Compact course notes

Combinatorics and Optimization 749, Winter 2013

Topological Methods in Graph Theory and Combinatorics

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0 Background

Definition 0.0.1. Let G be a graph and $P = V_1 \cup \cdots \cup V_m = V(G)$ be a partition of G. Then a set $S = \{v_1, \ldots, v_m\} \subseteq V(G)$ is termed an independent transversal of G with respect to P if:

1. S is independent (i.e. the graph $G[S]$ has no edges)
2. $v_i \in V_i$ for each i

Theorem 0.0.2. Let G be a graph and P a vertex partition of G. Suppose that $|V_i| = 2\Delta$ for all i, where $\Delta = \Delta(G)$, the maximum degree of G. Then G has an independent transversal with respect to P.

0.1 Elementary background

Definition 0.1.1. A set $\{v_0, \ldots, v_k\}$ of points in $\mathbb{R}^d$ is termed affinely dependent iff there exist $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ not all zero, such that $\alpha_0 v_0 + \cdots + \alpha_k v_k = 0$ and $\alpha_0 + \cdots + \alpha_k = 0$.

Example 0.1.2. Using the above definition, it follows that for $\{v_0, \ldots, v_k\}$ affinely dependent,

$$\begin{array}{ll} k=1 & \text{implies } v_0 = v_1 \\ k=2 & \text{implies } v_0, v_1, v_2 \text{ are on the same line} \\ k=3 & \text{implies } v_0, v_1, v_2, v_3 \text{ are on the same plane} \end{array}$$

A set is affinely independent iff it is not affinely dependent, or equivalently, if

· $\{v_1 - v_0, \ldots, v_k - v_0\}$ is linearly independent
· $\{(1, v_0), \ldots, (1, v_k)\} \subset \mathbb{R}^{d+1}$ is linearly independent

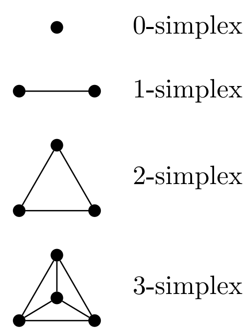
It follows that the maximum size of an affinely independent set in $\mathbb{R}^d$ is $d+1$.

Definition 0.1.3. Given a set of points $S = \{v_0, \ldots, v_k\}$, the convex hull of the given set is the set of points

$$\operatorname{conv}(S) = \{\gamma_0 v_0 + \cdots + \gamma_k v_k \ : \ \gamma_i \geqslant 0, \sum_{i=0}^{k} \gamma_i = 1\}$$

Definition 0.1.4. A simplex σ is the convex hull of a set $A = \{v_0, \ldots, v_k\}$ of affinely independent points in $\mathbb{R}^d$. We say that $v_0, \ldots, v_k$ are the vertices of σ, and k is the dimension of σ.

These are some of the fundamental simplices:



The empty set is termed *the* (-1)-simplex.

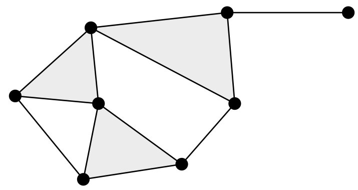
A face of σ is the convex hull of $B \subseteq A$. In particular, every simplex has the empty set as a face.

The relative interior of σ is the set of points $\sigma - \bigcup\{\tau \ : \ \tau \text{ is a proper face of } \sigma\}$. Hence every point in σ is in the relative interior of exactly one face of σ.

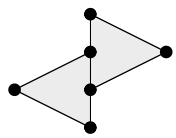
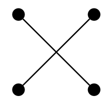
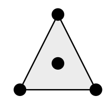
Definition 0.1.5. A set Σ of simplices is termed a geometric simplicial complex if for every $\sigma, \tau \in \Sigma$,

· every face of σ is in Σ
· $\sigma \cap \tau$ is a face of both σ and τ

Example 0.1.6. This is an example of a geometric simplicial complex in $\mathbb{R}^2$. The shaded simplices are 2-simplices.



