Calculus

PROOFS

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1. Five-Star Theorems

1.1. Fundamental Theorem of Calculus I. Assume that $f(x)$ is integrable on $[a, b]$. Let $F(x) = \int_a^x f(t)\,dt$, and let $c \in (a, b)$. If $f(t)$ is continuous at $t = c$, then $F(x)$ is differentiable at $x = c$, and $F'(c) = f(c)$.

· Let $\epsilon > 0$.
· Then there exists $\delta > 0$ such that if $|c - x| < \delta$, then $|f(c) - f(x)| < \epsilon$.
· We may assume that $\delta < \min\{c - a, b - c\}$.
· Let $0 < |h| < \delta$.
· Consider $\dfrac{F(c+h) - F(c)}{h}$

$$= \frac{\int_a^{c+h} f(t)\ dt - \int_a^c f(t)\,dt}{h}$$

$$= \frac{\int_a^{c+h} f(t)\ dt + \int_c^a f(t)\,dt}{h}$$

$$= \frac{1}{h}\int_c^{c+h} f(t)\,dt$$

· We know that if $t \in (c, c+h)$, then $|t - c| \leqslant |h| < \delta$
· This implies that $f(c) - \epsilon < f(t) < f(c) + \epsilon$
· Hence $f(c) - \epsilon < \dfrac{1}{h}\displaystyle\int_c^{c+h} f(t)\,dt < f(c) + \epsilon$
· This implies that $\left|\dfrac{F(c+h) - F(c)}{h} - f(c)\right| < \epsilon$
· Therefore $F'(c) = \lim\limits_{h \to 0}\left[\dfrac{F(c+h) - F(c)}{h}\right] = f(c)$

1.2. Fundamental Theorem of Calculus II. Assume that $f(t)$ is continuous on an interval I containing a, b. Assume that $F(t)$ is any antiderivative of $f(t)$ on I. Then $F(b) - F(a) = \int_a^b f(t)\,dt$.

· We may assume that $a < b$.
· Let $G(x) = \int_0^x f(t)\,dt$.
· Then by the Fundamental Theorem of Calculus I, $G'(x) = f(x)\ \forall\ x \in (a, b)$.
· Then by the Mean Value Theorem, there exists a constant c such that
  $F(x) = G(x) + c\ \forall\ x \in [a, b]$
· In particular,

$$\begin{aligned} F(b) - F(a) &= (G(b) + c) - (G(a) + c) \\ &= G(b) - G(a) \\ &= \int_0^b f(t)\,dt - \int_0^a f(t)\,dt \\ &= \int_a^b f(t)\,dt \end{aligned}$$

1.3. **Change of Variables Theorem.** Assume that $g(x)$ is continuously differentiable on $[a,b]$ and that $f(x)$ is continuous on $g([a,b])$. Then $\displaystyle\int_{g(a)}^{g(b)} f(u)\, du = \int_a^b f(g(x))g'(x)\, dx$.

- Let $\displaystyle F(t) = \int_{g(a)}^{t} f(u)\, du$.
- Then $F(u)$ is differentiable and defined on the interior of $g([a,b])$ and is continuous on $g([a,b])$ with $F'(t) = f(t)$ by the Fundamental Theorem of Calculus I.
- Let $H(x) = F(g(x))$.
- Then by the chain rule, $$\begin{aligned} H'(x) &= F'(g(x))g'(x) \text{ on } (a,b) \\ &= f(g(x))g'(x) \end{aligned}$$
- Then by the Fundamental Theorem of Calculus I,

$$\begin{aligned} \int_a^b f(g(x))g'(x)\, dx &= H(b) - H(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(u)\, du - \int_{g(a)}^{g(a)} f(u)\, du \\ &= \int_{g(a)}^{g(b)} f(u)\, du \end{aligned}$$

1.4. **Comparison Test for Series.** Let $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ be two positive series satisfying $0 \leqslant a_n \leqslant b_n$. Then

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

- Let $\displaystyle S_k := \sum_{n=1}^{k} a_n$ and $\displaystyle T_k := \sum_{n=1}^{k} b_n$ be the $k-$th partial sums of the two series. Let $\displaystyle T := \sum_{n=1}^{\infty} b_n$.
- Since T_k is nondecreasing, we have that $T_k \leqslant T \ \forall\, k$.
- Hence $\displaystyle S_k = \sum_{n=1}^{k} a_n \leqslant \sum_{n=1}^{k} b_n = T_k \leqslant T \ \forall\, k$
- Hence $\{S_k\}$ is bounded above by T.
- Since $\{S_k\}$ is nondecreasing, $\{S_k\}$ converges by the Monotone Convergence Theorem.
- Part **2.** is simply the contrapositive of the above and follows immediately.

