\Bbbk}{2}-n} &< n! \frac{\left(\frac{\Bbbk^2}{2}\right)^n}{n!}\left(\frac{1}{2}\right)^{\binom{\Bbbk}{2}-n} \\
&= \Bbbk^{2n} 2^{-\binom{\Bbbk}{2}} \\
&\leqslant n^{2n} 2^{-n^{3/2}} \\
&\xrightarrow{n\to\infty} 0
\end{aligned}$$

■

4.2 Large girth and chromatic number

Definition 4.2.1. The girth of a graph is the length of a shortest cycle.

Theorem 4.2.2. [Erdos 1959]
For all $k, \ell \in \mathbb{N}$, there is a graph with girth $\geqslant \ell$ and chromatic number $\geqslant k$.

Proof: (outline) Choose n, p so that for $G \in \mathcal{G}_{n,p}$,

- $P(\alpha(G) \geqslant \frac{n}{2k}) < \frac{1}{2}$, and
- With probability $< \frac{1}{2}$, G has more that $\frac{n}{2}$ cycles of length $\leqslant \ell$

So there exists $G \in \mathcal{G}_n$ such that $\alpha(G) < \frac{n}{2k}$, and G has $\leqslant \frac{n}{2}$ cycles of length ℓ. By deleting a vertex from each short cycle, we get a subgraph H with

- $\text{girth}(G) > \ell$

· $|V(H)| \geqslant \frac{n}{2}$
· $\alpha(H) < \frac{n}{2k}$

The last two facts together imply that $\chi(H) \geqslant k$. ■

To complete the proof, it is necessary to remark on some facts first.

1. If $0 < \epsilon < 1$, then $\lim_{n\to\infty} \left[\frac{\epsilon^n}{\log(n)}\right] = \infty$
2. If $0 < \epsilon < 1$, then $\lim_{n\to\infty} \left[n^{-n^{\epsilon}}\right] = 0$
3. If X is a non-negative random variable, then $P(X > 2\mathbb{E}(X)) < \frac{1}{2}$
4. For all $x \in \mathbb{R}$, $1 - x \leqslant e^{-x}$

With these facts we now complete the proof to (4.2.2).

<u>*Proof:*</u> Let $\epsilon = \frac{1}{2\ell}$ and choose $n \in \mathbb{N}$ suficiently large so that
i. $\ell\sqrt{n} < \frac{n}{4}$
ii. $n^{-\frac{1}{2}\lceil 3n^{1-\epsilon}\log(n)\rceil} < \frac{1}{2}$
iii. $k < \frac{n}{6\log(n)+2}$

Let $p = n^{\epsilon-1}$ and $t = \frac{3}{p}\log(n) + 1$. Consider $G \in \mathcal{G}_{n,p}$, and let x be the number of cycles of length $\leqslant \ell$ in G.

<u>Claim</u>: $P(x > \frac{n}{2}) < \frac{1}{2}$
This is a result of some routine calculations:

$$\mathbb{E}[X] = \sum_{i=3}^{\ell} \underbrace{\frac{i!}{2i}}_{\substack{\text{because of}\\ \text{symmetry}}} p^i \leqslant \sum_{i=3}^{\ell}(np)^i = \sum_{i=3}^{\ell} n^{\epsilon i} = \sum_{i=3}^{\ell} \sqrt{n}^{i/\ell} \leqslant \ell\sqrt{n} < \frac{n}{4}$$

This shows that $P(x > \frac{n}{2}) < \frac{1}{2}$, from application of fact **3.** above.

<u>Claim</u>: $P(\alpha(G) \geqslant t) < \frac{1}{2}$
This again is the result of a long calculation.

$$\begin{aligned} P(\alpha(G) \geqslant t) &\leqslant \binom{n}{t}(1-p)^{\binom{t}{2}} \\ &< n^t(1-p)^{t\left(\frac{t-1}{2}\right)} \\ &= \left(n(1-p)^{(t-1)/2}\right)^t \\ &\leqslant \left(ne^{-p\left(\frac{t-1}{2}\right)}\right)^t \\ &\leqslant \left(ne^{-3/2\cdot\log(n)}\right)^t && \left(\text{as } T = \frac{3}{p}\log(n) + 1\right) \\ &= n^{-t/2} && \left(\text{as } e^{\log(n)} = n\right) \\ &< \frac{1}{2} \end{aligned}$$

This proves the claim.

Now by the above claims there exists $G \in \mathcal{G}_{n,p}$ such that
a. G has $\leqslant \frac{n}{2}$ cycles of length ℓ
b. $\alpha(G) < t$

By deleting a vertex from each of the shortest cycles, there is an induced subgraph of G with girth$(G) > \ell$ and $|V(H)| \geqslant \frac{n}{2}$ and $\alpha(H) < t$. Then by applying **iii.**,

$$\chi(G) \geqslant \frac{|V(H)|}{\alpha(H)} > \frac{n}{2k} \geqslant \frac{n}{\frac{6}{p}\log(n) + 2} = \frac{n}{6n^{1-\epsilon}\log(n) + 2} \geqslant k$$

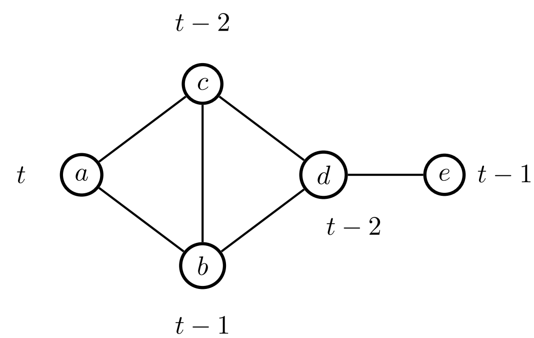
This completes the proof. ■

5 Flows

5.1 The chromatic and flow polynomials

Definition 5.1.1. Let G be a graph. Then $\lambda_G(t)$ is the chromatic polynomial of G, and indicates the number of t-colorings of G. The name implies it is a polynomial, a fact to be verified later.

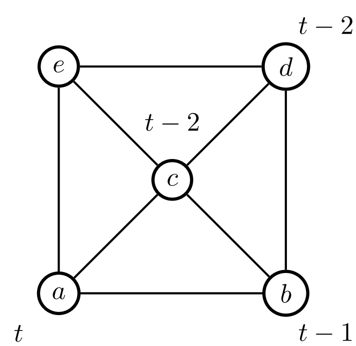
Example 5.1.2. To find the chromatic polynomial of a graph G, consider the following approach.



Choose a vertex, here we choose a. Keep choosing the next vertex that is adjacent to a previous vertex.

