Definitions & Theorems

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Contents

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1 Integration

1.1 Riemann sums

Definition 1.1.1. *Let $[a,b]$ be a closed interval with $a<b$. A* <u>*partition*</u> *of $[a,b]$ is a finite subset P of $[a,b]$ of the form $P=\{a=x_0<x_1<\cdots<x_i<\cdots<x_n=b\}$. Given such a P, let $\Delta x_i = x_i - x_{i-1}$. Note that $\sum_{i=i}^{n} \Delta x_i = b-a$.*

Definition 1.1.2. *The* <u>*norm*</u> *of P is $||P|| = \max\limits_{i=1,\dots,n}\{\Delta x_i\}$*

Definition 1.1.3. *Let $P=\{a=x_0<\cdots<x_n=b\}$ be a partition on $[a,b]$. For each $i=1,\dots,n$:*
- $M_i = \mathrm{lub}\{f(x) : x \in [x_{i-1}, x_i]\}$
- $m_i = \mathrm{glb}\{f(x) : x \in [x_{i-1}, x_i]\}$

The <u>*upper Riemann sum*</u> *for $f(x)$ with respect to the partition P is $U_a^b(P,f) = \displaystyle\sum_{i=1}^{n} M_i \Delta x_i$.*

The <u>*lower Riemann sum*</u> *for $f(x)$ with respect to the partition P is $L_a^b(P,f) = \displaystyle\sum_{i=1}^{n} m_i \Delta x_i$.*

The <u>*Riemann sum*</u> *for $f(x)$ with respect to the partition P is $S_a^b(P,f) = \displaystyle\sum_{i=1}^{n} f(c_i)\Delta x_i$ for $c_i \in [x_{i-1}, x_i]$.*

- *Note that $L(f,P) \leqslant S(f,P) \leqslant U(f,P)$.*

Theorem 1.1.4. *Assume that Q is a refinement of P. Then $L(f,P) \leqslant L(f,Q) \leqslant U(f,Q) \leqslant U(f,P)$.*

Definition 1.1.5. *Assume that $f(x)$ is bounded on the interval $[a.,b]$.*

The <u>*upper Riemann integral*</u> *of $f(x)$ on $[a,b]$ is $\displaystyle\overline{\int_a^b} f(x)\, dx = \mathrm{glb}\{U(f,P) : P \text{ is a partition}\}$*

The <u>*lower Riemann integral*</u> *of $f(x)$ on $[a,b]$ is $\displaystyle\underline{\int_a^b} f(x)\, dx = \mathrm{lub}\{L(f,P) : P \text{ is a partition}\}$*

Definition 1.1.6. *If $f(x)$ is bounded on $[a,b]$, then $f(x)$ is* <u>*Riemann integrable*</u> *if*

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx$$

Then the common value is denoted by $\displaystyle\int_a^b f(x)\, dx$, which is the <u>*Riemann integral*</u> *of $f(x)$ over $[a,b]$.*

Theorem 1.1.7. *A function $f(x)$ is integrable on $[a,b]$ if and only if for every $\epsilon>0$ there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\epsilon$.*

Definition 1.1.8. *If $f(x)$ is integrable over $[a,b]$ and P_n is the n-regular partition of $[a,b]$, then*

$$\lim_{n\to\infty}\big[L(f,P_n)\big] = \int_a^b f(x)\, dx = \lim_{n\to\infty}\big[U(f,P_n)\big]$$

Theorem 1.1.9. *If $f(x)$ is integrable over $[a,b]$, then for every $\epsilon>0$, there exists a $\delta>0$ such that if P is any partition of $[a,b]$, with $||P||<\delta$ and $S(f,P)$ is any Riemann sum associated with P, then*

$$\left|\int_a^b f(x)\, dx - S(f,P)\right| < \epsilon$$

Theorem 1.1.10. *Suppose $f(x)$ is bounded on $[a,b]$. Then $f(x)$ is Riemann integrable on $[a,b]$ if and only if $f(x)$ is continuous on $[a,b]$ except possibly on a set of Lebesgue measure zero.*