1.5. **Limit Comparison Test for Series.** Let $\{a_n\}$ and $\{b_n\}$ be positive sequences with $b_k \neq 0 \;\forall\, k$. Assume that $\lim_{n\to\infty}\left[\frac{a_n}{b_n}\right] = L$ where $L \in [0,\infty)$ or $L = \infty$. Then:

1. If $L \in (0,\infty)$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

3. If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

Proof for 1:

· Assume that $L \in (0,\infty)$.

· Since $\lim_{n\to\infty}\left[\frac{a_n}{b_n}\right] = L$, there exists an $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$\begin{aligned}
\left|\frac{a_n}{b_n} - L\right| &< \frac{L}{2}\\
-\frac{L}{2} < \frac{a_n}{b_n} - L &< \frac{L}{2}\\
\frac{L}{2} < \frac{a_n}{b_n} &< \frac{3L}{2}\\
\frac{L}{2}\cdot b_n < a_n &< \frac{3L}{2}\cdot b_n \qquad \text{for all } n \geqslant \mathbb{N}
\end{aligned}$$

· If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=N}^{\infty} a_n$ converges because it is a tail of the former.

· By the Comparison Test, $\sum_{n=N}^{\infty}\left[\frac{L}{2}\cdot b_n\right]$ converges.

· Since $L \neq 0$, we must have that $\sum_{n=1}^{\infty} b_n$ converges.

· Similarly, if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=N}^{\infty}\left[\frac{3L}{2}\cdot b_n\right]$.

· By the Comparison Test, $\sum_{n=N}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} a_n$ converges.

Proof for 2:

· If $L = 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$0 \leqslant \frac{a_n}{b_n} \leqslant 1 \quad \Longrightarrow \quad 0 \leqslant a_n \leqslant b_n \quad \text{for all } n \geqslant N$$

· By the comparison test and by series properties, if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Proof for 3:

· If $L = \infty$, then there exists an $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$\frac{a_n}{b_n} > 1 \quad \Longrightarrow \quad a_n > b_n > 0 \quad \text{for all } n \geqslant N$$

· By the comparison test and by series properties, if $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} b_n$.

1.6. **Integral Test.** Define $f(n) = a_n$ for each $n \in \mathbb{N}$. Assume that $f(x)$ is continuous on $[1, \infty)$, $f(x) \geqslant 0$ on $[1, \infty)$, and $f(x)$ is decreasing on $[1, \infty)$. If $S_k = \displaystyle\sum_{n=1}^{k} a_n$ and $S = \displaystyle\sum_{n=1}^{\infty} a_n$, then

1. $\displaystyle\int_1^{k+1} f(t)\, dt \leqslant S_k \leqslant \int_1^k f(t)\, dt + a_1 \quad$ for all $k \in \mathbb{N}$

2. $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\int_1^{\infty} f(t)\, dt$ converges.

3. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $0 \leqslant S - S_k \leqslant \displaystyle\int_k^{\infty} f(t)\, dt$

Proof for 1:

· Since $f(x)$ is decreasing, for all $k \in \mathbb{N}$ we have $\displaystyle\int_1^{k+1} f(t)\, dt \leqslant U_1^{k+1}(f, P_k)$, where P_k is the regular $k-$partition on $[1, k+1]$

· Then $U_1^{k+1}(f, P_k) = \displaystyle\sum_{n=1}^{k} f(n) = \sum_{n=1}^{k} a_n = S_k$

· SImilarly, we also have $\displaystyle\int_1^k f(x)\, dx \geqslant L_1^k(f, P_{k-1})$, where P_{k-1} is the regular $(k-1)-$partition on $[1, k]$.