Vertex a can be colored with any one of t colors.
Vertex b can be colored with any one of $t-1$ colors.
Vertex c can be colored with any one of $t-2$ colors.
Vertex d can be colored with any one of $t-2$ colors.
Vertex e can be colored with any one of $t-1$ colors.

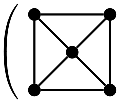
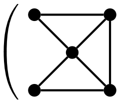
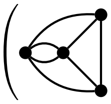
Then $\lambda_G(t) = t(t-1)^2(t-2)^2$. However, this approach fails in the following case, where it is not possible to determine whether e can be colored with any one of $t-2$ or $t-1$ colors, as a and e could have the same or different colors.



Proposition 5.1.3. Given a graph G, if $e \in E(G)$ is not a loop, then $\lambda_G(t) = \lambda_{G-e}(t) - \lambda_{G/e}(t)$.

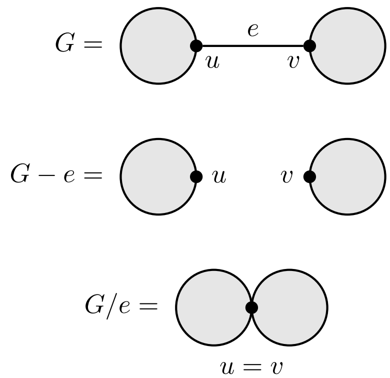
Proof: Each coloring of $G-e$ is either a coloring of G or gives a coloring of G/e. ■

This allows us to calculate the chromatic polynomial of the graph presented above. By slightly abusing notation,

λ  $= \lambda$  $- \lambda$ 

$$= t(t-1)(t-2)^3 - t(t-1)(t-2)(t-3)$$

Remark 5.1.4. If a graph G has a bridge $e = (u, v)$, then $\frac{1}{t}\lambda_{G-e}(t) = \lambda_{G/e}(t)$, hence $\lambda_G(t) = (1-t)\lambda_{G/e}(t)$. This is clear, as a component of $G - e$ has colorings where u (v, respectively) gets t different colors. In G/e, those colorings must have u and v in the same color, hence we can only take $\frac{1}{t}$ of all the colorings of $G - e$.



Proposition 5.1.5. Given a graph G, fix an edge $e \in E(G)$. Then the chromatic polynomial may be defined by a recursive formula:

$$\lambda_G(t) = \begin{cases} 0 & \text{if } e \text{ is a loop} \\ (t-1)\lambda_{G/e}(t) & \text{if } e \text{ is a bridge} \\ \lambda_{G-e}(t) - \lambda_{G/e}(t) & \text{if } e \text{ is any other type of edge} \\ t^{|V(G)|} & \text{if } E(G) = \emptyset \end{cases}$$

This shows that $\lambda_G(t)$ is a polynomial in t.

Recall that for a connected plane graph G and e not a loop or a bridge of G, we had

$$(G - e)^* = G^*/e$$
$$(G/e)^* = G^* - e$$

Also note that e is a loop of G iff e is a bridge of G^*. This leads us to the folowing definition.

Definition 5.1.6. Given a graph G, the flow polynomial $f_G(t)$ of G is defined recursively as

$$f_G(t) = \begin{cases} 0 & \text{if } e \text{ is a bridge} \\ (t-1)f_{G-e}(t) & \text{if } e \text{ is a loop} \\ f_{G/e}(t) - f_{G-e}(t) & \text{if } e \text{ is any other type of edge} \\ 1 & \text{if } E(G) = \emptyset \end{cases}$$

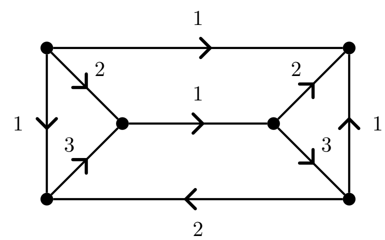
Comparing this and the chromatic polynomial, it emerges that, for G a connected graph, $\lambda_{G^*}(t) = tf_G(t)$.

5.2 Nowhere-zero flows

Definition 5.2.1. Let G be a graph and $\vec{G}$ an orientation of G, i.e. G with with direction assigned to every element of $E(G)$. Then $\vec{G}$ is termed a directed graph, or digraph.

Definition 5.2.2. Given a digraph $\vec{G}$ with values assigned to edges, a nowhere-zero flow over a finite abelian group Γ, is a set of assignments to edges of $\vec{G}$ such that for each vertex v of $\vec{G}$, the flow into v equals the flow out of v.

Example 5.2.3. This is an exapmle of a digraph $\vec{G}$ with a nowhere zero flow over the group $\Gamma = \mathbb{Z}_4$.



Definition 5.2.4. Let G be a graph, $\vec{G}$ an orientation, $X \subset V(G)$ and $f : E(G) \to \Gamma$ for Γ a finite abelian group. Then define

$$\mathsf{inflow}_f(X) = \sum_{\substack{e=uv\in E(\vec{G})\\ u\notin X, v\in X}} f(e) - \sum_{\substack{e=uv\in E(\vec{G})\\ u\in X, v\notin X}} f(e)$$

Proposition 5.2.5. For the definitions as above,

$$\mathsf{inflow}_f(X) = \sum_{x\in X} \mathsf{inflow}_f(x)$$

Proof: Consider the contributions of each edge. ■

Definition 5.2.6. With these new concepts, we may redefine some terms:

- A function f as above is termed a Γ-flow iff $\mathsf{inflow}_f(v) = 0_\Gamma$ for all $v \in V(G)$.
- A function f is termed nowhere-zero iff $f(e) \neq 0$ for all $e \in \vec{G}$.

Remark 5.2.7.

1. The existence of nowhere-zero flows is independent of the choice of orientation of $\vec{G}$.
2. By the previous proposition, if G has a bridge, then $\vec{G}$ cannot have a nowhere-zero flow.

Proposition 5.2.8. Given a graph G, if T is a spanning tree of G and $f' : E(G) - E(T) \to \Gamma$ is a function assigning weights, for Γ a finite abelian group, then there is a unique Γ-flow $f : E(G) \to \Gamma$ such that $f(e) = f'(e)$ for each $e \in E(G) - E(T)$.

Definition 5.2.9. Given a graph G and Γ a finite abelian group, let $F(G, \Gamma)$ denote the number of nowhere-zero Γ-flows. Note that if G has a bridge, then $F(G, \Gamma) = 0$ for any Γ.