Theorem 1.1.11. *If $f(x)$ is monotonic on $[a,b]$, then $f(x)$ is integrable on $[a,b]$.*

1.2 Properties of integrals

Theorem 1.2.1. *Assume that $f(x)$ and $g(x)$ are integrable over $[a, b] \subset \mathbb{R}$.*

i. If $c \in \mathbb{R}$, then $cf(x)$ is integrable over $[a, b]$ and $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$.

ii. $f(x) + g(x)$ is integrable over $[a, b]$ and $\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$.

Theorem 1.2.2. *Assume that $f(x)$ is integrable over $[a, b]$. Then $g(x) = |f(x)|$ is integrable over $[a, b]$. The converse is not necessarily true.*

Theorem 1.2.3. *Assume that $f(x)$ is bounded on $[a, b]$, and $c \in (a, b)$. Then $f(x)$ is integrable on $[a, b]$ if and only if $f(x)$ is integrable on $[a, c]$ and on $[c, b]$. Moreover, in this case*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Theorem 1.2.4. *Assume that $f(x)$ is integrable over $[a, b]$. Let*

$$\left.\begin{array}{rcl} m & = & \mathrm{glb}\{f(x) : x \in [a, b]\} \\ M & = & \mathrm{lub}\{f(x) : x \in [a, b]\} \end{array}\right\} \textit{Then } m \leqslant \frac{1}{b-a} \int_a^b f(x)\, dx \leqslant M.$$

Corollary 1.2.5. [Mean Value Theorem for Integrals]
Assume that $f(x)$ is continuous on $[a, b]$. Then there exists a $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$.

1.3 Fundamental Theorem of Calculus

Theorem 1.3.1. *Assume that $f(t)$ is such that over an interval I, $\int_x^y f(t)\, dt$ exists for each $x, y \in I$. Assume that $|f(t)| \leqslant M$ for all $t \in I$. Let $a \in I$. Let $F(x) = \int_a^x f(t)\, dt$. Then for any $x, y \in I$, $|F(x) - F(y)| \leqslant M|x - y|$.*

Theorem 1.3.2. [Fundamental Theorem of Calculus I]
Assume that $f(t)$ is integrable on $[a, b]$. Let $F(x) = \int_a^x f(t)\, dt$, and let $c \in (a, b)$. If $f(t)$ is continuous at $t = c$, then $F(x)$ is differentiable at $x = c$ with $F'(c) = f(c)$.

Theorem 1.3.3. [Extended Fundamental Theorem of Calculus]
Assume that $g(x), h(x)$ are differentiable, and that $f(x)$ is continuous on an open interval I. Let $F(x) = \int_{g(x)}^{h(x)} f(t)\, dt$. Then $F(x)$ is integrable on I with

$$F'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

Definition 1.3.4. *A function $F(x)$ is termed an <u>antiderivative</u> of $f(x)$ on I if $F'(x) = f(x)$ for all $x \in I$. The collection of all antiderivatives of $f(x)$ is denoted by $\int f(x)\, dx$, and termed the <u>indefinite integral</u>. The function $f(x)$ is the <u>integrand</u>.*

Corollary 1.3.5. *Assume that $F(x)$ is an antiderivative of $f(x)$ on I. Then $\int f(x)\, dx = F(x) + C$.*