The following are not examples of geometric simplicial complexes.

The first fails because the shared face is not a face of either. The second fails because the intersection of the two 1-simplicies does not form a face. The third fails because the intersection of the 2-simplex and 0-simplex, the 0-simplex, is not a face of the 2-simplex.

Definition 0.1.7. Given a geometric simplicial complex Σ, define the polyhedron and dimension of Σ by

$$\begin{aligned} \|\Sigma\| &= \bigcup\{\sigma \ : \ \sigma \in \Sigma\} \\ \dim(\Sigma) &= \max\{\dim(\sigma) \ : \ \sigma \in \Sigma\} \end{aligned}$$

For $k \leqslant \dim(\Sigma)$, the subcomplex of Σ consiting of all r-dimensional simplices, for $r \leqslant k$, is termed the k-skeleton of Σ, and denoted by $\Sigma^{\leqslant k}$.

Example 0.1.8. The 0-skeleton minus the empty set is the set of all vertices of Σ, termed the vertex set, and denoted by $V(\Sigma)$.

Note that a 1-dimensional simplicial complex in $\mathbb{R}^2$ consists of vertices and line segments, hence it is a plane graph (i.e. a planar graph with a planar embedding).

Definition 0.1.9. Given a simplicial complex Σ, the relative interiors of all simplices in Σ partition $\|\Sigma\|$. For $x \in \|\Sigma\|$, we call the unique σ that contains x in its relative interior, the support of x, denoted $\text{supp}(x)$.

Proposition 0.1.10. A simplex $\sigma \in \mathbb{R}^d$ together with all its faces forms a simplicial complex.

Proof: Clearly every face of σ is in σ, so it remains to show that the intersection property is satisfied.

Let $A = \{v_0, \ldots, v_k\}$ be the vertex set of σ. Let τ_1, τ_2 be faces of σ with vertex sets $B, C \subseteq A$, respectively. Hence we wish to show that $\text{conv}(B) \cap \text{conv}(C) = \text{conv}(B \cap C)$.

It is clear that $\text{conv}(B \cap C) \subseteq \text{conv}(B) \cap \text{conv}(C)$, so let $x \in \text{conv}(B) \cap \text{conv}(C)$. Therefore there exist β_i, γ_i such that

$$x = \sum_{v_i \in B} \beta_i v_i = \sum_{v_i \in C} \gamma_i v_i \quad \text{for} \ \sum_i \beta_i = \sum_i \gamma_i = 1$$

It follows that

$$\sum_{v_i \in B \setminus C} \beta_i v_i - \sum_{v_i \in C} \gamma_i v_i - \sum_{v_i \in B \cap C} (\gamma_i - \beta_i) v_i = 0$$

Note that the index set is $(B \cup C) \subseteq A$, and that A is affinely independent. Since $\sum \beta_i - \sum \gamma_i = 0$, we must have that $\beta_i = \gamma_i = 0$ for all i, unless $v_i \in B \cap C$. Hence $x \in \text{conv}(B \cap C)$. ■

0.2 Triangulations

Definition 0.2.1. Let X be a set of points in $\mathbb{R}^d$. A geometric simplicial complex Σ in $\mathbb{R}^d$ is termed a triangulation of X iff $\|\Sigma\| \cong X$, where $\cong$ denotes homeomorphism.

Example 0.2.2. These are examples of triangulations of 2-dimensional objects.

Proposition 0.2.3. Let Σ be a triangulation of a simplex σ in $\mathbb{R}^d$. Let τ be a $(d-1)$-simplex of Σ that is not contained in $\partial(\sigma)$, the boundary of σ. Then τ is a proper face of exactly 2 d-simplices in Σ. If τ is contained in $\partial\sigma$, then it is a proper face of exactly 1 d-simplex in Σ.

