Definitions & Theorems

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Contents

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1 Logic

1.1 Sets

Definition 1.1.1. *A set is a collection of objects. The universal set is X.*

Definition 1.1.2. *Let the empty set be the set that contains no elements. Denote this by $\emptyset$.*

Definition 1.1.3. *Let $A, B \subset X$. The set difference of B and A is defined as $B \setminus A = \{x \in B : x \notin A\}$.*

Definition 1.1.4. *Let $A, B \subset X$. The union of A and B is defined as $A \cup B = \{x : x \in A \text{ or } x \in B\}$.*

Definition 1.1.5. *Let $A, B \subset X$. The intersection of A and B is defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$.*
· *If $\{A_\alpha\}_{\alpha \in I}$ is a collection of sets, then*

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \ \forall \ \alpha \in I\}$$

Definition 1.1.6. *Let $A \subset X$. The complement of A is defined as $A^c = \overline{A} = \{x \in X : x \notin A\}$.*

Theorem 1.1.7. [DE MORGAN'S LAWS]
Let $A \subset X$. With respect to the above definitions,

· $\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$

· $\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$

Definition 1.1.8. *Define the following basic sets:*
· $\mathbb{N}$: *natural numbers* $= \{1, 2, 3, \dots\}$
· $\mathbb{Z}$: *integers* $= \{\dots, -2, -1, 0, 1, 2, \dots\}$
· $\mathbb{Q}$: *rational numbers* $= \{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a, b) = 1\}$
· $\mathbb{R}$: *real numbers*

Definition 1.1.9. *A subset $S \subset \mathbb{R}$ is termed an interval. For every $x, y \in S$, if $z \in \mathbb{R}$ is such that $x \leqslant z \leqslant y$, then $z \in S$. Note that the empty set is an interval.*

1.2 The Peano axioms

Axiom 1.2.1. [PRINCIPLE OF MATHEMATICAL INDUCTION]
If $S \subset \mathbb{N}$ is such that
1) $1 \in S$
2) *If $n \in S$, then $n + 1 \in S$*
Then $S = \mathbb{N}$.

Axiom 1.2.2. [PRINCIPLE OF STRONG MATHEMATICAL INDUCTION]
If $S \subset \mathbb{N}$ is such that
1) $1 \in S$
2) *If $\{1, \dots, n\} \subset S$, then $n + 1 \in S$*
Then $S = \mathbb{N}$.

Axiom 1.2.3. [WELL-ORDERING PRINCIPLE]
If $S \subset \mathbb{N}$ is non-empty, then S has a least element.

1.3 Properties of numbers

Definition 1.3.1. *Let $S \subset \mathbb{R}$.*

- *An element $\alpha \in \mathbb{R}$ is an <u>upper bound</u> for S if $x \leqslant \alpha \ \forall\, x \in S$.*
- *If S has an upper bound, then S is <u>bounded above</u>.*
- *An element $\beta \in \mathbb{R}$ is a <u>lower bound</u> for S if $x \geqslant \beta \ \forall\, x \in S$.*
- *If S has a lower bound, then S is <u>bounded below</u>.*
- *If S is bounded above and below, then S is <u>bounded</u>.*

Axiom 1.3.2. [Least upper bound property]
Every non-empty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound.

Corollary 1.3.3. [Greatest lower bound property]
Every non-empty subset $S \subset \mathbb{R}$ that is bounded below has a greatest lower bound.

Theorem 1.3.4. [Archimedean property I]
$\mathbb{N}$ is not bounded above.

Theorem 1.3.5. [Archimedean property II]
Let $\epsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$.

Corollary 1.3.6. *If $x \in \mathbb{R}$, then there exists some $z \in \mathbb{Z}$ such that $z < x \leqslant z+1$.*

Corollary 1.3.7. *If $x, y \in \mathbb{R}$ with $x < y$, then there exists some $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ such that $r, s \in (x, y)$.*

1.4 Functions

Definition 1.4.1. *A <u>function</u> is a rule that assigns to each element in a set X a single value y in a set Y.*

Definition 1.4.2. *A function f from a set X to a set Y is represented by $f : X \to Y$.*

- *A function $f : X \to Y$ is <u>$1-1$</u> is for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.*
- *A function $f : X \to Y$ is <u>onto</u> if for every $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.*
- *A function that is both $1-1$ and onto is termed a <u>bijection</u>.*

Definition 1.4.3. *Two sets X, Y are termed <u>equivalent</u> if there exists a bijection $f : X \to Y$. This is expressed $X \backsim Y$. Then f is termed an <u>isomorphism</u>.*

Remark 1.4.4. *$\mathbb{Q} \backsim \mathbb{N}$. This is given by $f : \mathbb{Q} \to \mathbb{N}$ defined by $f(\frac{m}{n}) = 2^n 3^m$.*

Definition 1.4.5. *A set X is <u>finite</u> if $X \backsim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, or $X \backsim \emptyset$. A set X is <u>infinite</u> if it is not finite.*

1.5 The absolute value

Definition 1.5.1. *The <u>absolute value</u> is a function* $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$

Remark 1.5.2. *The absolute value has the following properties:*

1. $|x| = |-x|$
2. $|x| \geqslant 0$ *and* $|x| = 0$ *if and only if* $|x| = 0$
3. $|xy| = |x||y|$

Theorem 1.5.3. [Triangle Inequality]
For every $x, y, z \in \mathbb{R}$, $|x-y| \leqslant |x-z| + |z-y|$.