Theorem 1.3.6. [Fundamental Theorem of Calculus II]
Assume that $f(t)$ in continuous on an interval I with $a, b \in I$. Assume that $F(t)$ is any antiderivative of $f(t)$ on I. Then

$$F(b) - F(a) = \int_a^b f(t)\, dt$$

1.4 Integral simplification

Theorem 1.4.1. [Change of Variables]
Assume that $g(x)$ is continuously differentiable on $[a,b]$, and $f(u)$ is continuous on $g([a,b])$. Then

$$\int_{g(a)}^{g(b)} f(u)\ du = \int_a^b f(g(x))g(x)\ dx$$

Theorem 1.4.2. [Integration by Parts]
Assume that $f(x), g(x)$ are continuously differentiable on $[a,b]$. Then

$$\int_a^b f(x)g(x)\ dx = f(x)\int_a^b g(x)\ dx - \int_a^b f'(x)\int g(x)\ dx\ dx$$

Definition 1.4.3. *A rational function is type I if $r(x) = \frac{p(x)}{q(x)}$ with*

- $deg(p(x)) < deg(q(x))$
- *if $q(a) = 0$, then $p(a) \neq 0$*
- $q(x) = c(x-a_1)(x-a_2)\cdots(x-a_n)$
  for $a_i \neq a_j$ if $i \neq j$

Then the partial fraction decomposition of $r(x)$ is that there exist $A_1, A_2, \ldots, A_n \in \mathbb{R}$ such that

$$r(x) = \frac{1}{c}\left[\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}\right] \qquad \text{and} \qquad A_i = \frac{p(a_i)}{\prod\limits_{i\neq j}(a_i - a_j)}$$

Definition 1.4.4. *A rational function is type II if $r(x) = \frac{p(x)}{q(x)}$ with*

- $deg(p(x)) < deg(q(x))$
- *if $q(a) = 0$, then $p(a) \neq 0$*
- $q(x) = c(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_n)^{m_n}$
  for $a_i \neq a_j$ if $i \neq j$, and at least one $m_i > 0$

Then every term $(x-a_i)^{m_i}$ contributes m_i terms to the decomposition in the form

$$\frac{A_{i1}}{(x-a_i)} + \frac{A_{i2}}{(x-a_i)^2} + \cdots + \frac{A_{im_i}}{(x-a_i)^{m_i}}$$

Definition 1.4.5. *A rational function is type III if $r(x) = \frac{p(x)}{q(x)}$ with*

- $deg(p(x)) < deg(q(x))$
- *if $q(a) = 0$, then $p(a) \neq 0$*
- $q(x) = c(x-a_1)^{m_1}\cdots(x-a_k)^{m_k}(x^2+b_{k+1}x+c_{k+1})^{m_{k+1}}\cdots(x^2+b_nx+c_n)^{m_n}$
  for $a_i \neq a_j$ if $i \neq j$, at least one $m_i > 0$, and $(x^2+b_ix+c_i)$ irreducible for $k+1 \leqslant i \leqslant n$

Then every term $(x-a_i)^{m_i}$ and $(x^2+b_jx+c_j)^{m_j}$ contributes to the decomposition as follows:

$$(x-a_i)^{m_i} \to \frac{A_{i1}}{(x-a_i)} + \cdots + \frac{A_{im_i}}{(x-a_i)^{m_i}}$$
$$(x^2+b_jx+c_j)^{m_j} \to \frac{B_{j1}x + C_{j1}}{(x^2+b_jx+c_j)} + \cdots + \frac{B_{jm_j}x + C_{jm_j}}{(x^2+b_jx+c_j)^{m_j}}$$

1.5 Applications of integration

Definition 1.5.1. *The area between two curves $f(x)$ and $g(x)$ over an interval $[a,b]$ is defined as*

$$A = \int_a^b |f(x) - g(x)| \, dx$$

Definition 1.5.2. *The moment is defined as the mass times distance. For an area bounded above by a function $g(x)$, below by $f(x)$, over an interval $[a,b]$, this becomes:*

$$M_x = \int_a^b x(g(x) - f(x)) \, dx \qquad \text{and} \qquad M_y = \int_a^b \frac{1}{2}\left(g(x)^2 - f(x)^2\right) dx$$

Definition 1.5.3. *The center of mass is given by the coordinate $\left(\frac{M_x}{A}, \frac{M_y}{A}\right)$.*

