Compact course notes

Math 247, Fall 2010

Multivariable Calculus

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Contents

<u>*Discretion:*</u> These theorem, remark, etc. numbers do not necessarily coincide with prof. Nica's.

File I
Lectures

1 Review

Operations with vectors
For $\vec{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) \in \mathbb{R}^n$, $\vec{y} = (y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$:

Addition: $\vec{x} + \vec{y} := (x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \ldots, x^{(n)} + y^{(n)})$
Scalar multiplication: $\alpha\vec{x} := (\alpha x^{(1)}, \alpha x^{(2)}, \ldots, \alpha x^{(n)})$
(Standard) inner product / dot product: $< x, y > := x \cdot y$
$:= x^{(1)}y^{(1)} + x^{(2)}y^{(2)} + \cdots + x^{(n)}y^{(n)}$
Norm (or length): $\|\vec{x}\| := \sqrt{< \vec{x}, \vec{x} >}$

Basic properties of the inner product
a. Bilinearity: $< \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2, \vec{y} >= \alpha_1 < \vec{x}_1, \vec{y} > +\alpha_2 < \vec{x}_2, \vec{y} >$
b. Symmetry: $< \vec{x}, \vec{y} >=< \vec{y}, \vec{x} >$
c. Positivity: $< \vec{x}, \vec{x} >\geqslant 0$ with $< \vec{x}, \vec{x} >= 0$ iff $\vec{x} = \vec{0}$

Theorem 1.1.$*$ [Cauchy-Bunyakovsky-Schwarz Inequality]
$| < \vec{x}, \vec{y} > | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$

Corollary 1.2. [Triangle Inequality]
$\|\vec{x} + \vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\| \ \forall \ \vec{x}, \vec{y} \in \mathbb{R}^n$

Definition 1.3. The distance between $\vec{x}$ and $\vec{y}$ is defined to be $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{\sum_{i=1}^{n} (x^{(i)} - y^{(i)})^2}$.

Corollary 1.4. For every $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $d(\vec{x}, \vec{z}) \leqslant d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$.

Definition 1.5. For $\vec{a} \in \mathbb{R}^n$ and $r > 0$, define a ball by $B(\vec{a}; r) = \{\vec{x} \in \mathbb{R}^n \mid d(\vec{x}, \vec{a}) < r\} = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < r\}$
An open ball has a strict inequality, whereas a closed ball may have $\|\vec{x} - \vec{a}\| = r$.

Remark 1.6. For all $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\|_\infty \leqslant \|\vec{x}\|_2 \leqslant \|\vec{x}\|_1 \leqslant n\|\vec{x}\|_\infty$.

2 Sequences in $\mathbb{R}^n$

Definition 2.1. A sequence in $\mathbb{R}^n$ is denoted by the following: $(\vec{x}_k)_{k=1}^{\infty}$

Definition 2.2. Suppose $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$ and $\vec{a} \in \mathbb{R}^n$. Then $(\vec{x}_k)_{k=1}^{\infty}$ converges to $\vec{a}$ when the following holds:
Given $\epsilon > 0$, there exists a $k_o \in \mathbb{N}$ such that $\|\vec{x}_k - \vec{a}\| < \epsilon$ for all $k \geqslant k_o$.

Remark 2.3. Instead of $\|\vec{x}_k\vec{a}\| < \epsilon$, we may also write $d(\vec{x}_k, \vec{a}) < \epsilon$ or $\vec{x}_k \in B(\vec{a}; \epsilon)$.

Definition 2.4. Given $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$, the sequence is a Cauchy sequence when the following holds:
Given $\epsilon > 0$, there exists a $k_o \in \mathbb{N}$ such that $\|\vec{x}_p - \vec{x}_q\| < \epsilon$ for all $p, q \geqslant k_o$.

Definition 2.5. Given $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$, the sequence is bounded when there exists $r > 0$ such that $\|\vec{x}_k\| \leqslant r$ for all $k \geqslant 1$. That is, $\vec{x}_k \in \overline{B}(\vec{0}; r)$ for all $k \geqslant 1$.

Definition 2.6. Given $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$, express explicitly $\vec{x}_k = \left(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(n)}\right)$ the k-th element.
From these we get n sequences in $\mathbb{R}$: $\left(\vec{x}_k^{(1)}\right)_{k=1}^{\infty}, \left(\vec{x}_k^{(2)}\right)_{k=1}^{\infty}, \ldots, \left(\vec{x}_k^{(n)}\right)_{k=1}^{\infty}$.
These are termed component sequences of the original sequence.

Proposition 2.7.* Let $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$ and $\vec{a} \in \mathbb{R}^n$. Then $(\vec{x}_k \xrightarrow[k\to\infty]{} \vec{a}$ in $\mathbb{R}^n)$ $\iff$ $\left(\begin{array}{ll} \vec{x}_k^{(1)} \xrightarrow[k\to\infty]{} \vec{a}^{(1)} & \\ \vdots & \text{in } \mathbb{R} \\ \vec{x}_k^{(n)} \xrightarrow[k\to\infty]{} \vec{a}^{(n)} & \end{array} \right)$

Proposition 2.8. Suppose $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$. Then the sequence is Cauchy $\iff$ each of the component sequences is Cauchy.

Similarly, the sequence is bounded $\iff$ each of the component sequences is bounded.

Theorem 2.9. [Cauchy Theorem]
Let $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$. Then it is convergent to some $\vec{a} \in \mathbb{R}^n$ if and only if it is Cauchy.

Theorem 2.10. [Bolzano-Weierstrass Theorem]
Suppose $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$ is bounded. Then there exist values $a < k(1) < k(2) < \cdots < k(p) < \cdots$ such that the subsequence $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ is convergent.