Corollary 1.5.4. *For every $x, y \in \mathbb{R}$, $|x+y| \leqslant |x| + |y|$.*

Corollary 1.5.5. *For every $x, y, z \in \mathbb{R}$, $||x| - |y|| \leqslant |x - y|$.*

Remark 1.5.6. *The absolute value has the following inequalities:*

1. $|x - a| < \delta \implies x \in (a - \delta, a + \delta)$

2. $0 < |x - a| < \delta \implies x \in (a - \delta, a + \delta) \setminus \{a\}$

3. $|x - a| \leqslant \delta \implies x \in [a - \delta, a + \delta]$

2 Sequences

Definition 2.0.1. *A sequence is an infinite ordered list of real numbers, denoted $\{a_n\}$.*

2.1 Limits of sequences

Definition 2.1.1. *A number L is the limit of a sequence $\{a_n\}$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geqslant N$, then $|a_n - L| < \epsilon$.*

Definition 2.1.2. *If a sequence has such a limit, then the sequence converges. If no such L exists, then the sequence diverges.*

Theorem 2.1.3. *Let $\{a_n\}$ be a sequence. Let $L, M \in \mathbb{R}$ such that $\lim_{n\to\infty} \left[a_n\right] = L$ and $\lim_{n\to\infty} \left[a_n\right] = M$. Then $M = L$.*

Theorem 2.1.4. *Every convergent sequence is bounded.*

Definition 2.1.5. *A sequence $\{a_n\}$ is monotonic if and only if it satisfies any one of the following:*

1. *$\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$*

2. *$\{a_n\}$ is non-decreasing if $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$*

3. *$\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$*

4. *$\{a_n\}$ is non-increasing if $a_n \geqslant a_{n+1}$ for all $n \in \mathbb{N}$*

Theorem 2.1.6. [Monotone Convergence Theorem]
If $\{a_n\}$ is non-decreasing and bounded above, then $\{a_n\}$ converges.

Remark 2.1.7. *The Least upper bound property and the Monotone convergence theorem are equivalent.*

Corollary 2.1.8. *A monotonic sequence $\{a_n\}$ converges if and only if it is bounded.*

Remark 2.1.9. *If $\{a_n\}$ is non-decreasing, then either*

1. *$\{a_n\}$ is bounded and hence converges.*

2. *$\{a_n\}$ diverges to ∞.*

Definition 2.1.10. *Given a sequence $\{a_n\}$ and an $N \in \mathbb{N}$, the set $\{a_N, a_{N+1}, a_{N+2}, \dots\}$ is termed a tail.*

Remark 2.1.11. *The following are equivalent:*

1. $\lim_{n\to\infty} \left[a_n\right] = L$

2. *For every $\epsilon > 0$, the open interval $(L - \epsilon, L + \epsilon)$ contains a tail of $\{a_n\}$.*

3. *For every $\epsilon > 0$, The open interval $(L - \epsilon, L + \epsilon)$ contains all but finitely many terms of $\{a_n\}$.*

4. *Every open interval (a, b) containing L contains a tail of $\{a_n\}$.*

5. *Every open interval (a, b) containing L contains all but finitely many terms of $\{a_n\}$.*

2.2 Series

Definition 2.2.1. *Let $\{a_n\}$ be a sequence. Then a* <u>*series*</u> *with terms given by the sequence $\{a_n\}$ is a formal sum of the form*

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

Definition 2.2.2. *For each $k \in \mathbb{N}$, $S_k = \sum_{n=1}^{\infty} a_n$ is termed the* <u>*k^{th} partial sum*</u> *of the series $\sum_{n=1}^{\infty} a_n$. The series converges if and only if $\{S_k\}$ converges as $k \to \infty$.*

Theorem 2.2.3. *If $\{S_k\}$ diverges, then $\{a_n\}$ diverges.*

Definition 2.2.4. *Let $r \in \mathbb{R}$. A* <u>*geometric series*</u> *with radius r is a series of the form*

$$1 + r + r^2 + \cdots + r^n + \cdots = \sum_{n=0}^{\infty} r^n$$

Theorem 2.2.5. *If $|r| \geqslant 1$, then the geometric series with radius r will diverge.*

Theorem 2.2.6. *The series $\displaystyle\sum_{n=0}^{\infty} ar^n$ converges to $\dfrac{a}{1-r}$ if and only if $|r| < 1$.*

Theorem 2.2.7. [Comparison Test]
Suppose that $\leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$.

1. *if $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.*
2. *If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.*

2.3 Subsequences

Definition 2.3.1. *Given a sequence $\{a_n\}$ and an increasing sequence $n_1 < n_2 < \cdots$ of the natural numbers, the sequence $b_k = a_{n_k} = \{a_{n_1}, a_{n_2}, \dots\}$ is termed a* <u>*subsequence*</u> *of $\{a_n\}$.*

Theorem 2.3.2. *If $\{a_n\}$ converges to L, then so does every subsequence of $\{a_n\}$.*

Definition 2.3.3. *A point L is termed a* <u>*limit point*</u> *of a sequence if there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\{a_{n_k}\}$ converges to L. The set of all limits points of a sequence $\{a_n\}$ is denoted by $\mathrm{Lim}\{a_n\}$.*

Definition 2.3.4. *Let $\{a_n\}$ be a sequence. An element $n_\circ \in \mathbb{N}$ is termed a* <u>*peak point*</u> *of $\{a_n\}$ if $a_n < a_{n_\circ}$ for all $n > n_\circ$.*

Theorem 2.3.5. [Peak Point Lemma]
Every sequence $\{a_n\}$ has a monotonic subsequence.

