Calculus

THEOREMS & PROOFS

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Contents

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1. The Peano axioms

1.1. WOP $\Rightarrow$ POMI.

· Let $S \subset \mathbb{N}$ such that

1: $1 \in S$
2: If $k \in S$, then $k+1 \in S$

· Let $T = \mathbb{N} \setminus S$
· Assume $T \neq \emptyset$
· By the Well Ordering Principle, T has a least element K_0.
· Now $K_0 \neq 1$, as $1 \in S$. Then we have that $K_0 - 1 \in \mathbb{N}$ and $K_0 - 1 < K_0$.
· Since K_0 is the least element in T, $K_0 - 1 \notin T$, which implies that $K_0 - 1 \in S$.
· Hence $K_0 = (K_0 - 1) + 1 \in S$ by **2**.
· Since this is not possible, we must have that $T = \emptyset$ and $S = \mathbb{N}$.

1.2. POMI $\Rightarrow$ POSI.

· Assume that $S \subset \mathbb{N}$ satisfies:

1: $1 \in S$
2: If $\{1, 2, \ldots k\} \in S$, then $k+1 \in S$

· Let $P(n)$ be the statement that $\{1, 2, \ldots k\} \in S$.
· If $P(n)$ holds for all $n \in \mathbb{N}$, then $S = \mathbb{N}$.
· Let $n = 1$. Clearly $1 \in S$, by the assumption **1**, so $P(1)$ holds.
· Assume that $P(k)$ holds. That is, $\{1, 2, \ldots k\} \in S$.
  · By the assumption **2**, $P(k+1) \in S$.
  · From this, $\{1, 2, \ldots k, k+1\} \in S$, and hence $P(k+1)$ holds.
· Therefore the Principle of Mathematical Induction shows that $P(n)$ holds for all $n \in \mathbb{N}$.

1.3. POSI $\Rightarrow$ WOP.

· Let $S \in \mathbb{N}$ be a set that does not have a least element. Also let
  $T = \mathbb{N} \setminus S = \{n \in \mathbb{N} \,|\, n \notin S\}$
· First, $1 \in T$, because if $1 \in S$, then it would be the least element of S.
· Assume that $\{1, 2, \ldots k\} \in T$.
· Since S does not have a least element, S cannot contain $k+1$.
· Therefore if $\{1, 2, \ldots k\} \in T$, then $k+1 \in T$.

· The Principle of Strong Induction shows that $T = \mathbb{N}$ and hence that $S = \emptyset$.
· If $S \in \mathbb{N}$ and S does not have a least element, then $S = \emptyset$.

· Therefore it follows that every nonempty subset of $\mathbb{N}$ has a least element.

2. Five-Star Theorems

2.1. **The Archimedean Property.** $\mathbb{N}$ is not bounded above.

· Assume that $\mathbb{N}$ is bounded above.
· Let $\alpha = lub(\mathbb{N})$
· Then $\alpha - 1$ is not an upper bound of $\mathbb{N}$.
· Then there exists an $n \in \mathbb{N}$ such that $\alpha - 1 < n \leqslant \alpha$.
· But then $\alpha = (\alpha - 1) + 1 < n + 1$
· This is impossible, as $\alpha = lub(\mathbb{N})$.
· Therefore $\mathbb{N}$ is not bounded above.

2.2. **The Monotone Convergence Theorem.** If $\{a_n\}$ is non-decreasing and bounded above, then $\{a_n\}$ converges.

· If $\{a_n\}$ is bounded above, then by the LUBP it has a least upper bound L.
· Let $\epsilon > 0$.
· Then $L - \epsilon < \epsilon$, so $L - \epsilon$ is not an upper bound for $\{a_n\}$.
· Hence there exists some $N_o \in \mathbb{N}$ with $L - \epsilon < a_{n_o}$
· If $n \geqslant N_o$, then $L - \epsilon < a_{n_o} \leqslant a_n \leqslant L \Rightarrow |a_n - L| < \epsilon$
· Hence $\lim\limits_{n\to\infty} [a_n] = L$.