Proposition 5.2.10. Given a graph G with $e \in E(G)$ a loop,

$$F(G, \Gamma) = (|\Gamma| - 1)\, F(G - e, \Gamma)$$

Proof: Any flow through e both enters and leaves the end of e, and hence the flow can take any non-zero value in Γ. ■

Proposition 5.2.11. Given a graph G with $e \in E(G)$ a non-loop edge,

$$F(G, \Gamma) = F(G/e, \Gamma) - F(G - e, \Gamma)$$

Proof: Each nowhere-zero flow in G/e is either a nowhere-zero Γ-flow of in $G - e$, or it extends uniquely to give a nowhere-zero Γ-flow in G. ■

Theorem 5.2.12. For the definitions as above,

$$F(G, \Gamma) = f_G(|\Gamma|)$$

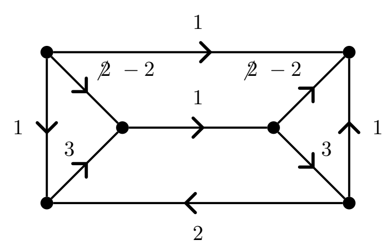
Proof: The result follows by induction. ■

Corollary 5.2.13. Let Γ_1, Γ_2 be abelian groups with $|\Gamma_1| = |\Gamma_2|$. Then G has a nowhere-zero Γ_1-flow iff G has a nowhere-zero Γ_2-flow.

Corollary 5.2.14. A connected plane graph G has a nowhere-zero $\mathbb{Z}_k$-flow iff G^* is k-colorable.

Definition 5.2.15. A $\mathbb{Z}_k$-flow is termed a $\underline{k\text{-flow}}$ iff $\mathsf{inflow}_f(v) = 0$ for all $v \in V$ and $f : E(G) \to \mathbb{Z}$ (i.e. it is a $\mathbb{Z}$-flow), and $|f(e)| < k$ for all $e \in E(G)$.

Example 5.2.16. Consider the previous example. This is a $\mathbb{Z}_4$-flow, but is not a 4-flow, as some vertices do not have 0 net flow. By replacing the edges with flow 2 by flow -2, we get a 4-flow.



Theorem 5.2.17. A graph has a nowhere-zero $\mathbb{Z}_k$-flow iff it has a nowhere-zero k-flow.

Proof: ($\Leftarrow$) Trivial.

($\Rightarrow$) Let f' be a nowhere-zero $\mathbb{Z}_k$-flow. Choose $f : E \to \mathbb{Z}$ such that for all $e \in E$,

- **i.** $|f(e)| < k$
- **ii.** $f(e) = f'(e) \pmod{k}$
- **iii.** $\sum_{v \in V} |\mathsf{inflow}_f(v)|$ is minimized subject to **i.**

By reorienting edges, we may assume that $f(e) > 0$ for all $e \in E$. Define

$$V^+ = \{v \in V \mid \mathsf{inflow}_f(v) > 0\}$$
$$V^- = \{v \in V \mid \mathsf{inflow}_f(v) < 0\}$$

Now suppose that f is not a k-flow, or equivalently, that $V^+ \cup V^- \neq \emptyset$. And since $\sum_{v \in V} \mathsf{inflow}_f(v) = 0$, we then have that $V^+ \neq \emptyset$ and $V^- \neq \emptyset$. Then either

- **I.** There is a dipath P from $u \in V^-$ to $v \in V^+$, or
- **II.** There is a partition (X^-, X^+) of V with $V^- \subset X^-$, $V^+ \subset X^+$ and $\{uv \in E \mid u \in X^-, v \in X^+\} = \emptyset$

<u>Case **I**</u>: Define $\tilde{f}(e) = \begin{cases} f(e)-k & \text{if } e \in P \\ f(e) & \text{else} \end{cases}$. Then for all $e \in E$, $\tilde{f}(e)$ satisfies **i.**, and

$$\sum_{v \in V} |\mathsf{inflow}_{\tilde{f}}(v)| < \sum_{v \in V} |\mathsf{inflow}_f(v)|$$

which contradicts **iii.**, so this case cannot hold.

Case **II**: Simply observe that, as $f > 0$ and no edge enters X^+,

$$0 \geqslant \sum_{v \in X^+} \mathsf{inflow}_f(v) = \sum_{v \in V^+} \mathsf{inflow}_f(v) > 0$$

As this is also a contradiction, case **II** cannot hold. Since none of the cases hold, the assumption that f is not a k-flow was false, hence f is a k-flow. ■

5.3 Flow conjectures and theorems

Definition 5.3.1. A graph $G = (V, E)$ is termed k-edge-connected iff for every $X \subset E$ of size less than k, $G - X$ is connected.

Conjecture 5.3.2. [5-FLOW CONJECTURE - TUTTE]
Every bridgeless graph has a nowhere-zero 5-flow.

Remark 5.3.3. Let G be a graph. Then

- G has a nowhere-zero 2-flow iff G is even
- G has a nowhere-zero $(\mathbb{Z}_2)^k$ flow iff G is the union of k even subgraphs
- A cubic graph has a nowhere-zero 4-flow iff it is 3-edge colorable

The last statement directly implies that the Petersen graph has no nowhere-zero 4-flow.

Conjecture 5.3.4. [4-FLOW CONJECTURE - TUTTE]
Every bridgeless graph with no Petersen graph minor has a nowhere-zero 4-flow.

This conjecture implies the four-color theorem, and has not been proven yet. Robertson, Seymour, and Thomas proved this for cubic graphs, by using the four-color theorem.

Conjecture 5.3.5. [3-FLOW CONJECTURE - TUTTE]
Every 4-edge-connected graph has a nowhere-zero 3-flow.

Theorem 5.3.6. [GROTZSCH]
Every triangle-free planar graph is 3-colorable.

The 3-flow conjecture, if true, implies Grotzsch's theorem.

Lemma 5.3.7. If T is a spanning tree of a graph G, then G has an even subgraph H with $E(G) = E(T) \cup E(H)$.

Proof: Let $f'(e) = 1$ for each $e \in E(G) - E(T)$ and apply a previous exercise. ■

Theorem 5.3.8. [NASH-WILLIAMS, TUTTE] Every 2^k-edge-connected graph has k edge-disjoint spanning trees.

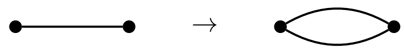
Proof: Apply the matroid intersection theorem. ■

Corollary 5.3.9. Every 4-edge-connected graph has a nowhere-zero 4-flow.