· Then $L_1^k(f, P_{k-1}) = \displaystyle\sum_{n=2}^{k} f(n) = \sum_{n=2}^{k} a_n = S_k - a_1$ for all $k \geqslant 2$, but also for $k = 1$.

· Combining, we have that for all $k \in \mathbb{N}$,

$$\int_1^{k+1} f(x)\, dx \leqslant S_k \leqslant a_1 + \int_1^k f(x)\, dx$$

Proof for 2:

· Assume that $\displaystyle\int_1^{\infty} f(t)\, dt$ converges.

· Then for each k, $S_k \leqslant \displaystyle\int_1^k f(t)\, dt + a_1 \leqslant \int_1^{\infty} f(t)\, dt + a_1$

· This implies that S_k is bounded and increasing.

· By the Monotone Convergence Theorem, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

· Assume that $\displaystyle\int_1^{\infty} f(t)\, dt$ diverges.

· Then $\left\{\displaystyle\int_1^k f(t)\, dt\right\}$ is unbounded.

· This implies that $\left\{\displaystyle\sum_{n=1}^{\infty} a_n\right\} = \{S_k\}$ is unbounded.

· Hence $\{S_k\}$ diverges to ∞.

Proof for 3:

· Assume that $\sum_{n=1}^{\infty} a_n$ converges. Then

$$\begin{aligned} 0 \quad \leqslant \quad S - S_k &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{k} a_n \\ &= \sum_{n=k+1}^{\infty} a_n \\ &= \lim_{j\to\infty} \left[\sum_{n=k+1}^{j} a_n \right] \qquad \text{for all } j \geqslant k+1 \\ &\leqslant \lim_{j\to\infty} \left[\int_1^j f(t)\, dt \right] \\ &\leqslant \int_1^{\infty} f(t)\, dt \end{aligned}$$

1.7. **Root Test.** Let $0 < a_n$ for all n. Then

1. If $\limsup\{\sqrt[n]{a_n}\} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\limsup\{\sqrt[n]{a_n}\} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof for 1:

· Assume that $L < 1$.

· Then we can find $0 \leqslant L < r < 1$.

· Moreover, there exists $N_\circ$ such that if $n \geqslant N_\circ$, then $\sqrt[n]{a_n} < r \Longrightarrow a_n < r^n$.

· This implies that $\sum_{n=1}^{\infty} r^n$ converges

· Then the Comparison Test shows that $\sum_{n=1}^{\infty} a_n$ converges.

Proof for 2:

· Assume that $\limsup\{\sqrt[n]{a_n}\} = L > 1$.

· Then there exists $1 < s < L$ for some s.

· Then there exists a subsequence $\{a_{n_k}\}$ with $s < \sqrt[n]{a_{n_k}}$ for each k.

· But then $a_{n_k} \geqslant s^n > 1$ for all k.

· Hence $\lim_{n\to\infty} a_n \neq 0$, and thus the series diverges by the Divergence Test.

1.8. **Ratio Test.** Let $\{a_n\}$ be a sequence with $a_n > 0 \; \forall \; n$ and let $L = \lim_{n\to\infty} \left[\frac{a_{n+1}}{a_n}\right]$. Then

1. If $0 < L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof for 1:

· Assume that $0 < L < 1$.

· Then there exists $L < r < 1$ for some r.

· Then there also exists $N_\circ \in \mathbb{N}$ with $\frac{a_{n+1}}{a_n} < r$ for all $n \geqslant N_\circ$.

· This implies that

$$\begin{aligned}
\frac{a_{N_\circ+1}}{a_{N_\circ}} < r &\Longrightarrow a_{N_\circ+1} < a_{N_\circ} r \\
\frac{a_{N_\circ+2}}{a_{N_\circ+1}} < r &\Longrightarrow a_{N_\circ+2} < a_{N_\circ+1} r < a_{N_\circ} r^2 \\
&\vdots \\
a_{N_\circ+k} < a_{N_\circ} r^k & \qquad \text{This step comes from induction.}
\end{aligned}$$

· Then since $0 < r < 1$, $\sum_{k=0}^{\infty} a_{N_\circ} r^k$ converges.

· This implies that $\sum_{k=0}^{\infty} a_{N_\circ+k}$ converges.

· This further implies that $\sum_{n=k}^{\infty} a_n$ converges, as it is a tail of $\sum_{k=0}^{\infty} a_{N\circ} + k$.