Theorem 2.11. [Operations with convergent sequences]
Let $(\vec{x}_k)_{k=1}^{\infty}$ and $(\vec{y}_k)_{k=1}^{\infty}$ be in $\mathbb{R}^n$ with $\lim\limits_{k\to\infty}[\vec{x}_k] = \vec{a}$ and $\lim\limits_{k\to\infty}[\vec{y}_k] = \vec{b}$. Then for $\alpha, \beta \in \mathbb{R}$,

1. $\lim\limits_{k\to\infty}[\alpha\vec{x}_k + \beta\vec{y}] = \alpha\vec{a} + \beta\vec{b}$
2. $\lim\limits_{k\to\infty}[\langle \vec{x}_k, \vec{y}_k \rangle] = \langle \vec{a}, \vec{b} \rangle$
3. $\lim\limits_{k\to\infty}[\|\vec{x}_k\|] = \|\vec{a}\|$
4. $\lim\limits_{k\to\infty}[d(\vec{x}_k, \vec{y}_k)] = d(\vec{a}, \vec{b})$

Theorem 2.12. [Banach Fixed Point Theorem]
Let $A \subseteq \mathbb{R}^n$ with $f : A \to A$ be a function given by $\|f(\vec{x}) - f(\vec{y})\| \leqslant \gamma\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in A$ and $\gamma \in (0, 1)$. Given any $\vec{x}_1 \in A$ and $\vec{x}_{k+1} = f(\vec{x}_k)$, the function f has a unique fixed point $\vec{p} \in A$ such that $\lim\limits_{k\to\infty}[\vec{x}_k] = \vec{p}$ and $f(\vec{p}) = \vec{p}$.

Definition 2.13. A subset $A \subseteq \mathbb{R}^n$ is said to be closed when it has the following property: If $(\vec{x}_k)_{k=1}^{\infty}$ is a sequence in A such that $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{b} \in \mathbb{R}^n$, then it follows that $\vec{b} \in A$.

3 Open and closed subsets of $\mathbb{R}^n$

Definition 3.1. Suppose $A \subseteq \mathbb{R}^n$ and $\vec{a} \in A$. Then $\vec{a}$ is an interior point of A when there exists an $r > 0$ such that $B(\vec{a}; r) \subseteq A$. The set of all interior points of A is termed the interior of A and is denoted by $int(A)$.

Definition 3.2. Suppose $A \subseteq \mathbb{R}^n$. If every $\vec{a} \in A$ is an interior point of A, then we say that A is open. Hence for every $A \in \mathbb{R}^n$, we have that $int(A) \subseteq A$, and A is open $\iff$ $int(A) = A$.

Definition 3.3. Given $C \subseteq \mathbb{R}^n$, the set C is closed when $\mathbb{R}^n \setminus C$ is open.

Remark 3.4. Most sets are neither open nor closed.

Definition 3.5. Let $\emptyset \neq C \in \mathbb{R}^n$. Then C has the cannot escape property when the following happens:
If $(\vec{x}_k)_{k=1}^{\infty} \in C$ and $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{b} \in \mathbb{R}^n$, then $\vec{b} \in C$.

Proposition 3.6.* Let $\emptyset \neq C \in \mathbb{R}^n$. Then C is closed $\iff$ C has the cannot escape property.

Remark 3.7. $\emptyset$ and $\mathbb{R}$ are clopen; they are both open and closed.

Definition 3.8. For $A \in \mathbb{R}^n$, define the closure of A to be the following set:
$cl(A) = \{\vec{b} \in \mathbb{R}^n \mid \text{ there exists } (\vec{x}_k)_{k=1}^{\infty} \in A \text{ such that } \vec{x}_k \xrightarrow[k\to\infty]{} \vec{b}\}$.

Remark 3.9. $A \subseteq cl(A)$ with $A = cl(A) \iff A$ is closed.

Proposition 3.10. For every $A \subseteq \mathbb{R}^n$, we have
i. $int(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus cl(A)$
ii. $cl(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus int(A)$

Definition 3.11. A set $B \in \mathbb{R}^n$ is said to be bounded when there exists $r > 0$ such that $\|\vec{x}\| \leqslant r$ for all $\vec{x} \in B$, and B is a subset of $\overline{B}(\vec{0}; r)$.

Definition 3.12. A set $K \in \mathbb{R}^n$ is said to compact when it is closed and bounded.

Theorem 3.13. Let $K \subseteq \mathbb{R}^n$ be compact, and let $(\vec{x}_k)_{k=1}^{\infty} \in K$. Then there exists a subsequence of $\vec{x}_k$ which converges to a limit in K.

Definition 3.14. For $A \in \mathbb{R}^n$, define the boundary of A to be $bd(A) = cl(A) \setminus int(A)$.

Remark 3.15. An alternative definition is $bd(A) = cl(A) \cap cl(\mathbb{R}^n \setminus A)$.

4 Continuous functions

Definition 4.1. Let $\emptyset \neq A \in \mathbb{R}^n$, and $f : A \to \mathbb{R}^n$ be a function. Let $\vec{a} \in A$. We say that f is continuous at $\vec{a}$ when the following holds:
For all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\vec{x} - \vec{a}\| < \delta$, then $\|f(\vec{x}) - f(\vec{a})\| < \epsilon$ for all $\vec{x} \in A$.
Moreover, f is continuous if f is continuous at every $\vec{x} \in A$.

Definition 4.2. For $\emptyset \neq A \in \mathbb{R}^n$, a function $f : A \to \mathbb{R}^n$, and $\vec{a} \in A$, we say that f respects sequences in A which converge to $\vec{a}$ when the following happens:
Whenever $(\vec{x}_k)_{k=1}^{\infty} \in A$ is such that $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{a}$, it follows that $f(\vec{x}_k) \xrightarrow[k\to\infty]{} f(\vec{b})$

Proposition 4.3.$*$ For $\emptyset \neq A \in \mathbb{R}^n$, a function $f : A \to \mathbb{R}^n$, and $\vec{a} \in A$, we have that
$$\begin{pmatrix} f \text{ respects sequences in} \\ A \text{ which converge to } \vec{a} \end{pmatrix} \iff \begin{pmatrix} f \text{ is continuous} \\ \text{at } \vec{a} \end{pmatrix}.$$

Definition 4.4. For $\emptyset \neq A \in \mathbb{R}^n$, a function $f : A \to \mathbb{R}^n$, for every $\vec{a} \in A$ write explicitly
$f(\vec{a}) = \big(f^{(1)}(\vec{a}), \ldots, f^{(n)}(\vec{a})\big) \in \mathbb{R}^n$
then get the functions $f^{(1)}, \ldots, f^{(n)} : A \to \mathbb{R}$, termed the components of f.

5 Continuity and compactness

Definition 5.1. Let $\emptyset \neq A \in \mathbb{R}^n$ with $f : A \to \mathbb{R}^n$. Then f is uniformly continuous on A when the following holds:
For all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|\vec{x} - \vec{a}\| < \delta$, then $\|f(\vec{x}) - f(\vec{a})\| < \epsilon$ for all $\vec{a}, \vec{x} \in A$.