Proof: There exist 2 edge-disjoint spanning trees T_1, T_2, and so there are two even subgraphs H_1, H_2 such that $E(G) = E(H_i) \cup E(T_i)$ for $i = 1, 2$. Since T_1 and T_2 are edge-disjoint, $E(G) = E(H_1) \cup E(H_2)$. ■

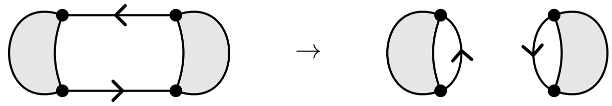
Corollary 5.3.10. [JAEGER]
Every 3-edge-connected graph has a nowhere-zero 8-flow.

Proof: It suffices to find 3 spanning trees such that no edge is contained in all three. Duplicate every edge, that is, apply the map



to every edge in G to get a 6-edge-connected graph. By the theorem, this new graph has 3 edge-disjoint spanning trees. ■

Remark 5.3.11. It is easy to deduce that every bridgeless graph has a nowhere-zero 8-flow. The only seeming problem is 2-edge cuts, but these can be taken care of as follows.



As flow values on the cut shown for the graph on the left are equal, the given problem is equivalent to finding flows on the two bridgeless connected components on the right.

Theorem 5.3.12. [6-FLOW THEOREM - SEYMOUR 1981]
Every bridgeless graph has a nowhere-zero 6-flow.

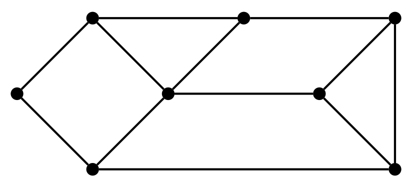
Proof: See below. ■

Definition 5.3.13. Given a bridgeless graph G, a 2-decomposition of G is a sequence $(H_0, \ldots, H_k)$ of bridgeless graphs such that

1. H_0 is even
2. $H_k = G$
3. For each $i = 1, \ldots, k$, H_{i-1} is a subgraph of H_i with $|E(H_i)| - |E(H_{i-1})| = 1$ or 2

Example 5.3.14. This is an example of a 2-decomposition.

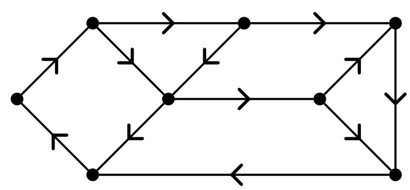
G =



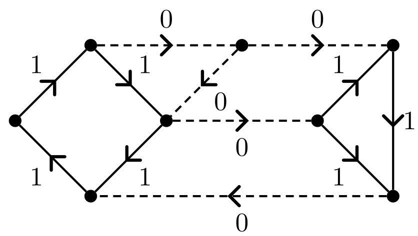
Lemma 5.3.15. Every graph with a 2-decomposition has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow.

Example 5.3.16. Consider the graph from the previous example with the above lemma, by applying an orientation to the edges of G.

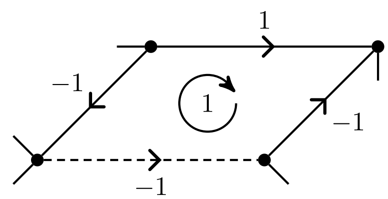
$\vec{G}$ =



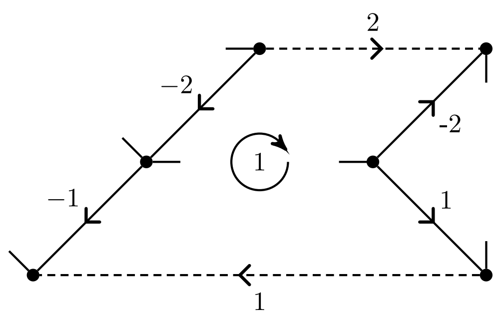
To get a $\mathbb{Z}_2$-flow, simply push a flow of 1 through every connected component of the base even subgraph H_0, and give a flow of 0 to all other edges.



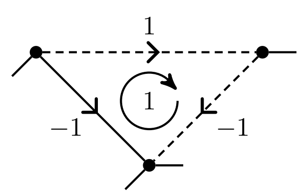
To get a $\mathbb{Z}_3$-flow, first assign all edges a flow of 0. To the last graph in the 2-decomposition of G, push a flow through a cycle containing the edge(s) added to the previous graph, so that (those) edge(s) has (have) non-zero flow. Here we choose a flow of 1 clockwise.



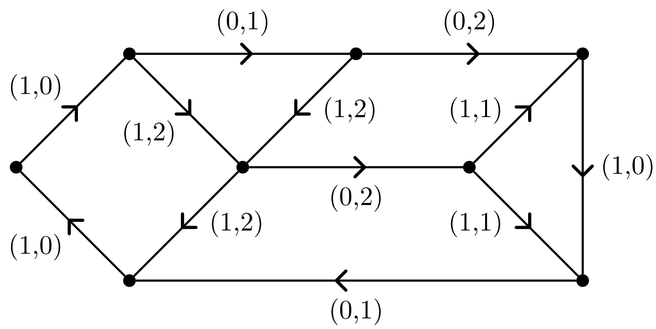
Keep the assigned flows on the edges and look at the previous graph in the decomposition, applying the same operation. Here we choose a flow of 1 clockwise, as a flow of 2 would give the added edge at the top a flow of 0.



Continue in the same manner, by applying a flow of 1 clockwise. A flow of 2 would again give one of the added edges zero flow.



The final flow is then found taking values of the edges most recently changed, and working backward. The result is a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow.



Theorem 5.3.17. [Menger]
If s, t are distinct vertices in a graph G, and $k \in \mathbb{Z}_+$, then either

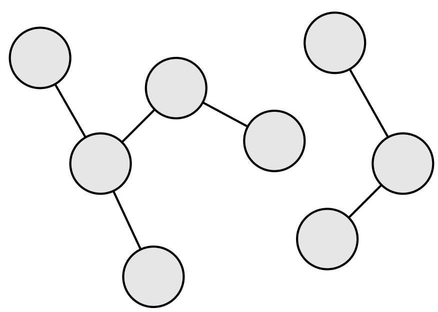
· there exist k edge-disjoint (s,t)-paths in G, or
· there is a partition (S,T) of $V(G)$ with $s \in S$, $t \in T$, and $\delta(S) < k$

Corollary 5.3.18. For any graph G and $k \in \mathbb{Z}_+$, there is a partition $\Pi_k(G)$ of $V(G)$ such that for $u, v \in V(G)$, there are k edge-disjoint (u,v)-paths in G iff u, v are in the same component of $\Pi_k(G)$.