· Hence $\sum_{n=1}^{\infty} a_n$ converges.

Proof for 2:

· Assume that $L > 1$.

· Then we can find $1 < s < L$ for some s.

· We can also find an $N_\circ$ such that if $n \geqslant N_\circ$, then $\frac{a_{n+1}}{a_n} > s$.

· This implies that

$$\begin{aligned}
\frac{a_{N_\circ+1}}{a_{N_\circ}} > s &\Longrightarrow a_{N_\circ+1} > a_{N_\circ} s \\
\frac{a_{N_\circ+2}}{a_{N_\circ+1}} > s &\Longrightarrow a_{N_\circ+2} > a_{N_\circ+1} s > a_{N_\circ} s^2 \\
&\vdots \\
a_{N_\circ+k} > a_{N_\circ} s^k & \qquad \text{This step comes from induction.}
\end{aligned}$$

· Since $s > 1$, $\lim_{k\to\infty} \left[s^k a_{N_\circ}\right] = \infty$.

· Hence by the Comparison Theorem for Sequences, $\lim_{k\to\infty} [a_{N_\circ+k}] = \infty$.

· This implies that $\lim_{n\to\infty} a_n = \infty \neq 0$, and so by the divergence test $\sum_{n=1}^{\infty} a_n$ diverges.

1.9. **Alternating Series Test.** Assume that $\{a_n\}$ satisfies the following conditions:

$$\left.\begin{array}{l}\textbf{1. } a_n \geqslant 0 \\ \textbf{2. } a_{n+1} \leqslant a_n \\ \textbf{3. } \lim\limits_{n\to\infty} a_n = 0\end{array}\right\} \text{ Then } \sum_{n=1}^{\infty}(-1)^{n-1}a_n \text{ converges.}$$

Moreover, if $S_k = \sum_{n=1}^{k}(-1)^{n-1}a_n$ and $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$, then $|S - S_k| \leqslant a_{k+1}$.

· First observe that

$$\begin{aligned} S_{2(k+1)-1} - S_{2k-1} &= S_{2k+1} - S_{2k-1} \\ &= \sum_{n=1}^{2k+1}(-1)^{n-1}a_n - \sum_{n=1}^{2k-1}(-1)^{n-1}a_n \\ &= (-1)^{2k-1}a_{2k} + (-1)^{(2k+1)-1}a_{2k+1} \\ &= -a_{2k} + a_{2k+1} \\ &\leqslant 0 \end{aligned}$$

· This shows that $\{S_{2k-1}\}$ is decreasing. Similarly,

$$\begin{aligned} S_{2(k+1)} - S_{2k} &= S_{2k+2} - S_{2k} \\ &= \sum_{n=1}^{2k+2}(-1)^{n-1}a_n - \sum_{n=1}^{2k}(-1)^{n-1}a_n \\ &= (-1)^{(2k+1)-1}a_{2k+1} + (-1)^{(2k+2)-1}a_{2k+2} \\ &= a_{2k+1} - a_{2k+2} \\ &\geqslant 0 \end{aligned}$$

· This shows that $\{S_{2k}\}$ is increasing. Now observe that

$$\begin{aligned} S_{2k-1} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2k-3} - a_{2k-2}) + a_{2k-1} \\ &\geqslant 0 + 0 + \cdots + 0 + a_{2k-1} \\ &\geqslant 0 \end{aligned}$$

· Hence $\{S_{2k-1}\}$ is bounded below by 0. Similarly,

$$\begin{aligned} S_{2k} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2k-2} - a_{2k-1}) - a_{2k} \\ &\leqslant a_1 - 0 - 0 - \cdots - 0 - a_{2k} \\ &\leqslant a_1 \end{aligned}$$

· Hence $\{S_{2k}\}$ is bounded above by a_1.

· By the Monotone Convergence Theorem, $\lim\limits_{k\to\infty} S_{2k-1} = L \in \mathbb{R}$ and $\lim\limits_{k\to\infty} S_{2k} = M \in \mathbb{R}$.

· Now let $\epsilon > 0$.

· We can choose a K so that we have $|a_{2K}| < \frac{\epsilon}{3}, |L - S_{2K-1}| < \frac{\epsilon}{3}$, and $|S_{2K} - M| < \frac{\epsilon}{3}$.