Remark 5.2. Uniform continuity gives us that:
1. The function f is continuous on A, i.e. f is continuous at every point in A.
2. The choice of δ in the epsilon-delta condition of continuity is the same for all $\vec{a} \in A$.

Theorem 5.3.$*$ Suppose $\emptyset \neq A \in \mathbb{R}^n$ is compact. If $f : A \to \mathbb{R}^n$ is continuous, then f is uniformly continuous on A.

Proposition 5.4.$*$ Suppose $\emptyset \neq A \in \mathbb{R}^n$ is compact. Let $f : A \to \mathbb{R}^n$ be continuous on A. Then the image set $B = f(A) \subseteq \mathbb{R}^n$, or $B = \{\vec{y} \in \mathbb{R}^n \,\big|\, \text{there exists } \vec{x} \in A \text{ such that } f(\vec{x}) = \vec{y}\}$ is a compact subset of $\mathbb{R}^n$.

Theorem 5.5.$*$ [Extreme Value Theorem]
Suppose $\emptyset \neq A \in \mathbb{R}^n$ is compact. Let $f : A \to \mathbb{R}$ be continuous. Then f has a minimum and maximum on A. That is, there exist $\vec{\gamma}_1, \vec{\gamma}_2 \in A$ such that $f(\vec{\gamma}_1) \leqslant f(\vec{x}) \leqslant f(\vec{\gamma}_2)$ for all $\vec{x} \in A$.

6 Integrable functions

Definition 6.1. A closed rectangle in $\mathbb{R}^n$ is a set of the form

$$P = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$
$$= \{\vec{x} \in \mathbb{R}^n \,\big|\, a_1 \leqslant x^{(1)} \leqslant b_1, a_2 \leqslant x^{(2)} \leqslant b_2, \ldots, a_n \leqslant x^{(n)} \leqslant b_n\}$$

Definition 6.2. For such P as above, define the volume of P by $vol(P) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$
the diameter of P by $diam(P) = \|\vec{b} - \vec{a}\|$
$diam_\infty(P) = \|\vec{b} - \vec{a}\|_\infty$

Definition 6.3. Let P be a closed rectangle in $\mathbb{R}^n$. A division of P is a collection of closed rectangles $\Delta = \{P_1, P_2, \ldots, P_k\}$ such that $P_1 \cup P_2 \cup \cdots \cup P_k = P$ with $int(P_i) \cap int(P_j) = \emptyset$ for $i \neq j$.

Definition 6.4. For P as above, define $\|\Delta\| = \max\limits_{1 \leqslant i \leqslant k} \{diam_\infty(P_i)\}$. Hence $\|\Delta\|$ is small implies that $diam_\infty(P_i)$ is small for all i.

Definition 6.5. For divisions $\Delta = \{P_1, P_2, \ldots, P_k\}$ and $\Gamma = \{Q_1, Q_2, \ldots, Q_\ell\}$ of P, we say that Γ refines Δ and write $\Gamma \prec \Delta$ if for every $1 \leqslant j \leqslant \ell$ there exists $1 \leqslant i \leqslant k$ such that $Q_j \subseteq P_i$.

Remark 6.6. If $\Gamma \prec \Delta$ as for above, then if $\Delta = \{P_1, P_2, \ldots, P_k\}$
$\Gamma = \{Q_{1_1}, \ldots, Q_{1_{m_1}}, Q_{2_1}, \ldots, Q_{2_{m_2}}, \ldots\ldots, Q_{k_1}, \ldots, Q_{k_{m_k}}\}$

we will have $P_i = \bigcup\limits_{j=1}^{m_i} Q_{i_j}$ for all $i \in [1, k]$.

Remark 6.7. If P is a closed rectangle in $\mathbb{R}^n$ and $\Delta_1 = \{P'_1, P'_2, \ldots, P'_k\}$ and $\Delta_2 = \{P''_1, P''_2, \ldots, P''_\ell\}$ are divisions of P, then we can find a division Γ such that $\Gamma \prec \Delta_1$ and $\Gamma \prec \Delta_2$. Then Γ is given by

$$\Gamma = \{P'_i \cap P''_j \,\big|\, 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell \text{ and } P'_i \cap P''_j \neq \emptyset\}.$$

Definition 6.8. Suppose $\emptyset \neq A \subseteq \mathbb{R}^n$ with $f : A \to \mathbb{R}$. Then f is bounded when there exists $c > 0$ such that $|f(x)| \leqslant c$ for all $\vec{x} \in A$. When f is bounded on A, we can talk about

$$\sup\{f(\vec{x}) \,\big|\, \vec{x} \in A\} = \sup_A\{f\}$$
$$\inf\{f(\vec{x}) \,\big|\, \vec{x} \in A\} = \inf_A\{f\}$$

Definition 6.9. Suppose P is a closed rectangle on $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ is bounded. Let $\Delta = \{P_1, P_2, \ldots, P_k\}$ be a division of P. Then $U(f, \Delta) = \sum\limits_{i=1}^{k} vol(P_i) \sup\limits_{P_i}\{f\}$

$$L(f, \Delta) = \sum_{i=1}^{k} vol(P_i) \inf_{P_i}\{f\}$$

These are termed the upper and lower Darboux sums of f over P.

Remark 6.10. $L(f, \Delta) \leqslant U(f, \Delta)$ for all Δ, f.

Lemma 6.11.$*$ Suppose P is a closed rectangle on $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ is bounded. Let Γ, Δ be divisions of P such that $\Gamma \prec \Delta$. Then $U(f, \Gamma) \leqslant U(f, \Delta)$ and $L(f, \Gamma) \geqslant L(f, \Delta)$.

Proposition 6.12. If P is a closed rectangle in $\mathbb{R}^n$ and $F : P \to \mathbb{R}$ is a bounded function, and Δ_1, Δ_2 are divisions of P, then $L(f, \Delta_1) \leqslant U(f, \Delta_2)$.

Definition 6.13. Let P be a closed rectangle in $\mathbb{R}^n$ and $F : P \to \mathbb{R}$ be bounded. Consider then

$$S = \{L(f, \Delta) \,\big|\, \Delta \text{ is a division of } P\}$$
$$T = \{U(f, \Delta) \,\big|\, \Delta \text{ is a division of } P\}$$

Then $s \leqslant t$ for every $s \in S$ and $t \in T$. Denote $\sup(S) = \underline{\int}_P f$ and $\inf(T) = \overline{\int}_P f$. Note that $\underline{\int}_P \leqslant \overline{\int}_P f$.