Remark 5.3.19. Consider $\Pi_2(G)$. Note that

· $e = uv \in E(G)$ is a bridge iff u, v are in different components of $\Pi_2(G)$
· For each $X \in \Pi_2(G)$, $G[X]$ is a 2-edge-connected subgraph

$\Pi_2(G)$ then resembles a forest with nodes that are connected graphs:

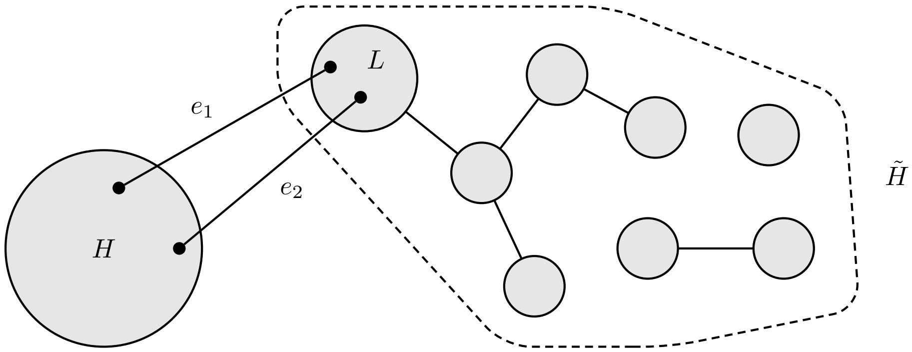


Lemma 5.3.20. Every non-empty graph G has a non-empty 2-edge-connected subgraph H with $|\delta_G(V(H))| \leqslant 1$.

Proof: Let G' be obtained from G by contracting all the non-bridge edges. Then G' is a non-empty forest, and thus has a vertex of degree at most 1. ■

Lemma 5.3.21. Every 3-edge connected graph has a 2-decomposition.

Proof: Let G be a 3-edge-connected graph, with H the largest 2-edge-connected subgraph of G that admits a 2-decomposition. Since H is connected by maximality, H is an induced subgraph of G. Let $(H_0, \ldots, H_k)$ be the 2-decomposition of H. Assume that $H \neq G$, hence $V(H) \neq V(G)$. Let $\tilde{H} = G - V(H)$. Let L be a 2-edge-connected subgraph of $\tilde{H}$, with $|\delta_{\tilde{H}}(V(L))| \leqslant 1$. As G is 3-edge-connected, there exist distinct edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ with $u_1, u_3 \in V(L)$ and $v_1, v_2 \in V(H)$.



Since L is 2-edge-connected, by Menger's theorem there exist 2 edge-disjoint (u_1, u_2)-paths P_1, P_2 in L. Note that $P_1 \cup P_2$ is an even subgraph of L. Then $(H_1 \cup P_1 \cup P_2, \ldots, H_k \cup P_1 \cup P_2, (H_k \cup P_1 \cup P_2) + \{e_1, e_2\})$ is a 2-decomposition. However, $(H_k \cup P_1 \cup P_2) + \{e_1, e_2\}$ is connected and larger than H, a contradiction. ■

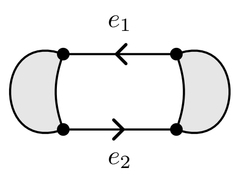
Definition 5.3.22. Let $G = (V, E)$ be a graph with an orientation $\vec{G}$. Then for each $v \in V$ define

- $\mathsf{indeg}_{\vec{G}}(v) = |\mathsf{IN}_{\vec{G}}(v)|$
- $\mathsf{outdeg}_{\vec{G}}(v) = |\mathsf{OUT}_{\vec{G}}(v)|$
- $\mathsf{outflow}_{\vec{G}}(v) = \mathsf{outdeg}_{\vec{G}}(v) - \mathsf{indeg}_{\vec{G}}(v)$

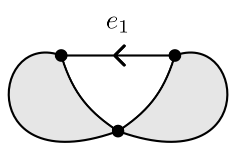
Here $\mathsf{IN}_{\vec{G}}(v)$ is the set of edges in E directed into v, and $\mathsf{OUT}_{\vec{G}}(v)$ is the set of edges directed out from v.

Now we return to prove (5.3.12).

Proof: (of (5.3.12)) Consider a counter-example G with $|E(G)|$ minimal. Thus G is bridgeless and has no nowhere-zero 6-flow. By minimality, G is connected. Moreover, G is bridgeless, so G is 2-edge-connected. Using the previous two lemmae, G is not 3-edge-connected. Hence there exists $X \subset V(G)$ with $\delta_G(X) = \{e_1, e_2\}$, so G resembles



Consider an orientation $\vec{G}$ of G. Note that $\vec{G}/e_2$ is bridgeless and smaller than G (in terms of edge count), so $\vec{G}/e_2$ has a nowhere-zero 6-flow f'.



Hence there is a unique $\mathbb{Z}_6$-flow f in $\vec{G}$ such that $f(e) = f'(e)$ for all $e \neq e_2$. Since G has no nowhere-zero $\mathbb{Z}_6$-flow, we must have that $f(e_2) = 0$. Then $0 = \mathsf{inflow}_f(X) = f(e_1) = f'(e_1) \neq 0$, which is a contradiction. ■

Remark 5.3.23. Consider a nowhere-zero flow f. By reorienting, we can get $f(e) = 1$ for all $e \in E(G)$. Hence G has a nowhere-zero 3-flow iff there is an orientation $\vec{G}$ of G such that $|\mathsf{IN}_{\vec{G}}(v)| = |\mathsf{OUT}_{\vec{G}}(v)| \pmod 3$ for all $v \in V(G)$.

Proposition 5.3.24. [Weak 3-flow conjecture - Jaeger]
There exists $k \in \mathbb{Z}$ such that each k-edge-connected graph has a nowhere-zero 3-flow.

Remark 5.3.25.

· The above conjecture has been proven for:
  $k = 8$ by Thomassen
  $k = 6$ by Lovasz, Zhang, and Thomassen
· The case $k = 5$ implies Tutte's 3-flow conjecture
· We will prove the case $k = 17$

Definition 5.3.26. Let $G = (V, E)$ be a graph. A $\underline{(p \mod 3)\text{-flow}}$ of G is an orientation $\vec{G}$ and a map $p : V \to \mathbb{Z}$ with $\mathsf{outflow}_{\vec{G}}(v) \equiv p(v) \pmod 3$ for all $v \in V$.

Note that G has a nowhere-zero 3-flow iff G has a $(p = 0 \mod 3)$-flow.

Further, for every $A \subset V$, let $p(A) = \sum_{v \in A} p(v)$.

Remark 5.3.27. Let $G = (V, E)$ be a graph. Then for each $v \in V$, specifying outflow is equivalent to specifying out-degree. This follows from the below calculation.