· Then we have

$$\begin{aligned} |L - M| &\leqslant |L - S_{2K-1}| + |S_{2K-1} - S_{2K}| + |S_{2K} - M| \\ &= |L - S_{2K-1}| + |(-1)^{2K-1}a_{2K}| + |S_{2K} - M| \\ &= |L - S_{2K-1}| + |a_{2K}| + |S_{2K} - M| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

· This shows that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n := \lim_{k\to\infty} S_k = S$, where $S := L = M$.

· Finally since S_{2k} is bounded above by S and S_{2k-1} is bounded below by S,

$$S_{2k} \leqslant S \leqslant S_{2k-1}$$

for all $k \in \mathbb{N}$.

· Therefore we have that

$$|S_k - S| \leqslant |S_k - S_{k+1}| = |(-1)^k a_{k+1}| = a_{k+1}$$

for all $k \in \mathbb{N}$.

1.10. **Weierstrass M-test.** Suppose that $\{f_n\} \subset C[a,b]$ and that $\displaystyle\sum_{n=1}^{\infty}||f_n||_\infty$ is convergent.

Define $S_k : [a,b] \to \mathbb{R}$ for each $k \in \mathbb{N}$ by $S_k(x) = \displaystyle\sum_{n=1}^{k} f_n(x)$ for $x \in [a,b]$.

Then $\{S_k\}$ converges uniformly on $[a,b]$ to some $f_\circ \in C[a,b]$ such that $f_\circ(x) = \displaystyle\sum_{n=1}^{\infty} f_n(x)$.

· Define $T_k = \displaystyle\sum_{n=1}^{k}||f_n||_\infty$ and $T = \displaystyle\sum_{n=1}^{\infty}||f_n||_\infty$.

· By the assumptions, $T_k \to T$ as $k \to \infty$, so T_k is Cauchy.

· Let $\epsilon > 0$.

· Then there exists $N \in \mathbb{N}$ such that if $k > j \geqslant N$, then $T_k - T_j = \displaystyle\sum_{n=j+1}^{k}||f_n||_\infty < \epsilon$.

· Then for all k, j satisfying $k > j \geqslant N$, for all $x \in [a,b]$ we have

$$\begin{aligned} |S_k(x) - S_j(x)| &= \left|\sum_{n=j+1}^{k} f_n(x)\right| \\ &\leqslant \sum_{n=j+1}^{k} |f_n(x)| \\ &\leqslant \sum_{n=j+1}^{k} ||f_n(x)||_\infty \\ &< \epsilon \end{aligned}$$

· Hence $||S_k(x) - S_j(x)||_\infty < \epsilon$ for all k, j satisfying $k > j \geqslant N$,

· Hence $\{S_k\}$ is Cauchy in $(C[a,b], d_\infty)$.

· Since $C[a,b]$ is complete, i.e. it is a Banach space, $\{S_k\}$ converges uniformly to $f_\circ \in C[a,b]$.

1.11. **Banach Contractive Mapping theorem.** Suppose that $\Gamma : C[a,b] \to C[a,b]$ is a contractive map. That is, suppose that Γ is such that there exists k satisfying $0 \leqslant k < 1$ with

$$||\Gamma(u) - \Gamma(v)||_\infty \leqslant k||u - v||_\infty$$

for all $u, v \in C[a,b]$. Then there exists a unique function $f \in C[a,b]$ such that $\Gamma(f) = f$.

· Let $f_\circ \in C[a,b]$.

· For each $n \in \mathbb{N}$, define $f_n := \Gamma(f_{n-1})$.