0.3 Abstract simplicial complices

Definition 0.3.1. An abstract simplicial complex is a set $\mathcal{A}$ of subsets of a (finite) set V, such that if $B \subset A \in \mathcal{A}$, then $B \in \mathcal{A}$.

· The sets $A \in \mathcal{A}$ are termed simplices
· The dimension of $A \in \mathcal{A}$ is $|A| - 1$
· $\mathcal{A}$ is termed pure iff every maximal simplex in $\mathcal{A}$ has the same dimension.

Unless $\mathcal{A} = \emptyset$, $\emptyset \in \mathcal{A}$. Moreover, $\emptyset$ is the unique simplex of dimension -1.

Example 0.3.2.
· If Σ is a geometric simplical complex, then all sets of the vertices of simplices in Σ form an abstract simplicial complex.
· A simple graph G with no isolated vertices is a pure simplicial complex of dimension 1
· A matroid is a pure simplicial complex (with more restrictions)

Remark 0.3.3. Note that a pure simplicial complex $\mathcal{A}$ of dimension d is determined by the set $\mathcal{A}^d$ of all d-dimensional simplices in $\mathcal{A}$. Here, $\mathcal{A}^d$ is the $(d+1)$-uniform hypergraph on V, meaning its elements are $(d+1)$-subsets of V.

Moreover, $\mathcal{A}$ is termed the (downward) closure of $\mathcal{A}^d$.

The boundary of a pure simplicial complex $\mathcal{A}$ of dimension d, expressed as $\partial\mathcal{A}$, is the closure of

$$\{B \subset V \ : \ |B| = d, |\{A \in \mathcal{A}^d \ : \ B \subset A\}| = 1 \pmod 2\}$$

If $\partial\mathcal{A} = \emptyset$, then $\mathcal{A}$ is termed a d-sphere.

Example 0.3.4. A set of vertices of even size is a 0-sphere. This folows as we need $\emptyset$ to be contained is an even number of them.

A convenient way to define $\partial\mathcal{A}$ is to create a matrix $\mathcal{A}[d, d+1]$, with rows indexed by the set of all d-subsets of V, and columns indexed by $\mathcal{A}^d$. Then the (B, A)-entry is 1 if $B \subset A$, and 0 otherwise. Moreover, $\partial\mathcal{A}$ is the closure of the d-uniform hypergraph whose characteristic vector contains elements that are all $\mathcal{A}[d, d+1]$ over $\mathbb{F}_2$.

0.4 Basic properties of spheres and boundaries

Lemma 0.4.1. Let $\mathcal{A}$ be a pure simplicial complex of dimension d. Then $\partial(\mathcal{A})$ is a $(d-1)$-sphere.

Proof: Let $v = \mathcal{A}[d, d+1] \cdot \mathbb{1}$. Then v is the characteristic vector of $(\partial(\mathcal{A}))^{d-1}$, the $(d-1)$-simplices of the boundary of $\mathcal{A}$. Let $\mathcal{K}$ denote the pure simplicial complex of dimension $d-1$ that is the closure of the complete d-uniform hypergraph on $V = V(\mathcal{A})$. Consider $\mathcal{K}[d-1, d] \cdot v$, which is represented as

$$\begin{array}{r} \text{all } (d-1)\text{-} \\ \text{subsets} \\ \text{of } V \end{array} \left[\begin{array}{c} \\ \\ \\ \end{array} \right.$$

Lemma 0.4.2. [Hole-filling lemma] Let $\mathcal{B}$ be a pure siplicial complex of dimension $d+1$, and let $B \in \mathcal{B} \setminus \partial\mathcal{B}$ be a simplex. Let $\mathcal{L} = \mathcal{L}_{\mathcal{B}}(B)$, and set $b = |B|$. Note that $\mathcal{L}$ is a $(d-b+1)$-sphere. Let $\mathcal{C}$ be a pure simplicial complex of dimension $d-b+2$ with $\partial\mathcal{C} = \mathcal{L}$.

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