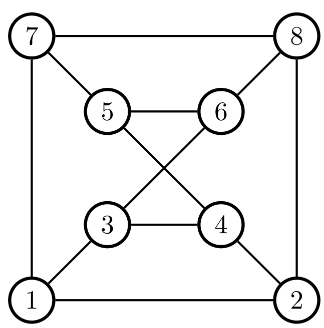
$$\begin{aligned} p(v) &\equiv \mathsf{outflow}_{\vec{G}}(v) \pmod 3 \\ &\equiv \mathsf{outdeg}_{\vec{G}} - \mathsf{indeg}_{\vec{G}}(v) \pmod 3 \\ &\equiv 2\mathsf{outdeg}_{\vec{G}}(v) - \mathsf{deg}_{\vec{G}}(v) \pmod 3 \\ &\equiv -\mathsf{outdeg}_{\vec{G}}(v) - \mathsf{deg}_{\vec{G}}(v) \pmod 3 \end{aligned}$$

Theorem 5.3.28. [Thomassen]
Let G be 17-edge-connected and $p : V(G) \to \mathbb{Z}$ with $p(v) \equiv 0 \pmod 3$. Then G has $(p \mod 3)$-flow.

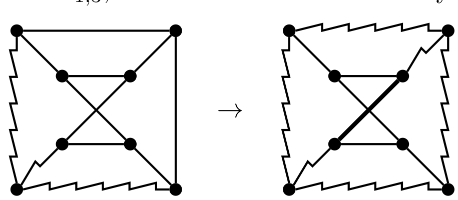
Eqivalently, if G is a 17-edge-connected graph with $d^+ : V \to \mathbb{Z}$ satisfying $\sum_{v \in V} d^+(v) \equiv |E(G)| \pmod 3$, then there is an orientation $\vec{G}$ of G with $\mathsf{outdeg}_{\vec{G}}(v) \equiv d^+(v) \pmod 3$ for each $v \in V$.

Corollary 5.3.29. If G is a 17-edge-connected graph with $|E(G)| \equiv 0 \pmod 3$, then $E(G)$ can be covered by edge-disjoint copies of $K_{1,3}$.

Example 5.3.30. The graph below is vertex-transitive, i.e. every pair of vertices is equivalent under some automorphism of the graph.



This graph cannot be covered by edge-disjoint copies of $K_{1,3}$, as using vertex 1 as the degree 3 vertex of $K_{1,3}$ means for vertex 2 either vertex 4 or 8 has to be the degree 3 vertex of $K_{1,3}$. Using 8, the edge $(3, 6)$ now has both ends taken by copies of $K_{1,3}$, and so cannot be covered by another edge-disjoint $K_{1,3}$.



Proposition 5.3.31. [Thomassen]
For each tree T, there exists $k \in \mathbb{N}$ such that for each k-edge-connected graph with $|E(G)| \equiv 0 \pmod{|E(T)|}$, there is a cover of $E(G)$ by edge-disjoint copies of T.

The above theorem has been proven for stars and for the path with 4 edges, and remains open.

5.4 The proof of the weakened 3-flow theorem

Definition 5.4.1. Let G be a graph and $p : V(G) \to \mathbb{Z})$ a map with $p(v) \equiv 0 \pmod 3$ for all $v \in V(G)$. For $A \subset V(G)$, define $\mathsf{deg}_G(A) = |\delta_G(A)|$. Note that if $\vec{G}$ is a $(p \mod 3)$-flow, then $\mathsf{outflow}_{\vec{G}}(A) \equiv p(A) \pmod 3$. Further, define $\mathsf{type}_G(A)$ to be the unique number $m \in \mathbb{Z}$ such that

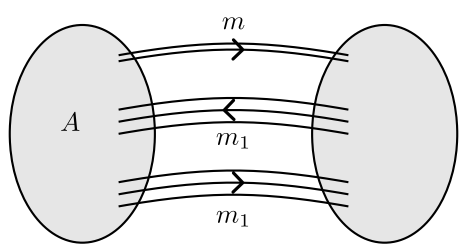
a. $-2 \leqslant m \leqslant 3$
b. $m \equiv \mathsf{deg}(A) \pmod 2$
c. $m \equiv p(A) \pmod 3$

The Chinese remainder theorem guarantees existence of such an m.

Lemma 5.4.2. If $\mathsf{deg}_G(A) \geqslant 3$, then there is an orientation $\vec{G}$ of G with

i. $-2 \leqslant \mathsf{outflow}_{\vec{G}}(A) \leqslant 3$, and
ii. $\mathsf{outflow}_{\vec{G}}(A) \equiv p(A) \pmod 3$

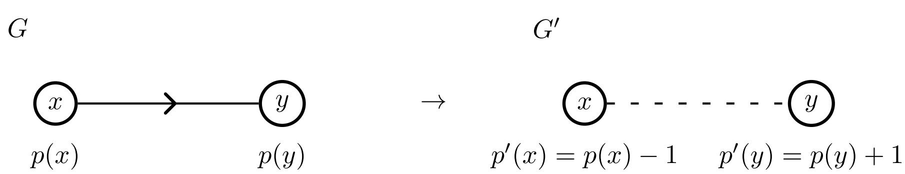
Moreover, for any such orientation, $\mathsf{outflow}_{\vec{G}}(A) = \mathsf{type}_G(A)$. Heuristically, the type is the measure of how unbalanced an orientation of $\delta(A)$ is. There exists some $m_1, m \in \mathbb{N}$ such that $\delta(A)$ has m_1 arcs oriented in both directions, and m additional arcs all in one direction. Then $\mathsf{type}_{\vec{G}}(A) = m$.



Theorem 5.4.3. Let $G = (V, E)$ be a 17-edge-connected graph. If $p : V \to \mathbb{Z}$ is such that $p(v) \equiv 0 \pmod 3$, then G has a $(p \mod 3)$-flow

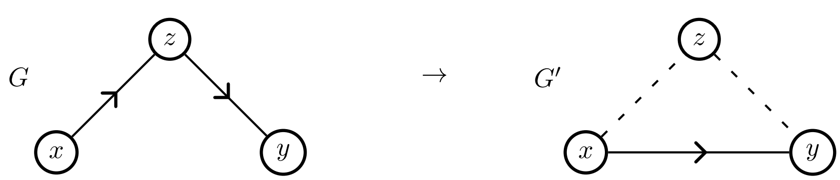
Remark 5.4.4. Before proceeding to the proof of this theorem, consider some reductions that will be applied:

1. Orient and delete an edge:



Now we have a $(p' \mod 3)$-flow in G', given a $(p \mod 3)$-flow in G, such that $\mathsf{type}_{G'}(\{x\}) > 0$ implies $\mathsf{type}_G(\{x\}) - 1$.