Definition 1.5.4. *The arc length of the graph of a function $f(x)$ over an interval $[a.b]$ is given by*

$$S = \int_a^b \sqrt{1 + f'(x)^2} \, dx$$

Definition 1.5.5. *The volume of revolution of a function $f(x)$ over an interval $[a,b]$ is given by*

$$V_x = \int_a^b \pi f(x)^2 \, dx \qquad \text{and} \qquad V_y = \int_a^b 2\pi x f(x) \, dx$$

1.6 Improper integrals

Definition 1.6.1. *Assume that $f(x)$ is integrable on $[a,b]$ for all $b > a$. Then*

$$\int_a^\infty f(x) \, dx = \lim_{b\to\infty} \left[\int_a^b f(x) \, dx\right]$$

This is termed an improper integral of the first kind. The improper integral converges if and only if the limit of the proper integral exists.

Theorem 1.6.2. [Comparison Test for Integrals]
Assume that $0 \leqslant f(t) \leqslant g(t)$ for all $x \in [a,\infty)$, and that $f(t), g(t)$ are integrable on $[a,b)$ for all $b \in [a,\infty)$. Then

1. *If $\int_a^\infty g(t) \, dt$ converges, then $\int_a^\infty f(t) \, dt$ converges.*

2. *If $\int_a^\infty f(t) \, dt$ diverges, then $\int_a^\infty g(t) \, dt$ diverges.*

Theorem 1.6.3. *Assume that $f(t)$ is integrable on $[a,b]$ for all $b \in [a,\infty)$. Assume that $\int_a^b |f(t)| dt$ converges. Then $\int_0^\infty f(t) \, dt$ converges.*

Theorem 1.6.4. [P-Test for Integrals]
$\displaystyle\int_a^\infty \frac{1}{x^p} \, dx$ converges if and only if $p > 1$.

2 Sequences and series

2.1 Convergence

Theorem 2.1.1. [Divergence Test]
If $\{a_n\}$ *is any sequence with* $\sum_{n=1}^{\infty} a_n$ *convergent, then* $\lim_{n\to\infty} [a_n] = 0$.

Definition 2.1.2. *Given* $\{a_n\} \subset \mathbb{R}$*, define* $\limsup\{a_n\} = \lim_{n\to\infty}\left[\sup_{k\geqslant n}\{a_k\}\right]$.

Remark 2.1.3. *The ratio test only detects rapidly convergent / divergent series.*

Definition 2.1.4. *A series of the from* $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *is termed an* <u>*alternating series*</u>*.*

Definition 2.1.5. *A series* $\sum_{n=0}^{\infty} a_n$ <u>*converges absolutely*</u> *if* $\sum_{n=0}^{\infty} |a_n|$ *converges.*

Definition 2.1.6. *A series* $\sum_{n=0}^{\infty} a_n$ <u>*converges conditionally*</u> *if* $\sum_{n=0}^{\infty} a_n$ *converges, but not* $\sum_{n=0}^{\infty} |a_n|$.

Theorem 2.1.7. *If a series converges absolutely, then it converges.*

Theorem 2.1.8. *If* $\sum_{n=0}^{\infty} a_n$ *converges absolutely, and if* $\varphi : \mathbb{N} \to \mathbb{N}$ *is one-to-one and onto, and if* $b_n = a_{\varphi(n)}$*, then* $\sum_{n=0}^{\infty} b_n$ *converges.*

Theorem 2.1.9. *If* $\sum_{n=0}^{\infty} a_n$ *is convergent conditionally, then there exists a bijection* $\varphi : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{n=0}^{\infty} a_{\varphi(n)} = \alpha$ *for any* $\alpha \in [-\infty, \infty]$.