Remark 6.14. If f is integrable, then f is bounded.

Proposition 6.15.$*$ Let P be a closed rectangle in $\mathbb{R}^n$ and $F : P \to \mathbb{R}$ be bounded. Then

$$\left(\ f \text{ is integrable }\right) \iff \left(\begin{array}{c}\text{for every } \epsilon > 0, \text{ there exists a division } \Delta \text{ of } P \\ \text{such that } U(f,\Delta) - L(f,\Delta) < \epsilon\end{array}\right)$$

Proposition 6.16.$*$ Let P be a closed rectangle in $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ be bounded. Then

$$\left(\ f \text{ is integrable }\right) \iff \left(\begin{array}{c}\text{there exists a sequence } \Delta_1, \Delta_2, \ldots, \Delta_k, \ldots \text{ of divisions} \\ \text{of } P \text{ such that } \lim\limits_{k\to\infty} [U(f,\Delta_k) - L(f,\Delta_k)] = 0\end{array}\right)$$

Also, if we let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be a sequence of divisions of P as above and f be integrable, then

$$\lim_{k\to\infty} [U(f,\Delta_k)] = \int_P f = \lim_{k\to\infty} [L(f,\Delta_k)]$$

Theorem 6.17. [Du Bois-Reymond Theorem]
Let P be a closed rectangle in $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ be bounded. Then

$$\left(\ f \text{ is integrable }\right) \iff \left(\begin{array}{c}\text{for every } \epsilon > 0, \text{ there exists a } \delta > 0 \text{ such that if} \\ \Delta \text{ is a partition of } P \text{ with } \|\Delta\| < \delta, \text{ then } U(f,\Delta) - L(f,\Delta) < \epsilon\end{array}\right)$$

7 Linearity of the integral

Lemma 7.1. Let P be a closed rectangle in $\mathbb{R}^n$ and $f, g : P \to \mathbb{R}$ be bounded. Consider $h = f + g$ with $h : P \to \mathbb{R}$ by $h(\vec{x}) = f(\vec{x}) + g(\vec{x})$. Then for every division $\Delta = \{P_1, P_2, \ldots, P_k\}$ of P we have

$U(f+g,\Delta) \leqslant U(f,\Delta) + U(g,\Delta)$

$L(f+g,\Delta) \geqslant L(f,\Delta) + L(g,\Delta)$

Proposition 7.2.$*$ Let P be a closed rectangle in $\mathbb{R}^n$ and $f, g : P \to \mathbb{R}$ be bounded and integrable on P. Then $f + g$ is integrable also with $\int_P (f+g) = \int_P f + \int_P g$.

Proposition 7.3. Let P be a closed rectangle in $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ is integrable and $\alpha \in \mathbb{R}$. Then $h = \alpha f$ is integrable also with $\int_P \alpha f = \alpha \int_P f$.

Theorem 7.4. Let P be a closed rectangle in $\mathbb{R}^n$. Let $\mathcal{F} = \{f : P \to \mathbb{R} \,\big|\, f \text{ is integrable}\}$. Then $\mathcal{F}$ is closed under linear combinations, and for $f, g, \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{R}$ we have $\int_P (\alpha f + \beta g) = \alpha \int_P f + \beta \int_P g$.

Remark 7.5. It is true that if $f, g \in \mathcal{F}$, then $fg \in \mathcal{F}$, but not necessarily true that $\int_P fg = \int_P f \int_P g$.

Remark 7.6. Let P be a closed rectangle in $\mathbb{R}^n$ and let $\Delta = \{P_1, \ldots, P_k\}$ be a division of P. If $f : P \to \mathbb{R}$ is integrable on P, then f is integrable on each of $P_1, \ldots, P_k$ and $\int_P f = \sum_{i=1}^k \int_{P_i} f$

8 Integration of functions modulo null sets

Definition 8.1. $C \subseteq \mathbb{R}^n$ is said to be a <u>null set</u> when the following happens:

For every $\epsilon > 0$, we can find a finite family of closed rectangles $Q_1, \ldots, Q_m \in \mathbb{R}^n$ such that

1. $Q_1 \cup \cdots \cup Q_m \supset C$

2. $vol(Q_1) + \cdots + vol(Q_m) < \epsilon$

Theorem 8.2.$*$ Let $P \subset \mathbb{R}^n$ be a closed ractangle and $f : P \to \mathbb{R}$ be bounded. Suppose we find subsets $B, G \subseteq P$ such that $B \cup G = P$ and such that

1. f is continuous at every $\vec{x} \in G$

2. B is a null set

Then f is integrable over P.

Corollary 8.3. Let $P \subseteq \mathbb{R}^n$ and $f : P \to \mathbb{R}$ be continuous. Then f is integrable over P.

Corollary 8.4.$*$ Let $A \subset \mathbb{R}^n$ be compact such that $bd(A)$ is a null set. Let $P \supseteq A$ be a closed rectangle and define $F : P \to \mathbb{R}$ by $f(\vec{x}) = \begin{cases} 1 & \vec{x} \in A \\ 0 & \text{else} \end{cases}$ Then f is integrable over P.

Remark 8.5. The above function f is termed the characteristic function of A, and denoted by χ_A. We can use χ_A to define the volume of A by $vol(A) = \int_P \chi_A$.

9 Theorem of Fubini

Remark 9.1. Let $n = p + q$ with $p, q \in \mathbb{N}$. Then
- For $A \in \mathbb{R}^p$ and $B \in \mathbb{R}^q$, define the Cartesian product $A \times B = \{(\vec{a}, \vec{b}) \mid \vec{a} \in A, \vec{b} \in B\} \subseteq \mathbb{R}^n$.
- Every closed rectangle $P = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ can be written as $P = M \times N$ with $M = [a_1, b_1] \times \cdots \times [a_p, b_p] \subseteq \mathbb{R}^p$ and $N = [a_{p+1}, b_{p+1}] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^q$.

Definition 9.2. Let $P = M \times N$ as above. Let $f : P \to \mathbb{R}$ be a function. For every $\vec{v} \in M$, define a partial function $f_{\vec{v}} : N \to \mathbb{R}$ by $f_{\vec{v}}(\vec{w}) = f(\vec{v}, \vec{w})$ for all $\vec{w} \in N$.

