Linear Algebra

CHEAP PROPS

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Contents

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1. Matrices and Vectors

1.1. **Basic principles.**

Proposition **1.1.1.** *If d is a lead column in an augmented matrix $[A|d]$, then there exists no solution to $Ax = d$ for a vector x.*

Proposition **1.1.2.** *If d is not a leading column, and all columns of A are leading, we have*

$$[A|d] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & \cdots & 0 & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & d_n \end{array}\right] \Longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

Proposition **1.1.3.** *For $[A|d]$, if $d = 0$, then $Ax = 0$ is homogeneous.*
- *If all columns of A are leading, the equation has the trivial solution.*
- *If some non-leading columns, non-trivial solution.*

Proposition **1.1.4.** *If a homogeneous system $Ax = 0$ has more unknowns than equations, then the system has a non-trivial solution.*

1.2. **The algebra of matrices.**

Proposition **1.2.1.** *If $A \xrightarrow[ops]{row} C$ is reduced echelon format and $C \neq I$, then A is not invertible.*

Proposition **1.2.2.** *If A is invertible, then A has only one inverse.*

Proposition **1.2.3.** *If A, B have inverses A^{-1}, B^{-1}, then AB has inverse $(AB)^{-1} = A^{-1}B^{-1}$.*

Proposition **1.2.4.** *If $E \in M_n(\mathbb{F})$ is elementary and B is any $n \times p$ matrix, then EB is the matrix obtained by applying to B the same row operations that were applied to I to get E.*

Proposition **1.2.5.** *Elementary matrices are invertible and have elementary matrices as inverses.*

Proposition **1.2.6.** *The following properties for an $n \times n$ matrix are equivalent:*
- *$BA = I$ for some $n \times n$ matrix B*
- *$Ax = 0$ only has the trivial solution*
- *$A \xrightarrow[ops]{row} I$*
- *A is a product of elementary matrices*
- *A^{-1} exists*

Proposition **1.2.7.** *If $B = A^{-1}$, then $BA = I = AB$.*

Definition **1.2.8.** *The transpose of a matrix $A = [a_{ij}]$ is $A^t = [a_{ji}]$. Also, $(AB)^t = B^t A^t$.*

Definition **1.2.9.** *The trace of an $n \times n$ matrix $A = [a_{ij}]$ is defined as $\sum_{j=1}^{n} a_{jj}$, or the sum of its diagonal values. The following properties hold for any equal size matrices A, B:*
- *$trace(AB) = trace(BA)$*
- *$trace(A + B) = trace(A) + trace(B)$*
- *$trace(\lambda A) = \lambda \cdot trace(A)$*

2. Vector Spaces

2.1. **Theory.**

Definition **2.1.1.** *A vector space is a set V closed under the following operations:*

$$
\begin{array}{llrcl}
\cdot\textit{Addition:} & +: & V \times V & \to & V \\
 & & (x,y) & \mapsto & x+y \\
\cdot\textit{Scaling}: & \cdot: & \mathbb{F} \times V & \to & V \\
 & & (\lambda, x) & \mapsto & \lambda x
\end{array}
$$

Proposition **2.1.2.** *If W is a subset of V over $\mathbb{F}$ with the following three closure properties:*
- $\cdot\ 0 \in W$
- $\cdot\ u, v \in W \Longrightarrow u + v \in W$
- $\cdot\ u \in W, \lambda \in \mathbb{F} \Longrightarrow \lambda u \in W$

Then W is a vector space over $\mathbb{F}$ also, and is termed a subspace of V.

Definition **2.1.3.** *Let A be any $m \times n$ matrix. Then $W = \{x \in \mathbb{F}^n : Ax = 0\}$ is a subspace of $\mathbb{F}^n$, termed the nullspace of A.*

Definition **2.1.4.** *If $v_1, \ldots, v_n \in V$, then the set of all linear combinations of $v_1, \ldots, v_n$ is a subspace of V, termed the span of $v_1, \ldots, v_n$. Thus $W = span\{v_i, \ldots, v_n\}$.*

Proposition **2.1.5.** *Let V be spanned by $v_1, \ldots, v_n$ and take any list $u_1, \ldots, u_m$ in V. If $m > n$, there exist scalars $\lambda_1, \ldots, \lambda_m$ not all zero such that $\lambda_1 u_1, \ldots, \lambda_m u_m = 0$.*