Theorem 2.3.6. [Bolzano-Weierstrass Theorem]
Every bounded sequence $\{a_n\}$ has a convergent subsequence.

Theorem 2.3.7. *Suppose that for sequences $\{a_n\}, \{b_n\}$ we have $\lim\limits_{n\to\infty} \big[a_n\big] = L$ and $\lim\limits_{n\to\infty} \big[b_n\big] = M$. Then*

1. $\lim\limits_{n\to\infty} \big[c \cdot a_n\big] = c \cdot L$ *for every $c \in \mathbb{R}$*
2. $\lim\limits_{n\to\infty} \big[a_n + b_n\big] = L + M$
3. $\lim\limits_{n\to\infty} \big[a_n \cdot b_n\big] = L \cdot M$
4. $\lim\limits_{n\to\infty} \left[\dfrac{1}{b_n}\right] = \dfrac{1}{M}$ *for $M \neq 0$*
5. $\lim\limits_{n\to\infty} \left[\dfrac{a_n}{b_n}\right] = \dfrac{L}{M}$ *for $M \neq 0$*

Theorem 2.3.8. *Suppose that for sequences* $\{a_n\}, \{b_n\}$*, the limit* $\lim\limits_{n\to\infty}\left[\dfrac{a_n}{b_n}\right]$ *exists, and* $\lim\limits_{n\to\infty}\left[b_n\right] = 0$*. Then* $\lim\limits_{n\to\infty}\left[a_n\right] = 0$*.*

Theorem 2.3.9. [Squeeze Theorem for Sequences]
Suppose that $\{a_n\}, \{b_n\}, \{c_n\}$ *are sequences with* $a_n \leqslant b_n \leqslant c_n$ *for all* $n \in \mathbb{N}$*. Also suppose that* $\lim\limits_{n\to\infty}\left[a_n\right] = L = \lim\limits_{n\to\infty}\left[c_n\right]$*. Then* $\lim\limits_{n\to\infty}\left[b_n\right]$ *exists and* $\lim\limits_{n\to\infty}\left[b_n\right] = L$*.*

2.4 Cauchy sequences

Definition 2.4.1. *A sequence* $\{a_n\}$ *is* Cauchy *if for every* $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that if* $n, m \geqslant N$*, then* $|a_n - a_m| < \epsilon$*.*

Theorem 2.4.2. *Every convergent sequence is Cauchy.*

Theorem 2.4.3. *Every Cauchy sequence is bounded.*

Theorem 2.4.4. *Suppose* $\{a_n\}$ *is Cauchy. Suppose that* $\{a_{n_k}\}$ *is a subsequence of* $\{a_n\}$ *with* $a_{n_k} \to L$*. Then* $a_n \to L$*.*

Theorem 2.4.5. [Completeness Theorem]
Every Cauchy sequence $\{a_n\} \subset \mathbb{R}$ *converges.*

Theorem 2.4.6. *The following are equivalent in* $\mathbb{R}$*:*

1. *Least upper bound property*
2. *Monotone convergence theorem*
3. *Bolzano-Weierstrass theorem*
4. *Completeness theorem*

3 Limits and continuity

3.1 Limits of functions

Definition 3.1.1. *A number* L *is the* limit *of a function* $f(x)$ *as* $x \to a$*, if for every* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that if* $0 < |x - a| < \delta$*, then* $|f(x) - L| < \epsilon$*. Then we denote the limit by* $\lim\limits_{x\to a}\left[f(x)\right] = L$*.*

Theorem 3.1.2. [Sequential Characterization of Limits]
The following are equivalent:

1. $\lim\limits_{x\to a}\left[f(x)\right] = L$
2. *Whenever* $\{x_n\}$ *is a sequence with* $x_n \to a$ *and* $x_n \neq a$*, then* $f(x_n) \to L$*.*

Theorem 3.1.3. *Suppose that for function* $f(x), g(x)$ *we have* $\lim\limits_{x\to a}\left[f(x)\right] = L$ *and* $\lim\limits_{x\to a}\left[g(x)\right] = M$*. Then*

1. $\lim\limits_{x\to a}\left[c \cdot f(x)\right] = c \cdot L$ *for every* $c \in \mathbb{R}$
2. $\lim\limits_{x\to a}\left[f(x) + g(x)\right] = L + M$
3. $\lim\limits_{x\to a}\left[f(x) \cdot g(x)\right] = L \cdot M$
4. $\lim\limits_{x\to a}\left[\dfrac{1}{g(x)}\right] = \dfrac{1}{M}$ *for* $M \neq 0$
5. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{M}$ *for* $M \neq 0$

Theorem 3.1.4. *Suppose that for functions $f(x), g(x)$, the limit* $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]$ *exists, and* $\lim_{x\to a}\left[g(x)\right] = 0$. *Then* $\lim_{x\to a}\left[f(x)\right] = 0$.