Theorem 9.3.$*$ [Theorem of Stolz-Fubini]
Let $P = M \times N$ as above, and let $f : P \to \mathbb{R}$ as above. Suppose that
i. f is integrable on P
ii. For every $\vec{v} \in M$, the function $f_{\vec{v}} : N \to \mathbb{R}$ is integrable over N.
Define $F : M \to \mathbb{R}$ by $F(\vec{v}) = \int_N f_{\vec{v}}$ for all $\vec{v} \in M$. Then F is integrable on M, and

$$\int_M F = \int_P f$$

Remark 9.4. With different notation, the above theorem states that:

$$\begin{aligned} \int_P f(\vec{x})\, d\vec{x} &= \int_M \left(\int_N f(\vec{v}, \vec{w})\, d\vec{w} \right) d\vec{v} \\ &= \int_P f(\vec{v}, \vec{w})\, d\vec{v}\, d\vec{w} \end{aligned}$$

This is termed the calculation of iterated integrals.

Lemma 9.5. Let $P = M \times N$ as above. Then for any division Δ of P, one can find divisions Φ for M and Ψ for N such that $\Phi \times \Psi$ always refines Δ.

Lemma 9.6. Let $P = M \times N$, and $f : P \to \mathbb{R}$ with $F(\vec{v}) = \int_N f_{\vec{v}}$ for all $\vec{v} \in M$ as above. Let $A \subseteq M$ be a closed rectangle. Let $\Psi = \{N_1, \ldots, N_s\}$ be a division of N. Then

$$\begin{aligned} \sup_A\{F\} &\leqslant \sum_{j=1}^{s} vol(N_j) \sup_{A \times N_j}\{f\} \\ \inf_A\{F\} &\geqslant \sum_{j=1}^{s} vol(N_j) \inf_{A \times N_j}\{f\} \end{aligned}$$

Lemma 9.7. Let $\Phi = \{M_1 \ldots, M_r\}$ be a division of M. Let $\Psi = \{N_1, \ldots, N_s\}$ be a division of N. Let $\Delta = \Phi \times \Psi$ be the division of P, such that $\Delta = \{M_i \times N_j \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$.
Then $U(F, \Phi) \leqslant U(f, \Delta)$ and $L(F, \Phi) \geqslant L(f, \Delta)$

10 Integration on more general domains

Lemma 10.1. Let $P \subseteq P' \subset \mathbb{R}^n$ be a closed rectangle, and $f : P \to P'$ be a bounded function such that $f(\vec{x}) = 0$ for all $\vec{x} \in P \setminus P'$. Then f is integrable on P if and only if f is integrable on P'.

Proposition 10.2. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be bounded. Let $f : A \to \mathbb{R}$ be a bounded function. Pick any closed rectangle $P \subseteq \mathbb{R}^n$ such that $P \supset A$, and extend f to $\tilde{f} : P \to \mathbb{R}$ by $\tilde{f} = \begin{cases} f(\vec{x}) & \vec{x} \in A \\ 0 & \vec{x} \in P \setminus A \end{cases}$

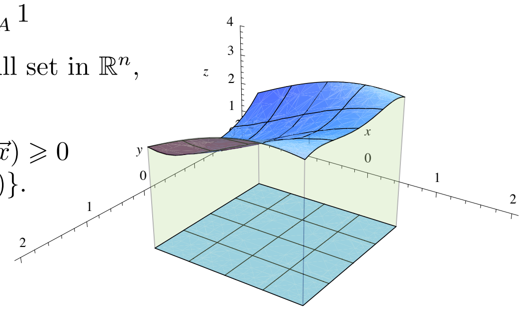
Then by definition f is integrable on A if and only if $\tilde{f}$ is integrable on P, and if f is integrable on A, then by definition $\int_A f = \int_P \tilde{f}$.

Proposition 10.3. Let $\emptyset \neq A \subset \mathbb{R}^n$ be a bounded set. Suppose $bd(A) = cl(A) \setminus int(A)$ is a null set. Let $f : A \to \mathbb{R}$ be a bounded, continuous function. Then f is integrable on A.

Definition 10.4. Let $\emptyset \neq A \subset \mathbb{R}^n$ be a bounded set. Consider $f : A \to \mathbb{R}$ with $f(\vec{x}) = 1$ for all $\vec{x} \in A$. If f is integrable on A, then A has <u>volume</u>, and define $vol(A) = \int_A f = \int_A 1$

Corollary 10.5. Let $\emptyset \neq A \subset \mathbb{R}^n$ be a bounded set. If $bd(A)$ is a null set in $\mathbb{R}^n$, then A has volume.

Definition 10.6. Let $\emptyset \neq A \subseteq \mathbb{R}^n$. Let $f : A \to \mathbb{R}$. Suppose that $f(\vec{x}) \geqslant 0$ for all $\vec{x} \in A$. Define the <u>graph</u> of f to be $\Gamma = \{(\vec{x}, t) \mid \vec{x} \in A, t = f(\vec{x})\}$.
A <u>subgraph</u> of f is $S = \{(\vec{x}, t) \mid \vec{x} \in A, 0 \leqslant t \leqslant f(\vec{x})\}$.
Note that $\Gamma \subset S \subset \mathbb{R}^{n+1}$.



Remark 10.7. The volume of a subset of $\mathbb{R}^n$ is its integral.

Proposition 10.8.$*$ Let $\emptyset \neq A \subset \mathbb{R}^n$ be bounded, and let $f : A \to \mathbb{R}$ be integrable, with $f(\vec{x}) \geqslant 0$ for all $\vec{x} \in A$. Let $S \subset \mathbb{R}^{n+1}$ be the subgraph of f. Then S has volume, and $vol(S) = \int_A f$.

11 Partial derivatives

Definition 11.1. Let $A \subseteq R^n$ be a set with $\vec{a} \in int(A)$ with $f : A \to \mathbb{R}$ a function. Let $\vec{v} \in \mathbb{R}^n$. If

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \left[\frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} \right] \in \mathbb{R} \text{ exists,}$$

then we say that f has <u>directional derivative</u> at $\vec{a}$ in direction $\vec{v}$ and denote the limit by $(\partial_{\vec{v}} f)(\vec{a})$.