2.2. **Linear independence.**

Proposition **2.2.1.** *A list of vectors $v_1, \ldots, v_n$ that has $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ not all zero and $\lambda_1 v_1, \ldots, \lambda_n v_n = 0$ is termed linearly dependent. If the only way to have $\lambda_1 v_1, \ldots, \lambda_n v_n = 0$ is for all $\lambda_j = 0$ for $1 \leqslant j \leqslant n$, then the list is linearly independent.*

Definition **2.2.2.** *A list $x_1, \ldots, x_n$ is a basis of V provided that*
- $\cdot\ x_1, \ldots, x_n$ *spans V*
- $\cdot\ x_1, \ldots, x_n$ *is linearly independent*

Proposition **2.2.3.** *If $x_1, \ldots, x_n$ is a basis for V and $x \in V$, then there is only one way to write $x = \lambda_1 x_1, \ldots, \lambda_n x_n$.*

Proposition **2.2.4.** *Let V be finite dimensional, and W a subspace of V, Then W is finite dimensional and $dim(W) \leqslant dim(V)$.*

Proposition **2.2.5.** *Let $dim(V) = n$, and take $x_1, \ldots, x_n \in V$. Then $x_1, \ldots, x_n$ is linearly independent if and only if $x_1, \ldots, x_n$ spans V.*

2.3. **Rank and basis of a space.**

Definition **2.3.1.** *Let $A = [x_1, \ldots, x_n]$ be an $m \times n$ matrix. The space spanned by the columns of A is termed the column space of A, and the dimension of the column space of A is termed the column rank of A.*

Proposition **2.3.2.** *The row rank of A is the column rank of A. This number is termed the rank.*

Proposition **2.3.3.** *Suppose $x_1, \dots, x_n$ is a basis for V and $y_1, \dots, y_n$ is another list in V. Express each $y_j = \sum_{i=1}^{n} p_{ij} x_i$ for $p_{ij} \in \mathbb{F}$.*

Then the matrix $P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$ is the matrix that writes $y_1, \dots, y_n$ in terms of $x_1, \dots, x_n$.

Proposition **2.3.4.** *With regard to above, the list $y_1, \dots, y_n$ is a basis for V if and only if P^{-1} exists.*

Proposition **2.3.5.** *Let $x_1, \dots, x_n$, $y_1, \dots, y_n$, and $z_1, \dots, z_n$ be 3 bases for V. Let P write the y_i in terms of the x_i, and let Q write the z_i in terms of the y_i. Then PQ writes the z_i in terms of the x_i.*

Proposition **2.3.6.** *Let V, W be vector spaces with $x_1, \dots, x_n$ a basis for V and $y_1, \dots, y_n$ any list in W. Then there is exactly one linear map $T : V \to W$ such that $T(x_j) = y_j$.*

Proposition **2.3.7.** [Lagrange Interpolation] *Suppose V is the vector space of polynomials in t of degree at most n, so $dim(V) = n + 1$. Given the standard basis $1, t, \dots, t^n$ of V and a list of distinct scalars $a_0, a_1, \dots, a_n \in \mathbb{F}$, another basis of V is*

$$\{p_n(t)\} \qquad \text{such that} \qquad p_j(t) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(t - a_i)}{(a_j - a_i)} \qquad \text{where} \qquad p_j(a_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Given another list of scalars $b_0, b_1, \dots, b_n \in \mathbb{F}$, the unique polynomial of degree at most $n - 1$ that satisfies $b_j = f(a_j)$ is

$$f(t) = b_0 p_0(t) + b_1 p_1(t) + \cdots + b_n p_n(t)$$

3. Linear Transformations

3.1. Kernel and range.

Definition **3.1.1.** *Let V, W be vector spaces and $T : V \to W$ linear. Then*

- *$ker(T) = \{x \in V : T(x) = 0\}$*
- *$range(T) = \{y \in W : y = T(x)$ for some $x \in V\}$*

Proposition **3.1.2.** *$T : V \to W$ is one-to-one if and only if $ker(T) = \{0\}$.*

Definition **3.1.3.** *A linear transformation that is one-to-one and onto is termed an isomorphism.*

Proposition **3.1.4.** [Dimension Theorem] *Let V, W be vector spaces and $T : V \to W$ linear. If V is finite dimensional, then $dim(V) = dim(ker(T)) + dim(range(T))$.*