2.3. **The Bolzano-Weierstrass Theorem.** Every bounded sequence $\{a_n\}$ has a convergent subsequence.

· By the Peak Point Lemma, $\{a_n\}$ has a monotonic subsequence $\{a_{n_k}\}$.
· Since $\{a_{n_k}\}$ is also bounded, it converges by the Monotone Convergence Theorem.

2.4. **The Completeness Theorem.** Every Cauchy sequence $\{a_n\} \subset \mathbb{R}$ converges.

· If $\{a_n\}$ is Cauchy, then $\{a_n\}$ is bounded.
· By the BWT, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to some L.
· Since $\{a_n\}$ is Cauchy, $\{a_n\}$ converges to L.

2.5. **The Squeeze Theorem for Sequences.** Assume that $\{a_n\} \leqslant \{b_n\} \leqslant \{c_n\}$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} [a_n] = L = \lim\limits_{n\to\infty} [c_n]$, then $\lim\limits_{n\to\infty} [b_n]$ exists and $\lim\limits_{n\to\infty} [b_n] = L$

· Let $\epsilon > 0$. Then we can find N_o so that if $n \geqslant N_o$, then
$$L - \epsilon < a_n \leqslant c_n < L + \epsilon$$
· If $n \geqslant N_o$, then
$$L - \epsilon < a_n \leqslant b_n \leqslant c_n < L + \epsilon$$
$$\Rightarrow |b_n - L| < \epsilon$$
· Hence the above statement holds.

2.6. **The Intermediate Value Theorem.** If $f(x)$ is continuous on $[a,b]$ and $f(a)<0$ and $f(b)>0$ Then there exists some $a<c<b$ such that $f(c)=0$).

· Let $S=\{x\in[a,b]|f(x)\leqslant 0\}$
· Since $a\in S$, $S\neq\emptyset$. And since S is bounded, it has a least upper bound.
· Let $c=lub(S)$.
· Then by the Sequential Characterization of Limits, there exists a sequence $\{x_n\}\subset S$ with $\{x_n\}\to c\in[a,b]$.
· Then $f(x_n)\to f(c)$ by continuity.
· Then since $f(x_n)\leqslant 0$, we have that $f(c)\leqslant 0$.
· Now let $\{y_n\}=c+\frac{b-c}{n}$
· Then $c<y_n\leqslant b$ and $\lim\limits_{n\to\infty}[y_n]=c$
· By continuity, $\lim\limits_{n\to\infty}[f(y_n)]=f(c)$
· Then since $f(y_n)<0$, we have that $f(c)\geqslant 0$.
· Hence $f(c)=0$.

2.7. **The Extreme Value Theorem.** If $f(x)$ is continuous on a finite closed interval $[a,b]$, then there exist $c,d\in[a,b]$ such that $f(c)\leqslant f(x)\leqslant f(d)$ for all $x\in[a,b]$.

· First we claim that $f(x)$ is bounded on $[a,b]$.
- Suppose that $f(x)$ is not bounded.
- Then for each $n\in\mathbb{N}$, we can find $\{x_n\}\subset[a,b]$ with $|f(x_n)|\geqslant n$.
- By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a subsequence x_{n_k} with $x_n\to x_o\in[a,b]$.
- By continuity, $f(x_{n_k})\to f(x_o)$.
- But from above we have that $|f(x_{n_k})|\geqslant n_k$, which implies that $\{f(x_{n_k})\}$ is not bounded.
- However, this contradicts the previous statement, so $f(x)$ is bounded.

· Let $T=\{f(x)|x\in[a,b]\}$
· Then T is bounded.
· Since T is nonempty, it has a least upper and greatest lower bound.
· Let $L=lub(T)$ and $M=glb(T)$
· There exist sequences $\{x_n\},\{y_n\}\subset[a,b]$ with
  i. $L-\frac{1}{n}\leqslant f(x_n)\leqslant L$
  ii. $M\leqslant f(y_n)\leqslant M+\frac{1}{n}$
· By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to x_o\in[a,b]$
· Choose $x_o=d$ so that $x_{n_k}\to d$
· By continuity, $f(x_{n_k})\to f(d)$
· By **i.** $f(x_{n_k})\to L$, hence $f(d)=L$
· Similarly we get a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ with $y_{n_k}\to c\in[a,b]$. Moreover, $f(c)=M$.

