Compact course notes

Combinatorics and Optimization 499 Winter 2012

Various topics

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March 14, 2012

Contents

1 Foundational ideas

1.1 Lie Groups and their algebras

Definition 1.1.1. A group G is termed a Lie group iff its elements may be interpreted as coordinates of a finite-dimensional C^2 manifold. Moreover, the group operations are C^2 maps.
$\mu : G \times G \to G$ given by $(x, y) \mapsto xy$
$\imath : G \to G$ given by $x \mapsto x^{-1}$

Definition 1.1.2. The tangent space of a group G at the identity element 1 or 1_G is the topological tangent space to the manifold equivalent of G, and is denoted by

$$\mathfrak{g} = T_1 G$$

Definition 1.1.3. Define several functions:

$$\begin{array}{rccc} \mathbf{Ad}\ x : & G & \to & G \\ & y & \mapsto & xyx^{-1} \end{array} \qquad \begin{array}{rccc} \mathrm{Ad}\ x : & \mathfrak{g} & \to & \mathfrak{g} \\ & x & \mapsto & T_1(\mathbf{Ad}\ x) \end{array}$$

1.2 Tensors

Necessary to define: symmetrization, symmetric function, permutation

Remark 1.2.1. In this section we assume Einstein notation, and sum over all repeated indeces. If an index is not repeated in an expression, it takes on all values.

Definition 1.2.2. An rank n tensor in m-dimensional space is a mathematical object with m^n components. Tensors are generalizations in rank, and we note that:
· scalars have rank 0
· vectors have rank 1
· matrices have rank 2

Definition 1.2.3. A tensor $\alpha_{a\ldots c}^{d\ldots f}$ with p lower and q upper indeces has valence $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ and rank $p + q$.

Definition 1.2.4. A covariant tensor has lowered indeces, and a contravariant tensor has raised indeces. Lowered indeces themselves are sometimes termed covariant, and raised indeces are termed contravariant.

Definition 1.2.5. There are certain special tensors.
· the Kronecker delta symbol $\delta_{ij} = \delta_i^j$ is a rank 2 tensor such that for any other rank 2 tensor α_i^j we have

$$\delta_i^j \alpha_i^j = \sum_i \alpha_i^i = \sum_j \alpha_j^j$$

· the Levi-Civita symbol ϵ_{ijk} is a rank 3 tensor such tha for any other rank 3 tensor α^{ijk} we have

$$\epsilon_{ijk}\alpha^{ijk} = \sum_{\substack{\text{even permutations} \\ \text{of } (ijk)}} \alpha^{ijk} + \sum_{\substack{\text{odd permutations} \\ \text{of } (ijk)}} \alpha^{ijk} + 0 \cdot \sum_{\substack{(ijk) \text{ has} \\ \text{repeated indeces}}} \alpha^{ijk}$$

Generalized in rank the above becomes the generalized Kronecker delta and the generalized Levi-Civita:

$$\delta_{i_1\ldots i_k}^{j_1\ldots j_k} \alpha_{i_1\ldots i_k}^{j_1\ldots j_k} = \sum_{i_1\ldots i_k} \alpha_{i_1\ldots i_k}^{i_1\ldots i_k} = \sum_{j_1\ldots j_k} \alpha_{j_1\ldots j_k}^{j_1\ldots j_k}$$

$$\epsilon_{i\ldots k}\alpha^{i\ldots k} = \sum_{\substack{\text{even permutations} \\ \text{of } (i\ldots k)}} \alpha^{i\ldots k} + \sum_{\substack{\text{odd permutations} \\ \text{of } (i\ldots k)}} \alpha^{i\ldots k} + 0 \cdot \sum_{\substack{(i\ldots k) \text{ has} \\ \text{repeated indeces}}} \alpha^{i\ldots k}$$

These two tensors are related, but it is not necessarily true that a single Levi-Civita tensor may be replaced by any sort of combination of Kronecker deltas, and vice versa.

$$\epsilon_{i_1\ldots i_k}\epsilon^{j_1\ldots j_k} = n!\delta_{[i_1}^{j_1}\delta_{i_2}^{j_2}\ldots\delta_{i_k]}^{j_k}$$

1.3 Index bracket notation

Indeces may have round or square brackets around them, sometimes spanning several tensors, and never partly overlapping. Round brackets denote a symmetrizer over the included indeces. Square brackets denote an anti-symmetrizer or skew-symmetrizer over the included indeces.