2.2 Power series

Definition 2.2.1. *A* <u>*power series*</u> *centered at* $x = a \in \mathbb{R}$ *is a formal series of the form* $\sum_{n=0}^{\infty} a_n(x-a)^n$*, where* $\{a_n\}$ *is a sequence of coefficients.*

Theorem 2.2.2. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series. Assume that* $\sum_{n=0}^{\infty} a_n x^n$ *converges for* $x_\circ \neq 0$*. If* $0 \leqslant |x_1| \leqslant |x_\circ|$*, then* $\sum_{n=0}^{\infty} a_n x_1^n$ *will converge absolutely.*

Definition 2.2.3. *The* <u>*radius of convergence*</u> *of a series is defined as* $R = \sup\left\{|x_\circ| : \sum_{n=0}^{\infty} a_n x_\circ^n \text{ converges}\right\}$.

Remark 2.2.4. *Given a series*$\{a_n\}$ *and its radius of convergence* R*:*
1. *If* $R = 0$*, then the series converges on* $\{0\}$.
2. *If* $R = \infty$*, then the series converges on all of* $\mathbb{R} = (-\infty, \infty)$.
3. *If* $0 < R < \infty$*, then the series converges on* $(-R, R)$ *or* $[-R, R)$ *or* $(-R, R]$ *or* $[-R, R]$.

Theorem 2.2.5. *Given a power series* $\sum_{n=0}^{\infty} a_n x^n$, *let* $\lim_{n\to\infty}\left[\,\left|\dfrac{a_{n+1}}{a_n}\right|\,\right] = L \in [0,\infty]$. *Then*

1. *If* $0 < L < \infty$, *then* $R = \frac{1}{L}$.
2. *If* $L = 0$, *then* $R = \frac{1}{0} = \infty$.
3. *If* $L = \infty$, *then* $R = \frac{1}{\infty} = 0$.

Theorem 2.2.6. *Given a series* $\sum_{n=0}^{\infty} a_n x^n$, *the radius of convergence is* $R = \dfrac{1}{\limsup\{\sqrt[n]{a_n}\}}$

Definition 2.2.7. *The set of values* x *on which a power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges is termed the interval of convergence.*

2.3 Sequences of functions

Definition 2.3.1. *A sequence of functions* $\{f_n\}$ *converges pointwise on* $S \subset \mathbb{R}$ *to* $f(x)$ *if for each* $x_\circ \in S$, $f(x_\circ) = \lim_{n\to\infty}\big[f_n(x_\circ)\big]$.

Theorem 2.3.2. *If* $\{f_n\}$ *is a sequence of continuous functions converging pointwise to* $f(x)$ *on* S, $f(x)$ *must have at least one point of continuity.*

Definition 2.3.3. *A sequence of functions* $\{f_n\}$ *converges uniformly on* $S \subset \mathbb{R}$ *to* $f(x)$ *if for every* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that if* $n \geqslant N$, *then* $|f_n(x) - f(x)| < \epsilon$ *for all* $x \in S$.

· *Note that uniform convergence implies pointwise convergence.*

Theorem 2.3.4. *If* $\{f_n\}$ *is a sequence of functions that converges uniformly on* S *to* $f(x)$, *and if each* $f_n(x)$ *is continuous at* $x_\circ$ *relative to* S, *then* $f(x)$ *is continuous at* $x_\circ$ *relative to* S.

Theorem 2.3.5. *Suppose that* $\{f_n\}$ *is a sequence of integrable functions on some interval* $[a,b]$. *Assume that* $\{f_n\}$ *converges uniformly to* $f(x)$ *on* $[a,b]$. *Then* $f(x)$ *is integrable, and*

$$\int_a^b f(x)\,dx = \int_a^b \lim_{n\to\infty}\big[f_n(x)\big] = \lim_{n\to\infty}\left[\int_a^b f_n(x)\,dx\right]$$

3 Normed Linear spaces

3.1 Norms

Definition 3.1.1. *Let* V *be a vector space over* $\mathbb{R}$. *A norm on* V *is a function* $||\cdot|| : V \to \mathbb{R}$ *such that*