Proposition **3.1.5.** *Let V, W be vector spaces and $T : V \to W$ linear. If $dim(V) = dim(W)$ are finite, then T is one-to-one if and only if T is onto.*

Proposition **3.1.6.** *Let $T : V \to W$ be linear with $x_1, \dots, x_n$ a basis for V, and $y_1, \dots, y_m$ a basis for W. Let A be the matrix of T using these bases.*

$$\left.\begin{array}{l} \text{Let } z \in V \text{ have coordinate vector } \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \text{ using } x_1, \dots, x_n. \\ \text{Let } T(z) \in W \text{ have coordinate vector } \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \text{ using } y_1, \dots, y_m. \end{array}\right\} \text{ Then } A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}.$$

Proposition **3.1.7.** *Let $T : V \to W$ be linear with $x_1, \ldots, x_n$ a basis for V, and $y_1, \ldots, y_m$ a basis for W. Let A be the matrix of T using these bases.*

· *For $x \in V$, $x \in ker(T) \iff$ the coordinate vector $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in$ the nullspace of A.*

· *For $y \in W$, $y \in range(T) \iff$ the coordinate vector $\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \in$ the column space of A.*

Observation **3.1.8.** *With reference to the above definitions:*
· *nullity(T) = dim(ker(T))*
· *rank(T) = dim(range(T))*

Definition **3.1.9.** [Composition] *Let $V \overset{T}{\to} W \overset{S}{\to} Z$ with T, S linear using bases $v_1, \ldots, v_n$ for V, $w_1, \ldots, w_m$ for W, and $z_1, \ldots, z_p$ for Z. Let A be the matrix of T using the given bases for V and W. Let B be the matrix for S using the given bases for W and Z. Then the matrix for $S \circ T$ is BA.*

3.2. **Linear operators.**

Definition **3.2.1.** *The set $\mathcal{L}(V, W)$ of linear operators $T : V \to W$ is a vector space. Then there exists a linear isomorphism $\phi : \mathcal{L}(V, W) \to M_{m \times n}(\mathbb{F})$ where m = dim(W) and n = dim(V).*
· *$\mathcal{L}(V, V) = \mathcal{L}(V)$ is a ring (and a vector space).*
· *$M_{n \times n}(\mathbb{F}) = M_n(\mathbb{F})$ is a ring.*

Definition **3.2.2.** *The space $\mathcal{L}(V, \mathbb{F}) = V^*$ of linear operators is termed the dual space of V. The linear maps $\alpha : V \to \mathbb{F}$ are termed linear functionals. Then the dual space is the space of linear functionals on V.*
· *A basis $\alpha_1, \ldots, \alpha_n$ of V^* is dual to a basis $x_1, \ldots, x_n$ of V.*

Proposition **3.2.3.** *Let V be a vector space with bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, and let $T : V \to V$.*

$$\left.\begin{array}{l} \text{Let } A \text{ be the matrix of } T \text{ using } x_1, \ldots, x_n. \\ \text{Let } B \text{ be the matrix of } T \text{ using } y_1, \ldots, y_n. \\ \text{Let } P \text{ be the matrix that writes } y_1, \ldots, y_n \text{ in terms of } x_1, \ldots, x_n. \end{array}\right\} \text{Then } B = P^{-1}AP.$$

Definition **3.2.4.** *A linear operator $T : V \to V$ is nilpotent of order n if $T^n(x) = 0$ but $T^{n-1}(x) \neq 0$ for any vector $x \in V$.*

Definition **3.2.5.** *A linear operator $T : V \to V$ is termed a projection when $T^2 = T$.*

Proposition **3.2.6.** *Every linear operator on a finite-dimensional space V over $\mathbb{F}$ is the root of some polynomial in $\mathbb{F}[t]$.*

4. Determinants

4.1. **Permutations.**

Definition **4.1.1.** *Let $L_n = \{1, 2, \ldots, n\}$ be the set of n letters. A permutation of L_n is a bijection $\sigma : L_n \to L_n$. A full description is $\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$.*

Definition **4.1.2.** *Every permutation can be factored into distinct cycles. Also, every permutation can be factored into 2-cycles, known as transpositions, which are not unique.*

Definition **4.1.3.** *With reference to the permutation σ on $p = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$, the parity of σ is even if $\sigma(p) = p$, and odd if $\sigma(p) = -p$.*