As an example, take the following:

$$\alpha_a^{b[cd}\beta_e^{f]} = \alpha_a^{bcd}\beta_e^f - \alpha_a^{bcf}\beta_e^d - \alpha_a^{bdc}\beta_e^f + \alpha_a^{bfc}\beta_e^d + \alpha_a^{bdf}\beta_e^c - \alpha_a^{bfd}\beta_e^c$$

1.4 Other base definitons

Definition 1.4.1. A function $\Delta : \underbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{n \text{ times}} \to \mathbb{F}$ is termed anti-symmetric iff it is

1. multilinear - linear in each variable
2. alternating - takes negative value if any two inputs switched
3. normalized - $\Delta(e_1, \ldots, e_n) = 1$

Definition 1.4.2. Define the q-binomial coefficient, or Gaussian coefficient to be the following:

$$\binom{m}{r}_q \circeq \begin{cases} \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & r \leq m \\ 0 & r > m \end{cases}$$

2 Algebras

Definition 2.0.3. A function $F : A \to B$ between groups is termed a homomorphism if for all $a, a' \in A$

$$F(a *_A a') = F(a) *_B F(a')$$

Definition 2.0.4. Let R be a ring. A ring A is termed an algebra over R if there exists a ring homomorphism $F : R \to Z(A)$ for $Z(A)$ the center of A, defined by

$$Z(A) \circeq \{a \in A \mid ab = ba \text{ for all } b \in A\}$$

Let V be a vector space over a field $\mathbb{F}$. Then define a map $p : V \otimes V \to V$ to be a product on V. We note that p is linear in each of its arguments.

Further, the pair $\mathfrak{A} = (V, p)$ is termed an algebra of the vector space V endowed with p. This algebra is termed commutative iff the following diagram commutes.

$$\begin{tikzcd} \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} \arrow[r, "\mathrm{id} \otimes p"] \arrow[d, "p \otimes \mathrm{id}"'] & \mathfrak{A} \otimes \mathfrak{A} \arrow[d, "p"] \\ \mathfrak{A} \otimes \mathfrak{A} \arrow[r, "p"'] & \mathfrak{A} \end{tikzcd}$$

An element $1_{\mathfrak{A}}$ of an algebra $\mathfrak{A}$ is termed a unit iff for all $a \in \mathfrak{A}$ it satisfies

$$m(a \otimes 1_{\mathfrak{A}}) = m(1_{\mathfrak{A}} \otimes a) = a$$

We note that if an algebra has a unit, then it must be unique. With such a unit, the algebra is then termed an algebra with unit.

2.1 The symmetric algebra

Definition 2.1.1. Given a vector space V over a field $\mathbb{F}$, the tensor algebra over V is defined to be

$$T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

Every element of this algebra is strictly within some $V^{\otimes n}$ for $n \in \mathbb{N} \cup \{0\}$.

Definition 2.1.2. The symmetric group, or permutation group of order n is the group with elements that are ordered permutations of the sequence $\{1, 2, \ldots, n\}$. It is denoted by S_n or $\mathfrak{S}_n$.

Definition 2.1.3. The symmetric algebra over a vector space V over a field $\mathbb{F}$ is defined to be the space

$$S(V) \doteq \bigoplus_{k \geqslant 0} S(V)^k = \bigoplus_{k \geqslant 0} \frac{T(V)^{\otimes k}}{\mathfrak{S}_n}$$

where $T(V)^{\otimes k}$ is the set of elements in $T(V)$ that may be expressed as k-dimensional tensors. The quotient of this space by the symmetric group ensures that the tensor product is commutative.