Definition 3.1.5. *A* <u>*rational function*</u> *is a function* $f(x) = \frac{p(x)}{q(x)}$, *where* $p(x), q(x)$ *are polynomials.*

Theorem 3.1.6. [ALGORITHM FOR RATIONAL FUNCTIONS]
Let $f(x) = \frac{p(x)}{q(x)}$. *Then*

1. *If* $q(a) \neq 0$, *then* $\lim_{x\to a}\left[f(x)\right] = \frac{\lim_{x\to a}\left[p(x)\right]}{\lim_{x\to a}\left[q(x)\right]} = \frac{p(a)}{q(a)} = f(a)$

2. *If* $q(a) = 0$ *and* $p(a) \neq 0$, *then* $\lim_{x\to a}\left[f(x)\right]$ *does not exist.*

3. *If* $q(a) = 0$ *and* $p(a) = 0$, *then* $\frac{p(x)}{q(x)} = \frac{(x-a)p_1(x)}{(x-a)q_1(x)}$, *and repeat* **1.** *with* $\frac{p_1(x)}{q_1(x)}$.

Theorem 3.1.7. [SQUEEZE THEOREM FOR FUNCTIONS]
Suppose that $f(x), g(x), h(x)$ are functions with $f(x) \leqslant g(x) \leqslant h(x)$ for all x in an open interval I containing $x = a$, except possibly at $x = a$. Also suppose that $\lim_{x\to a}\left[f(x)\right] = L = \lim_{x\to a}\left[h(x)\right]$. *Then* $\lim_{x\to a}\left[g(x)\right]$ *exists and* $\lim_{x\to a}\left[g(x)\right] = L$.

3.2 One-sided limits

Definition 3.2.1. *A number L is the limit of a function $f(x)$ as $x \to a$ from above (the right), if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - a| < \epsilon$. This implies that $x \in (a, a + \delta)$. This limit is denoted by* $\lim_{x\to a^+}\left[f(x)\right] = L$.

Definition 3.2.2. *A number L is the limit of a function $f(x)$ as $x \to a$ from below (the left), if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - a| < \epsilon$. This implies that $x \in (a - \delta, a)$. This limit is denoted by* $\lim_{x\to a^-}\left[f(x)\right] = L$.

Theorem 3.2.3. *The following are equivalent:*

1. $\lim_{x\to a}\left[f(x)\right] = L$

1. $\lim_{x\to a^+}\left[f(x)\right] = \lim_{x\to a^-}\left[f(x)\right] = L$

Remark 3.2.4. *One-sided limits have the same arithmetic rules , sequential characterization, and satisfy the squeeze theorem just as the two-sided limits.*

Definition 3.2.5. *A function $f : \mathbb{R} \to \mathbb{R}$ is termed* <u>*even*</u> *if*

· $f(x) = f(-x)$
· *The graph of f is symmetric about the y-axis.*

Definition 3.2.6. *A function $f : \mathbb{R} \to \mathbb{R}$ is termed* <u>*odd*</u> *if*

· $f(x) = -f(-x)$
· *The graph of f is symmetric about the origin.*

Remark 3.2.7. *With respect to the above definitions,*

· *If $f(x)$ is even, then* $\lim_{x\to 0}\left[f(x)\right]$ *exists if and only if* $\lim_{x\to a^+}\left[f(x)\right]$ *exists or* $\lim_{x\to a^-}\left[f(x)\right]$ *exists.*
· *If $f(x)$ is odd, then* $\lim_{x\to 0}\left[f(x)\right]$ *exists if and only if* $\lim_{x\to a^+}\left[f(x)\right] = \lim_{x\to a^-}\left[f(x)\right]$.

Theorem 3.2.8. [FUNDAMENTAL TRIGONOMETRIC LIMIT]
$\lim_{x\to 0}\left[\frac{\sin(x)}{x}\right] = 1$

Corollary 3.2.9. *If x is "small", i.e. $|x| \ll 1$, then $\sin(x) \approx x \approx \tan(x)$.*

Definition 3.2.10.

· $\lim\limits_{x\to a^+} \big[f(x)\big] = \infty$ *if for every $m > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $f(x) > m$.*

· $\lim\limits_{x\to a^+} \big[f(x)\big] = -\infty$ *if for every $m < 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $f(x) < m$.*

· $\lim\limits_{x\to a^-} \big[f(x)\big] = \infty$ *if for every $m > 0$, there exists a $\delta > 0$ such that if $0 < a - x < \delta$, then $f(x) > m$.*

· $\lim\limits_{x\to a^-} \big[f(x)\big] = -\infty$ *if for every $m < 0$, there exists a $\delta > 0$ such that if $0 < a - x < \delta$, then $f(x) < m$.*

Definition 3.2.11. *A function $f(x)$ has a vertical asymptote at $x = a$ if either one of the following hold:* $\lim\limits_{x\to a^+} \big[f(x)\big] = \pm\infty$ *or* $\lim\limits_{x\to a^-} \big[f(x)\big] = \pm\infty$

3.3 Continuity

Definition 3.3.1. *A function $f(x)$ is continuous at $x = a$ if*

1. $\lim\limits_{x\to a} \big[f(x)\big]$ *exists*

2. $\lim\limits_{x\to a} \big[f(x)\big] = f(a)$

Otherwise $f(x)$ is termed discontinuous.