Definition **4.1.4.** *The sign of σ is denoted as $sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$*
This function has the following properties:
- $sgn(\sigma \circ \tau) = sgn(\sigma) \circ sgn(\tau)$
- $sgn(\sigma^k) = (sgn(\sigma))^k$

Proposition **4.1.5.** *Every transposition is odd.*

Proposition **4.1.6.** *If A_n (an alternating group) is the set of all even permutations on L_n, and B_n is the set of all odd permutations on L_n, the number of elements in A_n is the number of elements in B_n, with $n > 1$. Also, $A_n + B_n = S_n$, the set of all permutations on n letters.*

Definition **4.1.7.** *If σ is a permutation on n letters, then the signed σ-product for some $n \times n$ matrix A is $sgn(\sigma) = a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}$. Then $det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}$.*

4.2. **Properties.**

Proposition **4.2.1.** *If an $n \times n$ matrix A is upper triangular, then $det(A)$ is the product of all the elements on the diagonal of A.*

Proposition **4.2.2.** [Multilinearity] *Let $r_1, r_2, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n$ be fixed row vectors in $\mathbb{F}^n$. Then $T_k : \mathbb{F}^n \to \mathbb{F}$, defined by $x \mapsto det \begin{bmatrix} r_1 \\ \vdots \\ r_{k-1} \\ x \\ r_{k+1} \\ \vdots \\ r_n \end{bmatrix}$ is linear.*

Proposition **4.2.3.** [Alternating] *If two rows of some $n \times n$ matrix A are equal, then $det(A) = 0$.*

Proposition **4.2.4.** *If $A \xrightarrow[ops]{row} B$, then for each row operation:*
- *If $R_i \leftrightarrow R_j$, then $det(A) = -detB$.*
- *If $\lambda R_i \to R_i$, then $det(A) = \frac{1}{\lambda} det(B)$.*
- *Else $det(A) = det(B)$.*

Proposition **4.2.5.** *A is invertible if and only if $det(A) \neq 0$.*

Proposition **4.2.6.** *For two square matrices A and B,* $\begin{aligned} det(AB) &= det(A)det(B) \\ det(A) &= det(A^t) \end{aligned}$

Definition **4.2.7.** *The (ij)-minor of some $n \times n$ matrix A is defined to be the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column.*

Definition **4.2.8.** *For some matrix A, the classical adjoint is $A^{adj} = [\alpha_{ij}]$ such that $\alpha_{ij} = (-1)^{i+j} det((A^t)_{ij})$, where A_{ij} denotes the (ij) minor of A. Then we have*
- $A^{-1} = \dfrac{1}{det(A)} A^{adj}$

Proposition **4.2.9.** [Laplace Expansion] *To find the determinant of an $n \times n$ matrix A, we can deconstruct it into smaller $(n-1) \times (n-1)$ determinants by the following formula:*

- $det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$

Proposition **4.2.10.** *If* $B = \begin{bmatrix} b_{11} & \cdots & b_{1(n-1)} & b_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ b_{n-1} & \cdots & b_{(n-1)(n-1)} & b_{(n-1)n} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$, *then* $det(B) = det(B_{nn})$.

Proposition **4.2.11.** *If two matrices A, B both represent some linear operator T, then* $det(XI - A) = det(XI - B)$.

5. Vector space characteristics

5.1. Similarity.

Definition **5.1.1.** *A matrix A is similar to a matrix B if there exists some invertible matrix P such that $B = P^{-1}AP$.*

Proposition **5.1.2.** *Similar matrices share the following properties:*

- *Determinant*
- *Rank*
- *Characteristic polynomial*
- *Eigenvalues*
- *Trace*
- *Minimal polynomial*
- *Eigenspace dimension of a common eigenvalue*

Proposition **5.1.3.** *Similar matrices represent the same linear operator with different bases.*

Proposition **5.1.4.** *Every $n \times n$ matrix over $\mathbb{C}$ is similar to an upper triangular matrix.*

5.2. Direct sums.

Definition **5.2.1.** *A vector space V is called the direct sum of U and W if U and W are subspaces of V such that $U \cap W = \{0\}$ and $U + W = V$. Then V as the direct sum of U and W is denoted by $V = U \oplus W$.*

Definition **5.2.2.** *A set W is a sum of $W_1, W_2, \ldots, W_k$ provided for each $x \in W$ there are unique $x_j \in W_j$ such that $x = x_1 + \cdots + x_k$.*