2. Splitting off:



Note that the flows along the oriented edges stay the same. So if G' has a (p mod 3)-flow, then so does G. Now we prove a technical lemma, from which the theorem (5.4.3) will follow trivially.

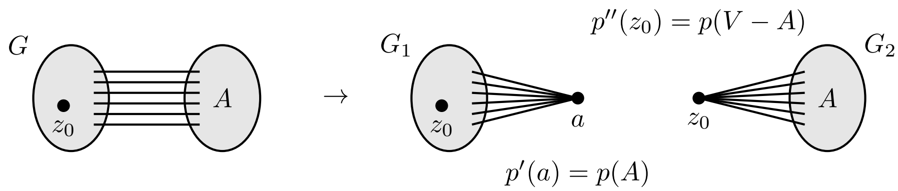
Lemma 5.4.5. Let $G = (V, E)$ be a loopless graph. If $p : V \to \mathbb{Z}$, $z_0 \in V$ are such that
i. $p(V) \equiv 0 \pmod 3$
ii. $\mathsf{deg}(z_0) \leqslant 25$
iii. For each $A \subset V - \{z_0\}$ with $1 \leqslant |A| \leqslant |V| - 2$, $\mathsf{deg}(A) \geqslant 14 + |\mathsf{type}_G(A)|$

Then each orientation of the edges incident with z_0 satisfying $\mathsf{outflow}_G(z_0) = p(z_0) \pmod 3$ extends to a (p mod 3)-flow of G.

Proof: (of (5.4.3)) Add an isolated vertex z_0 with $p(z_0) = 0$, and apply the technical lemma. ■

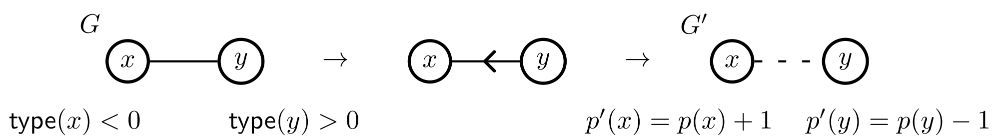
Proof: (of (5.4.5)) Consider a counter-example (G, p, z_0) with $|E(G)|+|V(G)|$ minimal. Consider the following claims.

Claim 1: For each $A \subset V(G) - \{z_0\}$ with $2 \leqslant A \leqslant |V| - 2$, $\mathsf{deg}_G(A) \geqslant 26$.
Proof sketch: Consider the following transformation by edge contraction:



Then G_1 is smaller than G, in terms of number of edges. Choose an orientation of the edges from z_0 to its neighbors in G_1, and extend that to an orientation of G_1. Using the same orientation of the oriented edges around a in G_1 for the edges around z_0 in G_2, orient G_2 completely. Now G may be oriented. This is essentially induction on (G_1, p', z_0) and (G_2, p''', z_0).

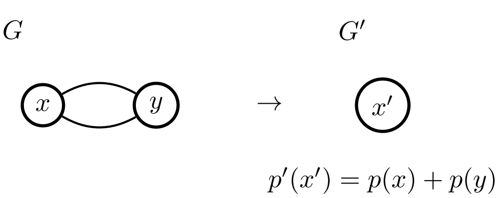
Claim 2: There is no edge $e = xy$ with $\mathsf{type}_G(x) < 0 < \mathsf{type}_G(y)$.
Proof sketch: If such a situation does arise, perform the orient and delete reduction to such an edge.



Then $|\mathsf{type}_{G'}(x)| < |\mathsf{type}_G(x)|$ and $|\mathsf{type}_{G'}(y)| > |\mathsf{type}_G(y)|$. Now apply induction to (G', p', z_0).

For $x, y \in V(G)$, let $\mathsf{mult}(x, y)$ be the number of edges whose ends are x, y.

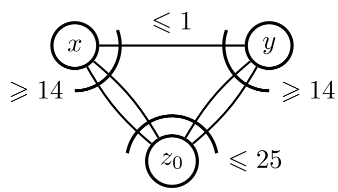
Claim 3: For each $x, y \in V(G) - \{z_0\}$, $\mathrm{mult}(x, y) \leqslant 1$.
Proof sketch: If such a situation does arise, apply the orientation and deletion reduction to each edge, and then identify x and y.



Since there are two edges to be oriented, there are 3 possible ways to orient them, and mod 3, it always possible to get what we want. Now apply induction to (G', p', z_0).

Claim 4: $|V(G)| \geqslant 10$, and for each $x \in V(G) - \{z_0\}$, $\mathsf{mult}(x, z_0) < \mathsf{deg}(x)/2$.

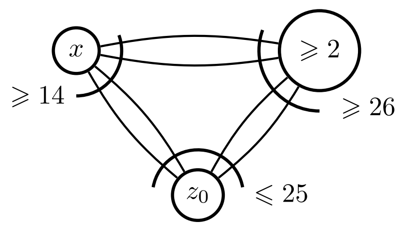
Proof: Clearly $|V(G)| \geqslant 2$. Suppose that $V(G) = \{z_0, x, y\}$. Visually, with the number of edges in each cut indicated, we have



Then it follows that

$$1 \geqslant \mathsf{mult}(x, y) = \frac{\mathsf{deg}(x) + \mathsf{deg}(y) - \mathsf{deg}(z_0)}{2} \geqslant \frac{14 + 14 - 25}{2} > 1$$

Hence $|V(G)| \geqslant 4$. WLOG we then group all the new vertices near y, so the situation looks like:



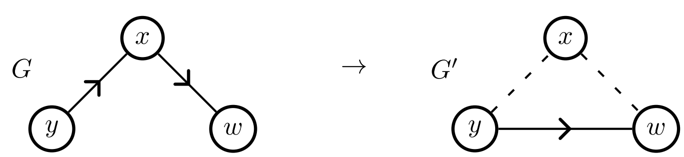
By Claim 1, $\mathsf{deg}(\{z_0, x\}) > \mathsf{deg}(z_0)$, so $\mathsf{mult}(x, z_0) < \frac{\mathsf{deg}(x)}{2}$. And by claim 3, $|V(G)| > |\{z_0, x\}| + \frac{1}{2}\mathsf{deg}(x) \geqslant 9$, so $|V(G)| \geqslant 10$.

Claim 5: For each $x \in V(G) - \{z_0\}$, $\mathsf{deg}(x) = 14 + |\mathsf{type}(x)|$.