Definition 3.3.2. *A function $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.*

Theorem 3.3.3. [Sequential Characterization of Continuity]

The following are equivalent:

1. *$f(x)$ is continuous at $x = a$*

2. *Whenever $\{x_n\}$ is a sequence with $x_n \to a$, then $f(x_n) \to f(a)$.*

Theorem 3.3.4. *Suppose that $f(x), g(x)$ are continuous at $x = a$. Then*

1. *$c \cdot f(x)$ is continuous at $x = a$ for all $c \in \mathbb{R}$*

2. *$f(x) + g(x)$ is continuous at $x = a$*

3. *$f(x) \cdot g(x)$ is continuous at $x = a$*

4. *$\frac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$*

Definition 3.3.5. *A function $f(x)$ is continuous on $[a, b]$ if*

1. *$f(x)$ is continuous at each $x_\circ \in [a, b]$*

2. $\lim\limits_{x\to a^+} \big[f(x)\big] = f(a)$

3. $\lim\limits_{x\to b^-} \big[f(x)\big] = f(b)$

Definition 3.3.6. *Let $f : S \to \mathbb{R}$ with $S \subset \mathbb{R}$. Then $f(x)$ is continuous on S (relative to S) if whenever $\{x_n\} \subset S$ with $x_n \to x_\circ$ for $x_\circ \in S$, then $f(x_n) \to f(x_\circ)$.*

3.4 Compositions of functions

Definition 3.4.1. *If the range of a function f is the subset of the domain of a function g, then define the composition of f and g by $h(x) = g(f(x))$.*

· *If $f : X \to Y$ and $g : Y \to Z$, then $h : X \to Z$.*

Theorem 3.4.2. *Suppose that $f(x)$ is continuous at $x = a$, and that $g(y)$ is continuous at $y = f(a)$. Then the function $g(f(x)) = g \circ f$ is continuous at $x = a$.*

Remark 3.4.3. *If $\lim\limits_{x\to a} \big[f(x)\big] = L$ and $\lim\limits_{y\to L} \big[g(y)\big] = M$, then it is not necessarily true that $\lim\limits_{x\to a} \big[g \circ f(x)\big] = M$. However, if $\lim\limits_{x\to a} \big[f(x)\big] = L$ and $g(y)$ is continuous at $y = L$, then $\lim\limits_{x\to a} \big[g \circ f(x)\big] = g(L)$.*

3.5 Discontinuity

Definition 3.5.1. *A function $f(x)$ has a* <u>*discontinuity*</u> *at $x = a$ of $f(x)$ is defined on an open interval I containing a except possibly at $x = a$, and $f(x)$ is not continuous at $x = a$. Let $D(f)$ denote the collection of all points of discontinuity of f.*

Definition 3.5.2. *There are two types of discontinuities:*

1. *If $\lim_{x \to a} \left[f(x) \right]$ exists, but either $f(a)$ is not defined, or $f(a) = \lim_{x \to a} \left[f(x) \right]$, then $x = a$ is termed a* <u>*removable*</u> *discontinuity.*

2. *If $a \in D(f)$such that $\lim_{x \to a} \left[f(x) \right]$ does not exist, then a is termed an* <u>*essential*</u> *discontinuity of f.*

$$\left.\begin{array}{l} \cdot \textit{ Jump discontinuity} \\ \cdot \textit{ Vertical asymptote discontinuity} \\ \cdot \textit{ Oscillatory discontinuity} \end{array}\right\} \textit{These are the types of essential discontinuities}$$

Remark 3.5.3. *A function with a removable (non-essential) discontinuity may be redefined at the point of discontinuity to create a continuous function, but a function with an essential discontinuity cannot be made continuous by redefining the function at the point of discontinuity.*

3.6 Value theorems

Theorem 3.6.1. [Intermediate Value Theorem]
Suppose that $f(x)$ is continuous on $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$. Then there exists some $c \in (a, b)$ such that $f(c) = 0$.

Corollary 3.6.2. *Suppose that $f(x)$ is continuous on $[a, b]$. If $f(a) < \alpha < f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = \alpha$.*

Corollary 3.6.3. *Suppose that $f(x)$ is continuous on an interval I. Then $f(I) = \{f(x) : x \in I\}$ is an interval.*

Definition 3.6.4. *A function $f(x)$ is termed* <u>*uniformly continuous*</u> *on $S \subset \mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for $x_a, x_2 \in S$ if $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.*

Theorem 3.6.5. [Sequential Characterization of Uniform Continuity]
Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$. Then the following are equivalent:

1. *$f(x)$ is uniformly continuous on S.*

1. *If $\{x_n\}, \{y_n\}$ are sequences in S, and $\lim_{n \to \infty} \left[x_n - y_n \right] = 0$, then $\lim_{n \to \infty} \left[f(x_n) - f(y_n) \right] = 0$.*

Theorem 3.6.6. *Suppose that $f(x)$ is uniformly continuous on $S \subset \mathbb{R}$. If $\{x_n\} \subset S$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.*

Theorem 3.6.7. *Suppose that $f(x)$ is continuous on $[a, b]$. Then $f(x)$ is uniformly continuous on $[a, b]$.*

Theorem 3.6.8. *Suppose that $f(x)$ is continuous on (a, b). Then $f(x)$ is uniformly continuous on (a, b) if and only if $\lim_{x \to a^+} \left[f(x) \right]$ and $\lim_{x \to b^-} \left[f(x) \right]$ exist.*

Theorem 3.6.9. [Extreme Value Theorem]
Suppose that $f(x)$ is continuous on $[a, b]$. Then there exist $c, d \in [a, b]$ such that $f(x) \leqslant f(x) \leqslant f(d)$ for all $x \in [a, b]$.