Proposition **5.2.3.** *A sum $W = W_1 + W_2 + \cdots + W_k$ of subspaces of some vector space V is direct if and only the only way to obtain $0 = w_1 + \cdots + w_k$ for $w_i \in W_i$ is by having all $w_i = 0$.*

Proposition **5.2.4.** *If $W = W_1 \oplus \cdots \oplus W_k$, and if the W_i have bases β_i, then the set $\beta_1, \ldots, \beta_k$ is a basis for W.*

Proposition **5.2.5.** *Let $T : V \to V$ be linear with V over $\mathbb{F}$, and $\lambda_1, \ldots, \lambda_k$ be distinct scalars in $\mathbb{F}$. then the sum $W = ker(T - \lambda_1 I) + \cdots + ker(T - \lambda_k I)$ is direct.*

5.3. **Eigenvalues.**

<u>*Definition*</u> **5.3.1.** *Given a linear operator $T : V \to W$, an eigenvalue for T is a scalar $\lambda \in \mathbb{F}$ such that $T - \lambda I$ has no inverse.*

<u>*Definition*</u> **5.3.2.** *If V is finite-dimensional over $\mathbb{C}$, and $T : V \to V$ is linear, then some matrix*

$$V = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

that is upper triangular, will represent T using somes basis for V. That is, there is a basis $x_1, \ldots, x_n$ of V such that

$T(x_1) = a_{11}$
$T(x_2) = a_{12}x_1 + a_{22}x_2$
$T(x_3) = a_{13}x_1 + a_{23}x_2 + a_{33}x_3$ *etc.*

<u>*Definition*</u> **5.3.3.** *The trace of a matrix is the sum of its eigenvalues.*

<u>*Proposition*</u> **5.3.4.** *Given $T : V \to V$ linear, and A the matrix representation of T, the following statements are equivalent:*

· *λ is an eigenvalue of T*
· *ker$(T - \lambda I)$ is non-zero*
· *The nullspace of $T - \lambda I$ is non-zero*
· *$T_\lambda I$ is not one-to-one*
· *$T - \lambda I$ has no inverse*
· *$A - \lambda I$ has no inverse*
· *det$(A - \lambda I) = 0$*

<u>*Definition*</u> **5.3.5.** *For an eigenvalue λ of T, ker$(T - \lambda I)$ is termed the eigenspace of λ.*

<u>*Proposition*</u> **5.3.6.** *If dim$(V) = n$, and $T : V \to V$ linear, and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues for T, then $k \leqslant n$.*

<u>*Proposition*</u> **5.3.7.** *Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $T : V \to V$. Then the following are equivalent:*

· *T has diagonal representation*
· *$V =$ ker$(T - \lambda_1 I) \oplus \cdots \oplus$ ker$(T - \lambda_k I)$*
· *dim$(V) =$ dim(ker$(T - \lambda_1 I)) + \cdots +$ dim(ker$(T - \lambda_k I))$*

<u>*Proposition*</u> **5.3.8.** *If $T : V \to V$ has dim(V) distinct eigenvalues, then T is diagonalizable.*

<u>*Proposition*</u> **5.3.9.** *Some operator $T : V \to V$ has diagonal representation if and only if the following two conditions hold:*

· *det$(XI - T)$ is a product of linear factors*
· *for each eigenvalue λ, the geometric multiplicity equals the algebraic multiplicity*

<u>*Proposition*</u> **5.3.10.** *An $n \times n$ matrix A over $\mathbb{F}$ is similar to a diagonal matrix D if and only if $\mathbb{F}^n$ has a basis made up of eigenvectors of A.*

5.4. **The characteristic polynomial.**

<u>*Definition*</u> **5.4.1.** *Let A be an $n \times n$ matrix over $\mathbb{F}$. Let X be indeterminate. Then the characteristic polynomial of A is det$(XI - A)$.*

Proposition **5.4.2.** *If matrices A and B both represent some operator $T : V \to V$ that is linear, then det$(XI - B)$ = det$(XI - A)$.*

Proposition **5.4.3.** *If $T : V \to V$ is linear and has characteristic polynomial det$(XI - T) = (X - \lambda)^\ell g(X)$, where $\ell \geqslant 1$ and $g(\lambda) \neq 0$, then dim(ker$(T - \lambda I)$) $\leqslant \ell$.*
- *dim(ker$(T - \lambda I)$) is the geometric multiplicity of λ*
- *ℓ is the algebraic multiplicity of λ*