Proof sketch: By definition,

$$\mathsf{deg}(x) \equiv \mathsf{type}(x) \pmod 2 \equiv 12 + |\mathsf{type}(x)| \pmod 2$$

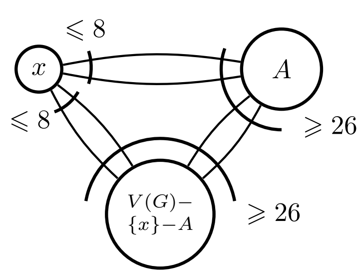
So if $\mathsf{deg}(x) \neq 14 + |\mathsf{type}(x)|$, then $\mathsf{deg}(x) \geqslant 14 + |\mathsf{type}(x)| + 2$, and then we may perform the splitting off reduction, which will reduce the degree of x by 2.



Now apply induction to (G', p, z_0).

Claim 6: For each $x \in V(G) - \{z_0\}$ and $A \subset V(G) - \{z_0\}$ with $2 \leqslant |A| \leqslant |V(G)| - 3$, $\mathsf{deg}_{G-\{x\}}(A) \geqslant 18$.

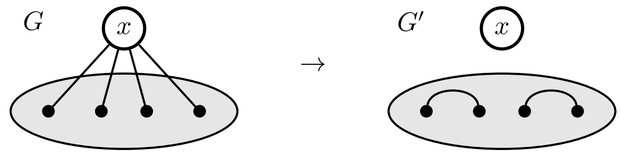
Proof: By claim 5, x has at most 8 edges going to A or to $V(G) - \{x\} - A$. However, by claim 1, $\mathsf{deg}(A) \geqslant 26$ and $\mathsf{deg}(A \cup \{x\}) \geqslant 26$.



This proves the claim.

Claim 7: No vertex $x \in V(G)$ has type 0.

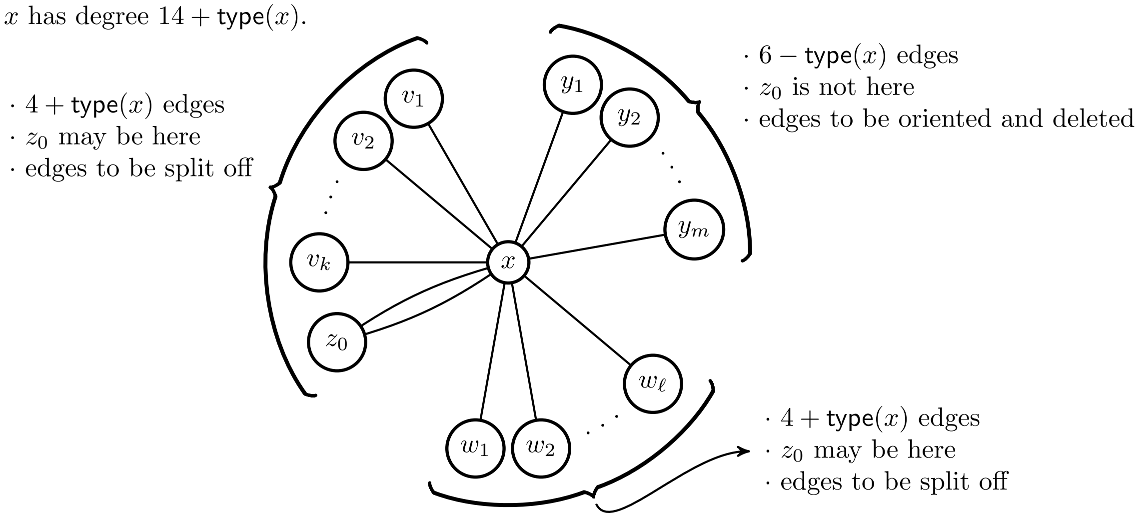
Proof sketch: If some vertex x has type 0, then $\mathsf{deg}(x) \equiv 0 \pmod 2$ and $p(x) \equiv 0 \pmod 3$. Perform a splitting off reduction, by splitting off all edges incident to x without creating loops.



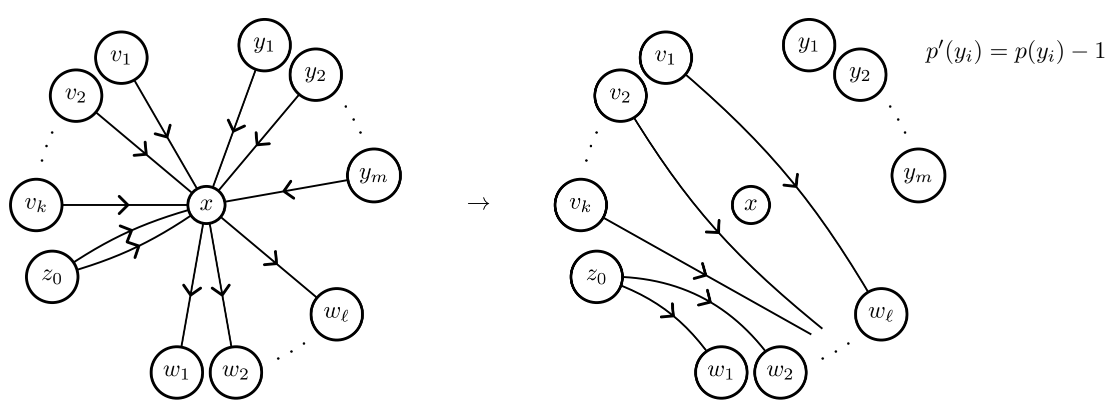
Such a loopless reduction is indeed possible, as the degree of x is even and its demand is zero (so no multiple edges). Now apply induction to (G', p', z_0).

Since $|V(G)| \geqslant 10$ and $\mathsf{deg}(z_0) \leqslant 25$, there exists $x \in V(G) - z_0$ with $\mathsf{mult}(x, z_0) \leqslant 2$. WLOG, we assume that $\mathsf{type}(x) > 0$, by possibly considering $(G, -p, z_0)$. By claims 2 and 7, $\mathsf{type}(y) > 0$ for each neighbor $y \neq z_0$ of x. Now we do a final reduction.

In G, x has degree $14 + \mathsf{type}(x)$.



Orient $y_1, \ldots, y_m$ and $v_1, \ldots, v_k$ toward x, and $w_1, \ldots, w_\ell$ out from x. Then delete $y_1x, \ldots, y_mx$, and split off the other edges. This results in G', where x has degree 0.



Now apply induction to (G', p', z_0). This concludes the proof of (5.4.5). ∎

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