Remark 11.2. With respect to the same notation as above:
· If $\vec{v} = \vec{0}$, then $(\partial_{\vec{v}} f)(\vec{a})$ exists and is equal to zero
· If $\vec{v} \neq \vec{0}$, then there exists $r > 0$ such that $B(\vec{a}; r) \subseteq A$. Then we may define $\varphi : \left(-\frac{r}{\|\vec{v}\|}, \frac{r}{\|\vec{v}\|}\right) \to \mathbb{R}$ by $\varphi(t) = f(\vec{a} + t\vec{v})$. Then the directional derivative may be expressed as

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \left[\frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} \right] = \lim_{\substack{t \to 0 \\ t \neq 0}} \left[\frac{\varphi(t) - \varphi(0)}{t - 0} \right] = \varphi'(0)$$

Proposition 11.3. Let $A \subseteq \mathbb{R}^n$ with $\vec{a} \in int(A)$ with $f : A \to \mathbb{R}$ a function, $\vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$. Suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists, Then for every $\alpha \in \mathbb{R}$, we have that $(\partial_{\alpha\vec{v}})(\vec{a})$ exists, and $(\partial_{\alpha\vec{v}} f)(\vec{a}) = \alpha(\partial_{\vec{v}} f)(\vec{a})$.

Definition 11.4. Let $A \subseteq \mathbb{R}^n$ with $\vec{a} \in int(A)$ and $f : A \to \mathbb{R}$. For every $1 \leqslant i \leqslant n$, consider the vector $\vec{e}_i$. If $(\partial_{\vec{e}_i} f)(\vec{a})$ exists, then we call it the <u>i-th partial derivative</u> of f at $\vec{a}$ and denote it by $(\partial_i f)(\vec{a})$.

12 C^1-functions

Definition 12.1. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set, and $f : A \to \mathbb{R}$ be a function. Let $\vec{v} \in \mathbb{R}^n$. Suppose that $(\partial_{\vec{v}} f)(\vec{a})$ exists for every $\vec{a} \in A$. Then we get a new function $\partial_{\vec{v}} f : A \to \mathbb{R}$ by $A \ni \vec{a} \mapsto (\partial_{\vec{v}} f)(\vec{a})$. The function $\partial_{\vec{v}} f$ is termed the directional derivative of f in the direction of $\vec{v}$.

Definition 12.2. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. A function $f : A \to \mathbb{R}$ is said to be a C^1-function when it has the following properties:
· f is continuous on A
· f has partial derivatives at every $\vec{a} \in A$.
· The new functions $\partial_i f : A \to \mathbb{R}$ for $1 \leqslant i \leqslant n$ are all continuous on A.
The collection of all C^1-functions from A to $\mathbb{R}$ is denoted $C^1(A, \mathbb{R})$.

Theorem 12.3.∗ Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. Let $f \in C^1(A, \mathbb{R})$. Then for every $\vec{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$, the directional derivative $\partial_{\vec{v}} f$ exists, and we have $\partial_{\vec{v}} f = v^{(1)}(\partial_1 f) + \cdots + v^{(n)}(\partial_n f)$.

Theorem 12.4.∗ Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. Let $\vec{a} \in A$ and $r > 0$ be such that $B(\vec{a}; r) \subseteq A$. Fix $i \in \{1, \ldots, n\}$ and let $\vec{x}, \vec{y} \in B(\vec{a}; r)$ be such that they only differ in component i. Let $f \in C^1(A, \mathbb{R})$. Then there exists $\vec{b} \in B(\vec{a}; r)$ such that $f(\vec{x}) - f(\vec{y}) = (y^{(i)} - x^{(i)})(\partial_i f)(\vec{b})$.

Lemma 12.5. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. Let $f \in C^1(A, \mathbb{R})$. Fix $\vec{a} \in A$. Then

$$\lim_{\substack{\vec{x} \to \vec{a} \\ \vec{x} \neq \vec{a}}} \left[\frac{f(\vec{x}) - f(\vec{a}) - \sum_{i=1}^{n} (x^{(i)} - a^{(i)}) \cdot (\partial_i f)(\vec{a})}{\|\vec{x} - \vec{a}\|} \right] = 0$$

Definition 12.6. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. Let $f \in C^1(A, \mathbb{R})$. For every $\vec{a} \in A$, the gradient vector of f at $\vec{a}$ is $(\nabla f)(\vec{a}) = ((\partial_1 f)(\vec{a}), \ldots, (\partial_n f)(\vec{a}))$.

Remark 12.7. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^1(A, \mathbb{R})$. Consider the graph $\Gamma = \{(\vec{a}, t) \mid \vec{a} \in A, t = f(\vec{a})\}$, with $\Gamma \subset \mathbb{R}^{n+1}$. Consider $\vec{p} \in \Gamma$, so $\vec{p} = (\vec{a}, f(\vec{a}))$. Then Γ has a tangent hyperplane at $\vec{a}$, which can be calculated by using $(\nabla f)(\vec{a})$.

Proposition 12.8. The vector tangent to Γ in the direction $\vec{v} \in \mathbb{R}^n$ at $\vec{p} = (\vec{a}, f(\vec{a}))$ is given by $(\vec{v}, (\partial_{\vec{v}} f)(\vec{a}))$.

Proposition 12.9. The vector in the normal direction to Γ at $\vec{p} = (\vec{a}, f(\vec{a}))$ is given by $(-(\nabla f)(\vec{a}), 1)$.

Remark 12.10. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open. The set of functions $C^1(A, \mathbb{R})$ is closed under algebraic operations. That is, for all $f, g \in C^1(A, \mathbb{R})$ and $\alpha \in \mathbb{R}$,
1. $f + g \in C^1(A, \mathbb{R}) \implies \partial_{\vec{v}}(f + g) = (\partial_{\vec{v}} f) + (\partial_{\vec{v}} g)$
2. $\alpha f \in C^1(A, \mathbb{R}) \implies \partial_{\vec{v}}(\alpha f) = \alpha(\partial_{\vec{v}} f)$
3. $fg \in C^1(A, \mathbb{R}) \implies \partial_{\vec{v}}(fg) = (\partial_{\vec{v}} f)g + (\partial_{\vec{v}} g)f$

Definition 12.11. Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Denote $\vec{v} = \vec{b} - \vec{a}$. The line segment $\mathrm{Co}(\vec{a}, \vec{b})$ from $\vec{a}$ to $\vec{b}$ is given by $\{\vec{a} + t\vec{v} \mid t \in [0, 1]\} = \{(1 - t)\vec{a} + t\vec{b} \mid t \in [0, 1]\}$. The vector $(1 - t)\vec{a} + t\vec{b}$ for $t \in [0, 1]$ is termed a convex combination of $\vec{a}$ and $\vec{b}$.