Corollary 3.6.10. *If $f(x)$ is continuous on $[a, b]$, then $f([a, b]) = \{f(x); s \in [a, b]\}$ is a closed interval. This is equal to $f(c), f(d)$.*

4 Differentiation

4.1 Derivatives

Definition 4.1.1. *Given $f(x)$ and a point $x = a \in \mathbb{R}$, the quantity $\frac{f(x)-f(a)}{x-a}$ is termed Newton's quotient for $f(x)$ centered at $x = a$.*

Definition 4.1.2. *A function $f(x)$ is termed differentiable at $x = a$ if $\lim\limits_{x \to a} \left[\dfrac{f(x) - f(a)}{x - a}\right]$ exists. Then it is expressed as $f'(a) = \lim\limits_{x \to a} \left[\dfrac{f(x) - f(a)}{x - a}\right]$. If the limit does not exist, then the function is not differentiable at $x = a$.*

Definition 4.1.3. *If a function $f(x)$ is differentiable at $x = a$, then the line $y = f(a) + f'(a)(x - a)$ is termed the tangent line to the graph of $f(x)$ through $(a, f(a))$. This is also termed the linear approximation of $f(x)$ at $x = a$, and denoted by $L_a(x)$.*

Theorem 4.1.4. *If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.*

Theorem 4.1.5. *Suppose that $f(x), g(x)$ are differentiable at $x = a$. Then*
1. *$c \cdot f(x)$ is differentiable at $x = a$ for all $c \in \mathbb{R}$ with $\frac{d}{dx} c \cdot f(x) = c \cdot \frac{d}{dx} f(x)$*
2. *If $h(x) = f(x) + g(x)$, then $h(x)$ is differentiable at $x = a$ with $h'(a) = f'(a) + g'(a)$*
3. *If $h(x) = f(x)g(x)$, then $h(x)$ is differentiable at $x = a$ with $h'(a) = f'(a)g(a) + f(a)g'(a)$*

Theorem 4.1.6. [Chain Rule]
Suppose $f(x)$ is defined upon an open interval I containing $x = a$, and $g(y)$ is defined upon an open interval J with $f(I) \subset J$. Suppose that $f(x)$ is differentiable at $x = a$, and that $g(y)$ is differentiable at $y = f(a)$. Then $h(x) = g \circ f(x)$ is differentiable at $x = a$ with $h'(a) = g'(f(a))f'(a)$.

Definition 4.1.7. *For a function $f(x)$, an element $x = c$ is termed a local maximum if there exists an open interval $I = (a, b)$ containing c such that $f(x) \leqslant f(c)$ for all $x \in (a, b)$. SImilarly, an element $x = d$ is termed a local minimum if there exists an open interval $I = (a, b)$ containing d such that $f(d) \leqslant f(x)$ for all $x \in (a, b)$. The two together are termed local extrema.*

Theorem 4.1.8. *Suppose that $x = c$ is either a local extremum for $f(x)$. If $f'(c)$ exists, then $f'(c) = 0$.*

Definition 4.1.9. *A point $x = c$ is termed a critical point of $f(x)$ if $f(x)$ is defined upon an open interval I containing c, and either*
1. *$f'(c) = 0$*
2. *$f(x)$ is not differentiable at $x = c$.*

Remark 4.1.10. *If $f(x)$ is continuous on $[a, b]$, then each of the global extrema (max/min) will be either at*
1. *an endpoint*
2. *a critical point in (a, b), i.e. an interior critical point*

4.2 Inverse functions

Definition 4.2.1. *Suppose that $f : X \to Y$ is a bijection. Define a function $g : Y \to X$ by $g(y) = x$ if and only if $y = g(x)$. Then the function $g(y)$ is termed the inverse of $f(x)$ and is denoted by $f^{-1}(x)$.*

Definition 4.2.2. *If $f : S \to \mathbb{R}$ with $S \subset \mathbb{R}$, then $f(x)$ is invertible on S if f is one-to-one on S. Then define $g : f(S) \to S$ by $g(y) = x$ if and only if $x \in S$ and $f(x) = y$.*

Definition 4.2.3. *A function $f : S \to \mathbb{R}$ is said to be (strictly) increasing on S if whenever $x_1, x_2 \in S$ with $x_1 < x_2$, then $\big(f(x_1) < f(x_2)\big)$ $f(x_1) \leqslant f(x_2)$. Similarly, f is (strictly) decreasing on S if whenever $x_1, x_2 \in S$ with $x_1 < x_2$, then $\big(f(x_1) > f(x_2)\big)$ $f(x_1) \geqslant f(x_2)$. If f is either (strictly) increasing or (strictly) decreasing on S, then f is monotonic on S.*

Remark 4.2.4. *If $f(x)$ is strictly increasing/decreasing on S, then f is $1-1$ on S, and hence invertible.*

Theorem 4.2.5. *Suppose that $f(x)$ is continuous on an interval I and also $1-1$ on I. Then either*
1. *$f(x)$ is strictly increasing on I*
2. *$f(x)$ is strictly decreasing on I*

Thus continuous functions tend to be precisely $1-1$ when they are monotonic.

Theorem 4.2.6. *Suppose that $f(x)$ is strictly increasing/decreasing on $[a,b]$. Then $f(x)$ is continuous on $[a,b]$ if and only if $f([a,b]) = [f(a), f(b)]$, i.e. if and only if $f([a,b])$ is an interval.*

Remark 4.2.7. *Suppose that $f : S \to \mathbb{R}$ is strictly increasing/ decreasing. Let $T = f(S)$. Then f is invertible on S. Let $g : T \to S$ be the inverse of f. Then $g(y)$ is strictly increasing/decreasing.*

Corollary 4.2.8. *Let I be an interval and $f : I \to \mathbb{R}$ be strictly increasing/decreasing with $J = f(I)$. Then $f(x)$ is continuous on I if and only if $g : J \to I$ (where $g = f^{-1}$) is also continuous.*