· Set $g_n := f_{n+1} - f_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
||g_1||_\infty &= ||f_2 - f_1||_\infty = ||\Gamma(f_1) - \Gamma(f_0)||_\infty \leqslant k||f_1 - f_0||_\infty = k^1||g_0||_\infty \\
||g_2||_\infty &= ||f_3 - f_2||_\infty = ||\Gamma(f_2) - \Gamma(f_1)||_\infty \leqslant k||f_2 - f_1||_\infty \leqslant k^2||g_0||_\infty \\
||g_3||_\infty &= ||f_4 - f_3||_\infty = ||\Gamma(f_3) - \Gamma(f_2)||_\infty \leqslant k||f_3 - f_2||_\infty \leqslant k^3||g_0||_\infty \\
&\vdots \\
||g_n||_\infty &= ||f_{n+1} - f_n||_\infty = ||\Gamma(f_n) - \Gamma(f_{n-1})||_\infty \leqslant k||f_n - f_{n-1}||_\infty \leqslant k^n||g_0||_\infty
\end{aligned}$$

· By induction we see that $||g_n||_\infty \leqslant k^n||g_0||_\infty$ for all $n \in \mathbb{N}$.

· Since $0 \leqslant k < 1$, by the geometric series test, $\displaystyle\sum_{n=0}^{\infty} k^n||g_0||_\infty = ||g_0||_\infty \sum_{n=1}^{\infty} k^n$ converges.

· Then by the comparison test $\displaystyle\sum_{n=1}^{\infty} ||g_n||_\infty$ converges.

· Then by the Weierstrass M-test, $\displaystyle\sum_{n=0}^{\infty} g_n$ converges uniformly to some $g \in C[a,b]$.

· Note that

$$\sum_{n=0}^{m} g_n = \sum_{n=0}^{m} f_{n+1} - f_n = (f_1 - f_0) + (f_2 - f_1) + \cdots + (f_{m+1} - f_m) = f_{m+1} - f_0$$

· Hence $f_{m+1} - f_0 \to g$ as $m \to \infty$ in d_∞, or $f_{m+1} \to g + f_0$.

· Let $f = \lim\limits_{m\to\infty} f_m = g + f_0$.

· <u>Claim:</u> $\Gamma(f) = f$. Observe that for each $n \in \mathbb{N}$,

$$0 \leqslant ||f_n - \Gamma(f)||_\infty = ||\Gamma(f_{n-1}) - \Gamma(f)||_\infty \leqslant k||f_{n-1} - f||_\infty$$

· Since $||f_{n-1} - f||_\infty \to 0$ as $n \to \infty$, by the squeeze theorem $\lim\limits_{n\to\infty} ||f_n - \Gamma(f)||_\infty = 0$.

· Hence $f_n \to \Gamma(f)$ as $n \to \infty$.

· From above, $f_n \to f$ as $n \to \infty$, and since limits are unique, $\Gamma(f) = f$.

· <u>Claim:</u> f is the only function that satisfies $\Gamma(f) = f$.

· Suppose that $h \in C[a,b]$ satisfies $\Gamma(h) = h$. Then

$$0 \leqslant ||h - f||_\infty = ||\Gamma(h) - \Gamma(f)||_\infty \leqslant k||h - f||_\infty$$

· Since $0 \leqslant k < 1$, we have $0 \leqslant (1-k)||h - f||_\infty \leqslant 0$.

· Then $||h - f||_\infty = 0$, and $h = f$.

2. Function Characteristics I

2.1. **Continuity implies integrability.** If $f(x)$ is continuous on $[a.b]$, then $f(x)$ is integrable on $[a,b]$.

· Let $\epsilon > 0$.
· Since $f(x)$ is uniformly continuous on $[a,b]$, we can find $\delta > 0$ such that if $|x-y| < \delta$ with $x, y \in [a,b]$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.
· Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition with $\|P\| < \epsilon$, so $\Delta x_i < \delta \ \forall i$.
· Let $M_i = \max\{f(x) : x \in [x_{i-1}, x_i]\}$
$m_i = \min\{f(x) : x \in [x_{i-1}, x_i]\}$
· By the Extreme Value Theorem, there exist $c_i, d_i \in [x_{i-1}, x_i]$ with $f(c_i) = m_i$ and $f(d_i) = M_i$.
· But $|c_i - d_i| \leqslant \|P\| < \delta$, which implies that $M_i - m_i = f(d_i) - f(c_i) < \frac{\epsilon}{b-a}$.
· Now
$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \\ &= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \\ &< \sum_{i=1}^{n} \frac{\epsilon}{b-a} \Delta x_i \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \\ &= \epsilon \end{aligned}$$

2.2. **Bounded conditional integrability.** A bounded function $f(x)$ is integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