1. $||v|| \geqslant 0$ *and* $||v|| = 0$ *if and only if* $v = 0$
2. $||\alpha v|| = |\alpha| \cdot ||v||$
3. $||v + w|| \leqslant ||v|| + ||w||$ [Triangle Inequality]

for any $\alpha \in \mathbb{R}$ *and* $u, v \in V$. *The norm is an abstract notion of length.*

Definition 3.1.2. *The Euclidean norm is defined for* $V = \mathbb{R}^n$ *and* $\vec{x} \in \mathbb{R}^n$ *such that* $\vec{x} = x_1, \ldots, x_n$:

$$||\vec{x}|| = ||x_1, \ldots, x_n|| = \sqrt{x_1^2 + \cdots + x_n^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

Definition 3.1.3. *The dot product of* $\vec{x}, \vec{y} \in \mathbb{R}^n$ *is defined as* $\vec{x} \bullet \vec{y} = x_1 y_1 + \cdots + x_n y_n$.

Theorem 3.1.4. [Cauchy-Schwarz Inequality] *For* $a, b \in V$ *for some vector space* V

$$\sum_{i=1}^{n} |\,a_i b_i\,| \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} \qquad \text{or, equivalently} \qquad \vec{a} \bullet \vec{y} \leqslant ||a|| \cdot ||b||$$

Definition 3.1.5. *The ordered pair $(V, ||\cdot||)$ is termed a* normed linear space*.*
For $V = \mathbb{R}^n$ and $v \in V$:

- *The 1-norm is defined as* $||v||_1 = ||v_1, \dots, v_n||_1 = |v_a| + \cdots |v_n| = \sum_{i=1}^{n} |v_i|$
- *The infinity norm is defined as* $||v||_\infty = ||v_1, \dots, v_n||_\infty = \max_{1 \leqslant i \leqslant n} \{v_i\}$
- *The p-norm for $1 < p < \infty$ is defined as* $||v||_p = ||v_1, \dots, v_n||_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$

It is always the case that $||v||_\infty \leqslant ||v||_p \leqslant ||v||_1$.

Definition 3.1.6. *The* inner product *on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that*

1. $\langle v, w \rangle = \langle w, v \rangle$
2. $\langle v + w, z \rangle = \langle v, z \rangle + \langle w, x \rangle$
3. $\langle 2v, z \rangle = 2\langle v, z \rangle$
4. $\langle v, v \rangle \geqslant 0$ *and* $\langle v, v \rangle = 0$ *if and only if* $v = 0$

for all $v, w, z \in V$. Then $(V, \langle \cdot, \cdot \rangle)$ is termed an inner product space*.*

Remark 3.1.7. *For the terms defined above, the norm can also be defined as $||v|| = \langle v, v \rangle^{1/2}$.*

3.2 Norms and continuous functions

Definition 3.2.1. *Define the vector space $V = C[a, b] = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous }\}$ as the set of all continuous functions on the interval $[a, b] \subset \mathbb{R}$.*

Remark 3.2.2. *The following are examples of norms on this vector space:*

- $||f||_1 = \int_a^b |f(x)| dx$
- $||f||_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$
- $||f||_\infty = \max_{x \in [a,b]} \{|f(x)| : x \in [a, b]\}$
- $\big\langle f(x), g(x) \big\rangle = \int_a^b f(x) g(x) \, dx$

Definition 3.2.3. *Let X be a vector space. Then a* metric *on X is a function $d : X \times X \to \mathbb{R}$ with*

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geqslant 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$
3. $d(x, y) \leqslant d(x, z) + d(z, y)$

for all $x, y, z \in X$.