Remark 4.2.9. *The graph of $f(x)$ and its inverse are essentially the same, as $(x, f(x)) \to (g(y), y)$. To graph $g(y)$ in the usual orientation, we "switch" x and y.*

Theorem 4.2.10. [Inverse Function Theorem]
Suppose that $f(x)$ is defined on a continuous and open interval I containing $x_\circ$. Suppose that $f(x)$ is either strictly increasing or strictly decreasing on I with inverse $g : J = f(I) \to I$. Then if $f(x)$ is differentiable at $x_\circ$ with $f'(x_\circ) \neq 0$, then $g(y)$ is differentiable at $y_\circ = f(x_\circ)$ with

$$g'(y_\circ) = \frac{1}{f'(x_\circ)}$$

Definition 4.2.11. *Let e be the unique base such that $\lim_{n\to 0}\left[\dfrac{e^n - 1}{n}\right] = 1$. Then $f(x) = e^x = f^{-1}(x)$.*

4.3 Mean value theorem

Theorem 4.3.1. [Rolle's Theorem]
If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$, then there exists a $c \in (a,b)$ with $f'(c) = 0$.

Theorem 4.3.2. [Mean value theorem]
If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists a $c \in (a,b)$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Theorem 4.3.3. *Suppose that $f(x), g(x)$ are continuous on some closed interval $[a,b]$ and differentiable on (a,b). If $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists some $m \in \mathbb{R}$ such that $g(x) = f(x) + m$ for all $x \in [a,b]$.*

Theorem 4.3.4. [Increasing Function Theorem]
Suppose that $f(x), g(x)$ are continuous on some closed interval $[a,b]$ and differentiable on (a,b) with $f'(x) > 0$ for all $x \in (a,b)$. Then $f(x)$ is strictly increasing on $[a,b]$.

Remark 4.3.5. *The increasing function theorem holds on the interval $[a,b]$ if there exist at most finitely many points $x_1, \ldots, x_n \in [a,b]$ such that $f(x_i) = 0$ for $1 \leqslant i \leqslant n$.*

Theorem 4.3.6. *Suppose that $f(x), g(x)$ are continuous on $[a,b]$ and differentiable on (a,b). Also suppose that $m \leqslant f'(x) \leqslant M$ for all $x \in (a,b)$. Then $f(a) + m(x-a) \leqslant f(x) \leqslant f(a) + M(x-a)$ for all $x \in (a,b)$.*

Theorem 4.3.7. *Suppose that $f(x)$ is defined on an interval I with $|f(x)| \leqslant M$ for all $x \in I$. Then $f(x)$ is uniformly continuous on I.*

Theorem 4.3.8. [First Derivative Test]
Suppose that c is a critical point of $f(x)$., and $f(x)$ is continuous at $x = c$.

1. *Suppose that there exists an open interval (a, b) containing c such that $f'(x) \leqslant 0$ for all $x \in (a, c)$ and $f'(x) \leqslant 0$ for all $x \in (c, b)$. Then c is a local min of $f(x)$.*

2. *Suppose that there exists an open interval (a, b) containing c such that $f'(x) \geqslant 0$ for all $x \in (a, c)$ and $f'(x) \leqslant 0$ for all $x \in (c, b)$. Then c is a local max of $f(x)$.*

4.4 Concavity

Definition 4.4.1. *Let $I \subset \mathbb{R}$ be an interval.*

1. *A function $f(x)$ is concave upward on I if for each $a, b \in I$ with $a < b$, the secant line joining $(a, f(a))$ and $(b, f(b))$ sits above the graph of $f(x)$ on $[a, b]$.*

2. *A function $f(x)$ is concave downward on I if for each $a, b \in I$ with $a < b$, the secant line joining $(a, f(a))$ and $(b, f(b))$ sits below the graph of $f(x)$ on $[a, b]$.*

Definition 4.4.2. *Suppose that $f(x)$ is differentiable on I with derivative function $f'(x)$. If $a \in I$ and $f'(x)$ differentiable at $x = a$, then the quantity*

$$\lim_{x \to a} \left[\frac{f'(x) - f'(a)}{x - a} \right] = f''(x)$$

is termed the second derivative at $x = a$. In general, $f^(n)(x) := \frac{d}{dx} f^{n-1}(x)$.

Theorem 4.4.3. *Suppose that $f(x)$ is such that $f''(x) > 0$ for all $x \in I$. Then $f(x)$ is concave upward on I. If $f''(x) < 0$ on I, then $f(x)$ is concave downward on I.*

Theorem 4.4.4. [Second Derivative Test]
Suppose that c is a critical point of $f(x)$, and $f(x)$ is continuous at $x = c$.

1. *Suppose that $f''(x)$ is continuous at $x = c$ and $f''(c) > 0$. Then c is a local min for $f(x)$.*

2. *Suppose that $f''(x)$ is continuous at $x = c$ and $f''(c) < 0$. Then c is a local max for $f(x)$.*

Definition 4.4.5. *A point $x = c$ is termed a point of inflection for $f(x)$ is $f(x)$ is continuous at $x = c$ and if there exists an open interval (a, b) containing c such that either*

1. *$f(x)$ is concave upward on (a, c) and concave downward on (c, b)*

2. *$f(x)$ is concave downward on (a, c) and concave upward on (c, b)*

Definition 4.4.6. *An extended real number is an element in the set of numbers $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$.*

Theorem 4.4.7. [L'Hôpital's Rule]
Suppose that $f, g : (a, b) \to \mathbb{R}$ for extended real numbers a, b and $a < b$. Suppose that f, g are differentiable on (a, b) with $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$.