2.8. **Rolle's Theorem.** If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists $c \in (a, b)$ with $f'(c) = 0$.

· By the Extreme Value Theorem, $f(x)$ attains its maximum (minimum) on $[a, b]$
· Since $f(a) = f(b)$, either **i.** $f(x)$ is constant on $[a, b]$, or
ii. $f(x)$ attains its maximum (minimum) at some point $c \in (a, b)$
· In case **i.** $f'(c) = 0$ for all $c \in [a, b]$
· In case **ii.** the point c is a local maximum (minimum) for $f(x)$
· Then since $f'(x)$ exists, $f'(c) = 0$

2.9. **The Mean Value Theorem.** If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ with $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

· Define $g(x)$ to be the linear curve such that $g(x) = f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$
· Hence $g(x)$ is the secant line from $f(a)$ to $f(b)$.
· Define $F(x) = f(x) - g(x)$
· Then $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b)
· Since $F(a) = F(b) = 0$, by Rolle's Theorem there exists $c \in (a, b)$ with $F'(c) = 0$
· Then we have that $F'(x) = f'(x) - \left[\dfrac{f(b) - f(a)}{b - a}\right]$

$$0 = F'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2.10. **The Increasing Function Theorem.** If $f(x)$ is continuous on $[a.b]$ and differentiable on (a, b) with $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is strictly increasing on $[a, b]$.

· Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$.
· Then the Mean Value Theorem can be applied to $[x_1, x_2]$.
· Then there exists $c \in (x_1, x_2)$ with $0 < f(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$
· This implies that $f(x_2) - f(x_1) = f(c)(x_2 - x_1) > 0$
· Finally we have that $f(x_2) > f(x_1)$ for all $x_2 > x_1 \in [a, b]$.

3. Statement-only Theorems

3.1. **Fundamental Trigonometric Limit.**

$$\lim_{\theta \to 0} \left[\frac{\sin(\theta)}{\theta} \right] \text{ exists and equals 1.}$$

3.2. **Fundamental Logarithmic Limit.**

$$\lim_{x \to \infty} \left[\frac{\ln(x)}{x} \right] \text{ exists and equals 0.}$$

3.3. **Inverse Function Theorem.** Assume that $f(x)$ is defined on a continuous and open interval I containing some x_0. Also assume that $f(x)$ is either strictly increasing or strictly decreasing on I with inverse $g : J = f(I) \to I$. Then if $f(x)$ is differentiable at x_0 with $f'(x_0) \neq 0$, then $g(y)$ is differentiable at $y_0 = f(x_0)$ with

$$g'(y_0) = \frac{1}{f'(x_0)}$$

3.4. **First Derivative Test.** Suppose that $f : S \to R$, where $S \subseteq R$, that there exists an open interval (a, b) containing some $c \in S$, and that $[a, b] \subset S$. Assume also that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) except possibly at $x = c$. Finally, assume that $x = c$ is a critical point for $f(x)$.

1. If $f'(x) \leqslant 0$ on (a, c) and $f'(x) \geqslant 0$ on (c, b), then c is a local minimum for $f(x)$.
2. If $f'(x) \geqslant 0$ on (a, c) and $f'(x) \leqslant 0$ on (c, b), then c is a local maximum for $f(x)$.

3.5. **Second Derivative Test.** Suppose that $f : S \to R$, where $S \subseteq R$, and that I is an open interval such that $I \subset J \subset S$ where J is an open interval. If $f(x)$ is twice differentiable at every $x \in I$ then we have the following:

1. If $f''(x) \geqslant 0$ for all $x \in I$, then $f(x)$ is concave upward on I.
2. If $f''(x) \leqslant 0$ for all $x \in I$, then $f(x)$ is concave downward on I.