Definition 12.12. A set $A \subseteq \mathbb{R}^n$ is said to be convex when for every $\vec{a}, \vec{b} \in A$ we have $\mathrm{Co}(\vec{a}, \vec{b}) \subseteq A$.

Proposition 12.13. [Mean Value Theorem]
Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^1(A, \mathbb{R})$. Suppose $\vec{a}, \vec{b} \in \mathbb{R}^n$ with $\vec{a} \neq \vec{b}$. Let $\vec{v} = \vec{b} - \vec{a}$. Then there exists $\vec{c} \in \mathrm{Co}(\vec{a}, \vec{b})$ with $\vec{c} \neq \vec{a}, \vec{c} \neq \vec{b}$ such that $f(\vec{b}) - f(\vec{a}) = \langle \vec{b} - \vec{a}, (\nabla f)(\vec{c}) \rangle = (\partial_{\vec{v}} f)(\vec{c})$.

13 The chain rule

Definition 13.1. Let $I \subseteq \mathbb{R}$ be an open interval. A function $\gamma : I \to \mathbb{R}^n$ has n components $\gamma^{(1)}, \ldots, \gamma^{(n)} : I \to \mathbb{R}$ and $\gamma(t) = (\gamma^{(1)}(t) \ \ldots \ \gamma^{(n)}(t)) \in \mathbb{R}^n$ for all $t \in I$. If every $\gamma^{(i)}$ is continuous on I, then we say that γ is a path in $\mathbb{R}^n$. If every $\gamma^{(i)}$ is differentiable on I, then we say that γ is a differentiable path in $\mathbb{R}^n$.

Definition 13.2. Let $\gamma : I \to \mathbb{R}^n$ be a differentiable path in $\mathbb{R}^n$. For every vector $t \in I$, the vector $\gamma'(t) = ((\gamma^{(1)})'(t), \ldots, (\gamma^{(n)})'(t)) \in \mathbb{R}^n$ is termed the velocity vector of γ at t.

Lemma 13.3. Suppose $f \in C^1(A, \mathbb{R})$. Let $K \subseteq A$ be compact and convex. Then there exists $c > 0$ such that $|f(\vec{x}) - f(\vec{y})| \leqslant c\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in K$. Here, c is termed the Lipschitz constant.

Theorem 13.4.$*$ [Chain rule 1]
Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open with $f \in C^1(A, \mathbb{R})$. Suppose $\gamma : I \to \mathbb{R}^n$ is a differentiable path such that $\gamma(t) \in A$ for all $t \in I$. Consider the composed function $\varphi = f \circ \gamma$, with $\varphi : I \to \mathbb{R}$ given by $\varphi(t) = f(\gamma(t))$ for all $t \in I$. Then φ is differentiable and we have $\varphi'(t) = \sum_{i=1}^{n} (\partial_i f)(\gamma(t))(\gamma^{(i)})'(t)$ for all $t \in I$.

The chain rule may also be given by $(f \circ \gamma)'(t) = \langle (\nabla f)(\gamma(t)), \gamma'(t) \rangle$.

14 Partial derivatives of higher order

Definition 14.1. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^1(A, \mathbb{R})$. Consider the partial derivatives

$$\partial_1 f : A \to \mathbb{R}, \partial_2 f : A \to \mathbb{R}, \ldots, \partial_n f : A \to \mathbb{R}$$

If $\partial_i f \in C^1(A, \mathbb{R})$ for every $1 \leqslant i \leqslant n$, then we say that $f \in C^2(A, \mathbb{R})$.

Definition 14.2. More generally, for every $p \in \mathbb{N}$, define $C^p(A, \mathbb{R}) = \{f : A \to \mathbb{R} \,\big|\, f$ has continuous partial derivatives up to order $p\}$

Definition 14.3. Functions $f \in C^\infty(A, \mathbb{R})$ are termed smooth for

$$C^\infty(A, \mathbb{R}) = \bigcap_{p=1}^{\infty} C^p(A, \mathbb{R}) = \{f : A \to \mathbb{R} \,\big|\, f \text{ has continuous partial derivatives for all orders } \}$$

Lemma 14.4. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open with $\vec{a} \in A$ and $f \in C^2(A, \mathbb{R})$.
· Fix two indices $i \neq j \in \{1, \ldots, n\}$.
· Fix $r > 0$ such that $B(\vec{a}; r) \subseteq A$.
· Let $\varphi \left(\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right) \to \mathbb{R}$ be defined by $\varphi(t) = f(\vec{a} + t\vec{e}_i + t\vec{e}_j) - f(\vec{a} + t\vec{e}_i) - f(\vec{a} + t\vec{e}_j) + f(\vec{a})$ for $t \in \left(\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right)$.
· Take a sequence $(t_n)_{n=1}^{\infty}$ in $\left(0, \frac{r}{\sqrt{2}}\right)$ such that $t_n \xrightarrow[n\to\infty]{} 0$. Then

1. $\lim\limits_{n\to\infty} \left[\dfrac{\varphi(t_n)}{t_n^2} \right] = (\partial_i(\partial_j f))(\vec{a})$
2. $\lim\limits_{n\to\infty} \left[\dfrac{\varphi(t_n)}{t_n^2} \right] = (\partial_j(\partial_i f))(\vec{a})$

Theorem 14.5. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open with $f \in C^2(A, \mathbb{R})$. Then for every $1 \leqslant i, j \leqslant n$, $\partial_i(\partial_j f) = \partial_j(\partial_i f)$

Definition 14.6. Let $f \in C^2(A, \mathbb{R})$ and $\vec{a} \in A$. The below matrix is termed the Hessian matrix of f at $\vec{a}$:

$$(Hf)(\vec{a}) = \begin{bmatrix} (\partial_1^2 f)(\vec{a}) & (\partial_1 \partial_2 f)(\vec{a}) & \cdots & (\partial_1 \partial_n f)(\vec{a}) \\ (\partial_2 \partial_1 f)(\vec{a}) & (\partial_2^2 f)(\vec{a}) & \cdots & (\partial_2 \partial_n f)(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ (\partial_n \partial_1 f)(\vec{a}) & (\partial_n \partial_2 f)(\vec{a}) & \cdots & (\partial_n^2 f)(\vec{a}) \end{bmatrix}$$

Remark 14.7. Note that $(Hf)(\vec{a}) \in M_{n\times n}(\mathbb{R})$, and the (i,j) entry of $(Hf)(\vec{a})$ is $(\partial_i\partial_j f)(\vec{a})$. Also, by the above theorem, the matrix is symmetric.