Definition 3.2.4. *With respect to the above definition, the ordered pair (X, d) is termed a* metric space*.*

3.3 Convergence in a metric space

Definition 3.3.1. *Given a metric $d : X \times X \to \mathbb{R}$, a sequence $\{x_n\} \subset X$* converges *to $x_\circ \in X$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geqslant N$, then $d(x_\circ, x_n) < \epsilon$. Therefore $\lim_{n \to \infty} \big[d(x_\circ, x_n) \big] = 0$.*

Definition 3.3.2. *A sequence $\{x_n\} \subset (X, d)$ is* Cauchy *if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $k, m \geqslant N$, then $d(x_k, x_m) < \epsilon$.*

Theorem 3.3.3. *If $\{x_n\}$ is convergent in (X, d), then $\{x_n\}$ is Cauchy.*

Remark 3.3.4. *If $X = \mathbb{Q}$ and $d(x,y) = |x-y|$, then there exist Cauchy sequences in (X,d) that do not converge.*

Definition 3.3.5. *A set $A \subset (X,d)$ is* <u>*bounded*</u> *if there exists $x_\circ \in X$ and $M > 0$ such that $d(x_\circ, x) \leqslant M$ for all $x \in A$.*

Theorem 3.3.6. *If $\{x_n\} \subset (X,d)$, then $\{x_n\}$ is bounded.*

Definition 3.3.7. *A metric space is* <u>*complete*</u> *if every Cauchy sequence in the metric space converges.*

Theorem 3.3.8. [Completeness theorem for C[a,b]]
The metric space $C[a,b]$ is complete.

Remark 3.3.9. *A sequence $\{f_n\} \subset C[a,b]$ converges to $f \in C[a,b]$ in $||\cdot||_\infty$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geqslant N$, then $||f_n - f||_\infty = \max\{|f_n(x) - f(x)| < \epsilon : x \in [a,b]\}$ if and only if f_n converges to f uniformly.*

Theorem 3.3.10. *Suppose $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Let $0 \leqslant x_1 < R$. Let $f_k(x) = \displaystyle\sum_{n=0}^{k} a_n x^n$. Then $\{f_k\}$ converges uniformly on $[-x_1, x_1]$ to $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.*

3.4 Differentiability and integrability

Theorem 3.4.1. *If a power series has an interval of convergence I, then the series is continuous on I.*

Corollary 3.4.2. *Suppose that a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Assume that $[a,b] \subset (-R,R)$. If $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, then*

$$\int_a^b f(x)\ dx = \int_a^b \sum_{n=0}^{\infty} a_n x^n\ dx = \sum_{n=0}^{\infty} \left(\int_a^b a_n x^n\ dx \right) = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \Bigg|_a^b$$

Corollary 3.4.3. *Suppose that a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Assume that $[a,b] \subset (-R,R)$. If $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, then*

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(\frac{d}{dx} a_n x^n \right) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

Definition 3.4.4. *Given a series $\sum_{n=0}^{\infty} a_n x^n$:*
- *The series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ is termed the* <u>*formal derivative*</u> *of the given series*
- *The series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is termed the* <u>*formal integral*</u> *of the given series*

Theorem 3.4.5. *Suppose $\{F_n\} \subset C[a,b]$ with $\lim\limits_{n\to\infty} [F_n(a)] = a_\circ$. If $\{F_n\}$ has continuous derivatives $F_n'(x) = f_n(x)$, such that $\{f_n\}$ converges uniformly on $[a,b]$ to $g(x) \in C[a,b]$, then $\{F_n\}$ converges uniformly to a continuous function $G \in C[a,b]$ such that $G'(x) = g(x)$ for all $x \in (a,b)$.*

Corollary 3.4.6. *If $f(x)$ is represented by a power series $\sum_{n=0}^{\infty} a_n x^n$ on $(-R,R)$ for $R > 0$, then $f(x)$ is infinitely differentiable on $(-R,R)$ with*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \prod_{i=0}^{k-1} (n-i)\ a_n x^{n-k}$$

Theorem 3.4.7. [Uniqueness of Representation]
Assume that $f(x)$ has power series representations $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ centered at $x = a$ where each series has a positive radius of convergence. Then $a_n = b_n = \frac{f^{(n)}(a)}{n!}$.