· Assume that $f(x)$ is integrable.
· Let $\epsilon > 0$.
· We can find partitions P_1, P_2 such that
$$\underline{\int_a^b} f(x)\, dx - \frac{\epsilon}{2} < L(f, P_1) \leqslant U(f, P_2) < \overline{\int_a^b} f(x)\, dx$$
· Let $Q = P_1 \cup P_2$.
· Then
$$\begin{aligned} \int_a^b f(x)\, dx - \frac{\epsilon}{2} &= \underline{\int_a^b} f(x)\, dx - \frac{\epsilon}{2} \\ &< L(f, P_1) \leqslant L(f,Q) \leqslant U(f,Q) \leqslant U(f, P_2) \\ &< \overline{\int_a^b} f(x)\, dx + \frac{\epsilon}{2} \\ &= \int_a^b f(x)\, dx + \frac{\epsilon}{2} \end{aligned}$$
· This implies that $U(f,Q) - L(f,Q) < \epsilon$.
· Now assume that for each $\epsilon > 0$ we can find P with $U(f,P) - L(f,P) < \epsilon$.

· Let $\epsilon > 0$.

· Choose P as above, then

$$L(f,P) \leqslant \underline{\int_a^b} f(x)\,dx \leqslant \overline{\int_a^b} f(x)\,dx \leqslant U(f,P)$$

· This implies that

$$\left|\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx\right| \leqslant U(f,P) - L(f,P) < \epsilon$$

· Since ϵ is arbitrary, we have $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$.

· Therefore $f(x)$ is integrable on $[a,b]$.

2.3. **Absolute convergence implies convergence.** If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then is converges.

· Assume that $\sum_{n=1}^{\infty} |a_n|$ converges.

· Let $T_k = \sum_{n=1}^{k} |a_n|$. Note that T_k is Cauchy.

· Let $S_k = \sum_{n=1}^{k} a_n$. We claim that S_k is Cauchy.

· Let $\epsilon > 0$.

· We can find $N \in \mathbb{N}$ such that if $N \leqslant k < j$, then

$$|T_j - T_k| = \sum_{n=k+1}^{j} |a_n| < \epsilon$$

· Let $N \leqslant k < j$. Then

$$\left|\sum_{n=1}^{j} a_n - \sum_{n=1}^{k} a_n\right| \leqslant \sum_{n=k+1}^{j} |a_n| = T_j - T_k < \epsilon$$

· This implies that $\{S_k\}$ is Cauchy.

· This implies that $\{S_k\}$ converges.

3. Function Characteristics II

3.1. **Power series radius of convergence.** If a power series $\sum_{n=1}^{\infty} a_n x^n$ converges at $x_\circ \neq 0$, then it also converges absolutely at any x_1 with $0 \leqslant |x_1| < |x_\circ|$.

- Since $\sum_{n=1}^{\infty} a_n x_\circ^n$ converges, $\lim_{n\to\infty} [a_n x_\circ^n] = 0$.
- In particular, there exists an M such that $|a_n x_\circ^n| \leqslant M$ for all n.
- Suppose that $|x_1| < |x_\circ|$.
- Hence
$$\begin{aligned} |a_n x_1^n| &= |a_n| \cdot \left|\frac{x_1}{x_\circ}\right|^n \cdot |x_\circ|^n \\ &\leqslant M \left|\frac{x_1}{x_\circ}\right|^n \end{aligned}$$
- By the geometric series test and the comparison test, $\sum_{n=1}^{\infty} |a_n x_1^n|$ converges.

3.2. **Power series uniform convergence.** Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Let $0 \leqslant x_1 < R$. Let $f_k(x) = \sum_{n=0}^{k} a_n x^n$. Then $\{f_k\}$ converges uniformly on $[-x_1, x_1]$ to $\sum_{n=0}^{\infty} a_n x^n$.