3.6. **Taylor's Theorem.** Suppose that $f : S \to R$, where $S \subseteq R$, and that $I \subset S$ is an open interval containing some $a \in S$. Suppose also that $f(x)$ is $n + 1$ times differentiable on I. Let $R_{n,a}(x)$ be the nth Taylor remainder of $f(x)$ centered at $x = a$. For each $x \in I$, there exists some $c := c_x \in I$with $x < c_x < a$ such that

$$\begin{aligned} R_{n,a}(x) &:= f(x) - P_{n,a}(x) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \end{aligned}$$

4. Function Characteristics

4.1. **Uniform Continuity.** If $f(x)$ is continuous in $[a, b]$, then it is uniformly continuous on $[a, b]$.

· Assume that $f(x)$ is not uniformly continuous on $[a, b]$.
· That is, assume $\{x_n\}, \{y_n\} \in [a, b]$ with $\lim_{n\to\infty} [x_n - y_n] = 0$, but $\lim_{n\to\infty} [f(x_n) - f(y_n)] \neq 0$
· Since $\lim_{n\to\infty} [f(x_n) - f(y_n)] \neq 0$, replacing $\{x_n\}, \{y_n\}$ with subsequences if necessary, we can assume that for some $\epsilon_0 > 0$ we have $|f(x_{n_k}) - f(y_{n_k})| \geqslant \epsilon_0$
· Since $\{x_n\} \in [a, b]$, by the Bolzano-Weierstrass Theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0 \in [a, b]$
· Since $\lim_{n\to\infty} [x_{n_k} - y_{n_k}] = 0$, we have that $y_{n_k} \to y_0 \in [a, b]$
· By continuity, $f(x_{n_k}) \to f(x_0)$ and $f(y_{n_k}) \to f(y_0)$
· Then $f(x_{n_k}) - f(y_{n_k}) \to 0$, which is impossible, since $\lim_{n\to\infty} [f(x_{n_k}) - f(y_{n_k})] \neq 0$.
· Hence we must have that $f(x)$ is uniformly continuous on $[a, b]$.

4.2. **Local Extrema.** If $f(x)$ has a local maximum or minimum at some $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

· Assume that $f(x)$ has a local maximum at $x = c$.
· Then there exists an open interval (a, b) containing c with $f(x) \leqslant f(c)$ for all $x \in (a, b)$.
· If $a < x < c$ then $\dfrac{f(x) - f(c)}{x - c} \geqslant 0$
· Hence $f'(c) = \lim_{x\to c^-} \left[\dfrac{f(x) - f(c)}{x - c}\right] \geqslant 0$
· If $c < x < b$ then $\dfrac{f(x) - f(c)}{x - c} \leqslant 0$
· Hence $f'(c) = \lim_{x\to c^+} \left[\dfrac{f(x) - f(c)}{x - c}\right] \leqslant 0$
· Therefore $f'(c) = 0$.
· A similar procedure can be applied if c is a local minimum.

4.3. **Functions and Big-O.** If $f(x)$ is $n+1$ times differentiable on some open interval $I \supset [-1,1]$ and $f^{(n+1)}(x)$ is continuous on $[-1,1]$, then $f(x) = P_{n,0}(x) + O(x^{n+1})$ as $x \to 0$.

· By the Extreme Value Theorem, $f^{(n+1)}(x)$ is bounded on $[-1,1]$.
· Choose $M \in \mathbb{R}$ such that $|f^{(n+1)}(x)| \leqslant M$ for all $x \in [-1,1]$.
· Taylor's Theorem implies that for any $x \in [-1,1]$ there exists $0 < c_x \leqslant x$ such that

$$\begin{aligned} |f(x) - P_{n,0}(x)| = \left| \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1} \right| &\leqslant \left| \frac{M}{(n+1)!} x^{n+1} \right| \\ &= \frac{M}{(n+1)!} \left| x^{n+1} \right| \\ &= O(x^{n+1}) \end{aligned}$$

· Hence we have that

$$\begin{aligned} f(x) - P_{n,0}(x) &= O(x^{n+1}) \\ f(x) &= P_{n,0}(x) + O(x^{n+1}) \end{aligned}$$