Definition 3.4.8. *Assume that $f(x)$ is n times differentiable at $x = a$. Then the k-th degree Taylor polynomial for $f(x)$ at $x = a$ is*

$$P_{k,a}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Definition 3.4.9. *Assume that $f(x)$ is n times differentiable at $x = a$. Then the n-th degree error term in using $P_{n,a}(x)$ to approximate $f(x)$ is*

$$R_{n,a}(x) = f(x) - P_{n,a}(x)$$

Theorem 3.4.10. *Assume that $f(x)$ is $n+1$ times differentiable on an open interval I containing a. Let $x \in I$ with $x \neq a$. Then there exists a $c \in (x, a)$ such that*

$$R_{n,a}(x) - P_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Remark 3.4.11. *Assume that $x_1 > 0$ and there exists M such that $|f^{(n)}(x)| < M$ for all $n \in \mathbb{N} \cup \{0\}$ and for all $x \in [-x_1, x_1]$. Then the Taylor series summation $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n$ converges uniformly to $f(x)$ on $[-x_1, x_1]$.*

Theorem 3.4.12. *If two functions have power series representations $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then $h(x) = f(x)g(x)$ is represented by the power series*

$$h(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

Remark 3.4.13. *Let $f, g \in (C[a,b], ||\cdot||_\infty)$. Note that if $x \in [a,b]$, then $|f \cdot g(x)| \leqslant ||f||_\infty ||g||_\infty$. Hence $||f \cdot g||_\infty \leqslant ||f||_\infty ||g||_\infty$.*

Theorem 3.4.14. *If $\{f_n\}, \{g_n\} \in C[a,b]$ such that $f_n \to f_\circ, g_n \to g_\circ$ for $f_\circ, g_\circ \in C[a,b]$, that is, both f_n an g_n converge uniformly, then $f_n g_n \to f_\circ g_\circ$ in $C[a,b]$.*

Theorem 3.4.15. [Weierstrass Approximation Theorem]
If $f \in C[a,b]$, then there exists a sequence $\{p_n\}$ of polynomials such that $p_n(x) \to f(x)$ uniformly on $[a,b]$.

Theorem 3.4.16. [Weierstrass M-test I]
Suppose that $\{f_n\}$ is a sequence in $(C[a,b], ||\cdot||_\infty)$. If $\sum_{n=1}^{\infty} ||f_n||_\infty$ converges, then $\sum_{n=1}^{\infty} f_n$ converges in $(C[a,b], ||\cdot||_\infty)$.

Theorem 3.4.17. [Weierstrass M-test II]
Let $(V, ||\cdot||_\infty)$ be a normed linear space. Then the following are equivalent:
- *$(V, ||\cdot||_\infty)$ is complete*
- *If $\{v_n\} \in V$ is such that $\sum_{n=1}^{\infty} ||v_n||_\infty < \infty$, then $\sum_{n=1}^{\infty} v_n$ converges.*

Theorem 3.4.18. [Banach Contractive Mapping Theorem]
Suppose that $\Gamma : C[a,b] \to C[a,b]$ is a contractive map. That is, suppose that Γ is such that there exists k satisfying $0 \leqslant k < 1$ with

$$\Gamma(u) - \Gamma(v)_\infty \leqslant ku - v_\infty$$

for all $u, v \in C[a,b]$. Then there exists a unique function $f \in C[a,b]$ such that $\Gamma(f) = f$.

4 Differential equations

4.1 Separable equations

Definition 4.1.1. *A differential equation is an equation of the form* $f(x, y, y', y'', \ldots, y^{(n)}) = 0$*.*

Definition 4.1.2. *The order of the differential equation is the order of the highest derivative in the differential equation.*

Definition 4.1.3. *A first order ordinary differential equation is said to be separable if it can be written in the form* $y' = f(x)g(y)$*.*