- Since $0 \leqslant x_1 < R$, the sum $\sum_{n=0}^{\infty} a_n x_1^n$ converges absolutely.
- Let $\epsilon > 0$.
- Define $T_k := \sum_{n=0}^{k} |a_n x_1^n|$.
- Since $\{T_k\}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that if $m > j \geqslant N$, then $T_m - T_j = \sum_{n=j+1}^{m} |a_n x_1^n| < \epsilon$.
- Let $x \in [-x_1, x_1]$. Then
$$\begin{aligned} |f_m(x) - f_j(x)| &= \left| \sum_{n=j+1}^{m} a_n x^n \right| \\ &\leqslant \sum_{n=j+1}^{m} |a_n x^n| \\ &\leqslant \sum_{n=j+1}^{m} |a_n x_1^n| \\ &< \epsilon \end{aligned}$$
- Hence $||f_m - f_j||_\infty < \epsilon$.
- Hence $\{f_k\}$ is Cauchy on $(C[-x_1, x_1], ||\cdot||_\infty)$.
- Hence $\{f_k\}$ converges uniformly on $[-x_1, x_1]$.

3.3. **Uniform convergence and differentiation.** Suppose $\{F_n\} \subset C[a,b]$ with $\lim_{n\to\infty} [F_n(a)] = a_\circ$. If $\{F_n\}$ has continuous derivatives $F_n'(x) = f_n(x)$, such that $\{f_n\}$ converges uniformly on $[a,b]$ to $g(x) \in C[a,b]$, then $\{F_n\}$ converges uniformly to a continuous function $G \in C[a,b]$ such that $G'(x) = g(x)$ for all $x \in (a,b)$.

- For every $n \in \mathbb{N}$ and $x \in [a,b]$, FTCII states that $F_n(x) = \displaystyle\int_a^x f_n(t)\ dt + F_n(a)$.
- For each $x \in [a,b]$, define $G : [a,b] \to \mathbb{R}$ by $G(x) = \displaystyle\int_a^x g(t)\ dt + a_\circ$.
- Let $\epsilon > 0$.
- Then there exists an $N \in \mathbb{N}$ such that if $n \geqslant N$, then $||f_n - g||_\infty < \frac{\epsilon}{2(b-a)}$ and $|F_n(a) - a_\circ| < \frac{\epsilon}{2}$.
- Hence for $n \geqslant N$ and $x \in [a,b]$:

$$
\begin{aligned}
|F_n(x) - G(x)| &= \left|\left(\int_a^x f_n(t)\ dt + F_n(a)\right) - \left(\int_a^x g(t)\ dt + a_\circ\right)\right| \\
&= \left|\int_a^x (f_n(t) - g(t))\, dt + (F_n(a) - a_\circ)\right| \\
&\leqslant \left|\int_a^x (f_n(t) - g(t))\, dt\right| + |F_n(a) - a_\circ| \\
&\leqslant \int_a^x |f_n(t) - g(t)|\, dt + |F_n(a) - a_\circ| \\
&\leqslant \int_a^x ||f_n(t) - g(t)||_\infty\ dt + |F_n(a) - a_\circ| \\
&= (x-a)||f_n - g||_\infty + |F_n(a) - a_\circ| \\
&\leqslant (b-a)||f_n - g||_\infty + |F_n(a) - a_\circ| \\
&< (b-a)\frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

- Hence $||F_n - G||_\infty < \epsilon$ for all $n \geqslant N$.
- Hence $\{F_n\} \to G$ as $n \to \infty$ with respect to d_∞.
- So by FTCI, $G(x)$ is differentiable on (a,b) with $G'(x) = g(x)$ for each $x \in (a,b)$.

3.4. **Uniform continuous convergence.** If $\{f_n\} \subset C[a,b]$ converges uniformly on S to $f(x)$, and if each f_n is continuous at $x_\circ$ relative to S, then $f(x)$ is continuous at $x_\circ$ relative to S.

- Let $\epsilon > 0$.
- Then there exists some $N \in \mathbb{N}$ such that if $n \geqslant N$, then $|f(x) - f_n(x)| < \frac{\epsilon}{3}$.
- Since $f_N(x)$ is continuous at $x_\circ$ relative to S, we can find $\delta > 0$ such that if $|x - x_\circ| < \delta$ and $x \in S$, then $|f_N(x) - f_N(x_\circ)| < \frac{\epsilon}{3}$.
- Let $|x - x_\circ| < \delta$ for $x \in S$. Then

$$\begin{aligned} |f(x) - f(x_\circ)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_\circ) + f_N(x_\circ) - f(x_\circ)| \\ &\leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(x_\circ)| + |f_N(x_\circ) - f(x_\circ)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$