Proposition **5.4.4.** [Cayley-Hamilton Theorem] *If A is an $n \times n$ matrix over $\mathbb{C}$, and $f(X) =$det$(XI - A)$, then $f(A) = 0$.*

5.5. **T-invariant subspaces.**

Definition **5.5.1.** *If $T : V \to V$ is linear and W is a subspace of V, then W is $T-$invariant when $x \in W \Longrightarrow T(x) \in W$.*

Proposition **5.5.2.** *If $S \circ T = T \circ S$, then ker(T) and range(S) are $T-$invariant.*

Proposition **5.5.3.** *If W is a $T-$invariant subspace of $T : V \to V$, then the restriction of T to W is the operation $T|_W = S : W \to W$ defined by $x \mapsto T(x)$.*

Proposition **5.5.4.** *If $T : V \to V$ is linear and W is $T-$invariant, and $T|_W = S : W \to W$, then det$(XI - S)$ divides det$(XI - T)$. That is, the characteristic polynomial of the restriction divides the characteristic polynomial of the operator that it restricts.*

Proposition **6.0.0.** *The following diagram commutes.*

$$\begin{tikzcd}
z=\sum \lambda_j x_j \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & T(z)=\sum \mu_i y_i \arrow[ddd, mapsto] \\
& V \arrow[r, "T"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] & \\
& \mathbb{F}^n \arrow[r, "L_A"'] & \mathbb{F}^m & \\
\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix} \arrow[rrr, mapsto] & & & A\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}=\begin{pmatrix}\mu_1\\ \vdots\\ \mu_m\end{pmatrix}
\end{tikzcd}$$

Where:
$T : V \to W$ is linear
$x_1, \ldots, x_n$ is a basis for V
$y_1, \ldots, y_m$ is a basis for W
$A = [a_{ij}]$ is the matrix for T using the given bases
φ, ψ are coordinate isomorphisms

6. Proofs

Theorem 6.1. [Dimension Theorem] *If V over $\mathbb{F}$ is finite-dimensional and $T : V \to W$ is linear, then dim(V) = dim(ker(T)) + dim(range(T)).*

Proof:

Let $x_1, \dots x_n$ be a basis for ker(T).
Extend this to a basis for V, $x_1, \dots, x_n, y_1, \dots, y_m$. Then dim($V$) $= n + m$.
Check that $T(y_1), \dots, T(y_m)$ is a basis for range(T).
Check that $T(y_1), \dots, T(y_m)$ is linearly independent.
Let $\lambda_1 T(y_1) + \cdots + \lambda_m T(y_m) = 0$ for $\lambda_i \in \mathbb{F}$ for $1 \leqslant i \leqslant m$.

$$T(\lambda_1 y_1 + \cdots + \lambda_m y_m) = 0$$

$$\lambda_1 y_1 + \cdots + \lambda_m y_m \in \mathit{ker}(T)$$

Hence we can write $\lambda_1 y_1 + \cdots + \lambda_m y_m = \mu_1 x_1 + \cdots + \mu_n x_n$ for some $\mu_i \in \mathbb{F}$ for $1 \leqslant i \leqslant n$.
Then $\lambda_1 y_1 + \cdots + \lambda_m y_m - \mu_1 x_1 - \cdots - \mu_n x_n = 0$
Since $x_1, \dots, x_n, y_1, \dots, y_m$ is a basis for V, all $\lambda_i = 0$ and all $\mu_i = 0$.
Hence $T(y_1), \dots, T(y_m)$ is linearly independent.
Check that $T(y_1), \dots, T(y_m)$ spans range(T).
Let $z \in$ range(T).
Then $z = T(x)$ for some $x \in V$.
Then $x = \lambda_1 y_1 + \cdots + \lambda_m y_m + \mu_1 x_1 + \cdots + \mu_n x_n$
Then

$$\begin{aligned} z &= T(x) \\ &= \lambda_1 T(y_1) + \cdots + \lambda_m T(y_m) + T(\mu_1 x_1 + \cdots + \mu_n x_n) \\ &= \lambda_1 T(y_1) + \cdots + \lambda_m T(y_m) \end{aligned}$$

Theorem 6.2. *Let $V \xrightarrow{T} W \xrightarrow{S} Z$ with T, S linear using bases $v_1, \dots, v_n$ for V, $w_1, \dots, w_m$ for W, and $z_1, \dots, z_p$ for Z. Let $A = [a_{ij}]$ be the matrix of T using the given bases for V and W.*
Let $B = [b_{ki}]$ be the matrix of S using the given bases for W and Z.
Then the matrix for $S \circ T$ is BA.