1. *Suppose that $\lim\limits_{x \to a^+} \big[f(x)\big] = 0 = \lim\limits_{x \to a^+} \big[g(x)\big]$. Then if $\lim\limits_{x \to a^+} \left[\frac{f'(x)}{g'(x)}\right] = L$ for $L \in \mathbb{R}^*$, then $\lim\limits_{x \to a^+} \left[\frac{f(x)}{g(x)}\right] = L$*

2. *Suppose that $\lim\limits_{x \to b^-} \big[f(x)\big] = 0 = \lim\limits_{x \to b^-} \big[g(x)\big]$. Then if $\lim\limits_{x \to b^-} \left[\frac{f'(x)}{g'(x)}\right] = L$ for $L \in \mathbb{R}^*$, then $\lim\limits_{x \to b^-} \left[\frac{f(x)}{g(x)}\right] = L$*

3. *Suppose that $\lim\limits_{x \to a^+} \big[f(x)\big] = \pm\infty$ and $\lim\limits_{x \to a^+} \big[g(x)\big] = \pm\infty$.*

Then if $\lim\limits_{x \to a^+} \left[\frac{f'(x)}{g'(x)}\right] = L$ for $L \in \mathbb{R}^*$, then $\lim\limits_{x \to a^+} \left[\frac{f(x)}{g(x)}\right] = L$

4. *Suppose that $\lim\limits_{x \to b^-} \big[f(x)\big] = \pm\infty$ and $\lim\limits_{x \to b^-} \big[g(x)\big] = \pm\infty$.*

Then if $\lim\limits_{x \to b^-} \left[\frac{f'(x)}{g'(x)}\right] = L$ for $L \in \mathbb{R}^*$, then $\lim\limits_{x \to b^-} \left[\frac{f(x)}{g(x)}\right] = L$

Theorem 4.4.8. [Cauchy Mean Value Theorem]
Assume that $f(x), g(x)$ are continuous on $[a, b]$ and differentiable on (a, b) with $g(x) \neq 0$ for all $x \in (a, b)$. Then there exists a $c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{b - a}$.

5 Taylor's Theorem

Definition 5.0.1. *Suppose that $f(x)$ is n times differentiable at $x = a$. Then the n-th degree Taylor polynomial for $f(x)$ centered at $x = a$ is*

$$P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k$$

Definition 5.0.2. *Suppose that $f(x)$ is n times differentiable at $x = a$. Then the error term in approximating $f(x)$ using the n-th degree Taylor polynomial is*

$$R_{n,a}(x) = f(x) - P_{n,a}(x)$$

Theorem 5.0.3. [Taylor's Theorem]
Suppose that $f(x)$ is n times differentiable on an open interval I containing $x = a$. Then for every $x \in I$ and $x \neq a$, there exists some $c \in (x, a)$ such that

$$R_{n,a}(x) = f(x) - P_{n,a}(x) = \frac{f^{(n+1)}(c)}{n+1}(x-a)^{n-1}$$

5.1 Big O notation

Definition 5.1.1. *A function $f(x)$ is big O of $g(x)$ as $x \to a$ if there exists a $\delta > 0$ and $m > 0$ such that $|f(x)| \leqslant M|g(x)|$ for all $x \in (a-\delta, a+\delta)$, except possibly at $x = a$. This is expressed as $f(x) = \mathcal{O}(g(x))$ as $x \to a$.*

Definition 5.1.2. *For functions $f(x), g(x)$, the equation $f(x) = g(x) + \mathcal{O}(x^n)$ holds if $f(x) - g(x) = \mathcal{O}(x^n)$.*

Theorem 5.1.3. *Suppose that $f(x) = \mathcal{O}(x^n)$ and $g(x) = \mathcal{O}(x^m)$. Then*

1. $c \cdot f(x) = \mathcal{O}(x^n) \implies c \cdot \mathcal{O}(x^n) = \mathcal{O}(x^n)$
2. $f(x) + g(x) = \mathcal{O}(x^n) + \mathcal{O}(x^m) = \mathcal{O}(x^k)$ *such that* $k = \min\{n, m\}$
3. $f(x)g(x) = \mathcal{O}(x^n)\mathcal{O}(x^m) = \mathcal{O}(x^{n+m})$
4. $x^k f(x) = x^k \mathcal{O}(x^n) = \mathcal{O}(x^{k+n})$
5. *If $n \leqslant m$, then $g(x) = \mathcal{O}(x^n)$*
6. *If $n \geqslant 1$, then $\frac{1}{x}f(x) = \frac{1}{x}\mathcal{O}(x^n) = \mathcal{O}(x^{n-1})$*

Theorem 5.1.4. *Suppose that $f^{(n+1)}(x)$ is continuous on $[-\delta, \delta]$. Then $f(x) = P_{n,a}(x) + \mathcal{O}(x^{n+1})$.*

Theorem 5.1.5. *If $f^{(n+1)}(x)$ is continuous on $[-\delta, \delta]$ and if $p(x)$ is a polynomial of degree n or less with $f(x) = p(x) + \mathcal{O}(x^{n+1})$, then $p(x) = P_{n,0}(x)$.*

Theorem 5.1.6. *If $p(x)$ is a polynomial of degree $\leqslant n$ for $n \in \mathbb{N} \cup \{0\}$ and $p(x) = \mathcal{O}(x^{n+1})$, then $p(x) = 0$ for all x.*