Proof:

$$\begin{aligned} S \circ T(v_j) &= S(T(v_j)) \qquad \text{for } 1 \leqslant j \leqslant n \\ &= S\left(\sum_{i=1}^{m} a_{ij} w_i\right) \\ &= \sum_{i=1}^{m} a_{ij} \left(S(w_i)\right) \\ &= \sum_{i=1}^{m} a_{ij} \left(\sum_{k=1}^{p} b_{ki} z_k\right) \\ &= \sum_{k=1}^{p} \left(\sum_{i=1}^{m} a_{ij} b_{ki}\right) z_k \end{aligned}$$

This is then the matrix for $S \circ T$ using the given bases for V, Z.
Then we see that BA is the matrix for $S \circ T$.

Theorem 6.3. [Alternating] *If two rows of some $n \times n$ matrix A are equal, then det$(A) = 0$.*

Proof:
For sanity of notation, say row 1 = row 2.

Let $A = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_n \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Let A_n be the set of even permutations, of which there are $\frac{n!}{2}$.
Let $\tau = (1\ 2)$.
As σ runs over A_n, $\sigma \circ \tau$ runs over the odd permutations.

$$\begin{aligned} \text{So } det(A) &= \left(\sum_{\sigma \in A_n} b_{\sigma(1)} b_{\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}\right) - \left(\sum_{\sigma \notin A_n} b_{\sigma(1)} b_{\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}\right) \\ &= \left(\sum_{\sigma \in A_n} b_{\sigma(1)} b_{\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}\right) - \left(\sum_{\sigma \in A_n} b_{\sigma(\tau(1))} b_{\sigma(\tau(2))} a_{3\sigma(\tau(3))} \cdots a_{n\sigma(\tau(n))}\right) \\ &= \sum_{\sigma \in A_n} b_{\sigma(1)} b_{\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} - \sum_{\sigma \in A_n} b_{\sigma(2)} b_{\sigma(1)} a_{3\sigma(3)} \cdots a_{n\sigma(n)} \\ &= 0 \end{aligned}$$

Theorem 6.4. *An $n \times n$ matrix A is invertible if and only if its determinant is nonzero.*

Proof:
If A is invertible, then $A \xrightarrow[ops]{row} I$.
Since det$(I) = 1 \neq 0$, then det$(A) \neq 0$.

If A is not invertible, then $A \xrightarrow[ops]{row} B$, where B is upper triangular such that its last row is 0.
Since det$(B) = 0$, det$(A) = 0$.

Theorem 6.5. *Let $T : V \to V$ be linear and $\lambda_1, \ldots, \lambda_n$ be distinct scalars in $\mathbb{F}$. Then the sum $W = ker(T - \lambda_1 I) + \cdots + ker(T - \lambda_n I)$ is direct.*

Proof:
This proof is by induction on n.
Evidently, this works for $n = 1$.
Suppose that the sum of $ker(T - \lambda_j I)$'s is direct when fewer than k distinct λ_j's are used.
Let $W = ker(T - \lambda_1 I) + \cdots + ker(T - \lambda_k I)$.
Let $0 = x_1 + \cdots + x_k$ for $x_j \in ker(T - \lambda_j I)$ for all j such that $1 \leqslant j \leqslant k$.
Apply T to the above equation: $0 = T(x_1) + \cdots + T(x_k)$
$= \lambda_1 x_1 + \cdots + \lambda_k x_k$
Multiply the equation two lines above by λ_1 to get: $0 = \lambda_1 x_1 + \cdots + \lambda_1 x_k$
Subtract to get $(\lambda_2 - \lambda_1)x_2 + \cdots + (\lambda_k - \lambda_1)x_k = 0$.
By the induction hypothesis, conclude that all $(\lambda_j - \lambda_1)x_j = 0$ for $j = 2, 3, \ldots, k$.
Since the λ_j's are distinct, conclude that $x_j = 0$ for $j = 2, 3, \ldots, k$.
Look at the initial equation to deduce that $x_1 = 0$.