Remark 14.8. Let $H = H^t \in M_{n\times n}(\mathbb{R})$. Then all eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. If $\lambda_1, \dots, \lambda_n \geqslant 0$, then H is positive definite. When it is positive definite, then the corresponding linear transformation $T_H : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\langle T_H(\vec{v}), \vec{v}\rangle > 0$ for all $\vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$.

Remark 14.9. Let $\varphi : (a,b) \to \mathbb{R}$ be twice differentiable. Let $t \in (a,b)$ such that $\varphi'(t) = 0$ and $\varphi''(t) > 0$. Then t is a local minimum.

Definition 14.10. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^1(A,\mathbb{R})$ with $\vec{a} \in A$. If $(\nabla f)(\vec{a}) = \vec{0}$, then $\vec{a}$ is a stationary point for f.

Remark 14.11. Let $f \in C^1(A,\mathbb{R})$. If $\vec{a} \in A$ is a local extremum for f, then $\vec{a}$ is a stationary point.

Lemma 14.12. Let $f \in C^2(A,\mathbb{R})$, $\vec{a} \in A$ such that $(Hf)(\vec{a})$ is positive definite. Then there exists $r > 0$ such that $B(\vec{a};r) \subseteq A$ and such that $(Hf)(\vec{x})$ is positive definite for every $\vec{x} \in B(\vec{a};r)$.

Lemma 14.13. Let $f \in C^2(A,\mathbb{R})$, $\vec{a} \in A$ a stationary point and $r > 0$ such that $B(\vec{a};r) \subseteq A$. Suppose that $\vec{b} \in B(\vec{a};r)$, and denote $\vec{v} = \vec{b} - \vec{a}$. Then there exists $t_o \in (0,1)$ and $\vec{c} \in \operatorname{Co}(\vec{a},\vec{b})$ such that $f(\vec{b}) - f(\vec{a}) = t_o\langle T_{(Hf)(\vec{c})}(\vec{v}), \vec{v}\rangle$.

Theorem 14.14.∗ Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^2(A,\mathbb{R})$ with $\vec{a} \in A$ a stationary point for f. If the Hessian matrix $(Hf)(\vec{a})$ is positive definite, then $\vec{a}$ is a local minimum for f.

Remark 14.15. There are analogies for the above theorem and lemmas for $\vec{a}$ a local maximum. Replace f with $-f$ and have $(Hf)(\vec{a})$ be negative definite.

15 Functions in $C^1(A,\mathbb{R}^m)$

Definition 15.1. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open, and $f : A \to \mathbb{R}^m$ a function for $m \geqslant 1$. Write $f = (f^{(1)}, \dots, f^{(m)})$ with $f^{(i)} : A \to \mathbb{R}$ for $1 \leqslant i \leqslant m$. If each $f^{(i)}$ is in $C^1(A,\mathbb{R})$, then we say that $f \in C^1(A,\mathbb{R}^m)$.

Definition 15.2. Let $f \in C^1(A,\mathbb{R}^m)$ and $\vec{a} \in A$. The below matrix is the Jacobian matrix of f at $\vec{a}$:

$$(Jf)(\vec{a}) = \begin{bmatrix} (\partial_1 f^{(1)})(\vec{a}) & (\partial_2 f^{(1)})(\vec{a}) & \cdots & (\partial_n f^{(1)})(\vec{a}) \\ (\partial_1 f^{(2)})(\vec{a}) & (\partial_2 f^{(2)})(\vec{a}) & \cdots & (\partial_n f^{(2)})(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ (\partial_1 f^{(m)})(\vec{a}) & (\partial_2 f^{(m)})(\vec{a}) & \cdots & (\partial_n f^{(m)})(\vec{a}) \end{bmatrix} = \begin{bmatrix} (\nabla f^{(1)})(\vec{a}) \\ (\nabla f^{(2)})(\vec{a}) \\ \vdots \\ (\nabla f^{(m)})(\vec{a}) \end{bmatrix}$$

Proposition 15.3.∗ [Chain rule 2]
Let $A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m$ be open nonempty sets. Let $f \in C^1(A,\mathbb{R}^m)$ such that $f(\vec{x}) \in B$ for all $\vec{x} \in A$. Suppose also $g \in C^1(B,\mathbb{R}^p)$. Consider $h = g \circ f : A \to \mathbb{R}^p$ defined by $h(\vec{x}) = g(f(\vec{x}))$ for $\vec{x} \in A$. Then $(Jh)(\vec{x}) = (Jg)(f(\vec{x})) \cdot (Jf)(\vec{x})$ for all $\vec{x} \in A$.

Remark 15.4. The above may be restated as $(\partial_j h^{(k)})(\vec{x}) = \displaystyle\sum_{i=1}^{m} (\partial_i g^{(k)})(f(\vec{x})) \cdot (\partial_j f^{(i)})(\vec{x})$

Remark 15.5. With respect to the above defined functions, we have $h^{(k)}(\vec{x}) = g^{(k)}(f^{(1)}(\vec{x}), \dots, f^{(m)}(\vec{x}))$

Proposition 15.6. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^1(A,\mathbb{R}^m)$. For $\vec{a} \in A$ and $\vec{x} \approx \vec{a}$, we have

$$f(\vec{x}) = f(\vec{a}) + T_{(Jf)(\vec{a})}(\vec{x} - \vec{a}) + D(\vec{x})$$

where the error term D is given by $D(\vec{x}) = f(\vec{x}) - f(\vec{a}) - T_{(Jf)(\vec{a})}(\vec{x} - \vec{a})$ and $T_{(Jf)(\vec{a})}$ is the matrix transformation associated with the Jacobian of f at $\vec{a}$.

16 Inverse function theorem

Theorem 16.1. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open and $f \in C^1(A, \mathbb{R}^n)$. Let $\vec{a} \in A$ be such that $(Jf)(\vec{a})$ is invertible. Let $f(\vec{a}) = \vec{b} \in \mathbb{R}^n$. Then there exist open sets $U, V \subseteq \mathbb{R}^n$ such that

1. $\vec{a} \in U, \vec{b} \in V \subseteq f(A)$
2. f maps bijectively U onto V, that is, $f(U) = V$ and f is one-to-one on U
3. The function $g : V \to U$ which inverts f is a C^1-function with $(Jg)(\vec{b}) = ((Jf)(\vec{a}))^{-1}$

Lemma 16.2.$*$ Let $M = [\alpha_{ij}] \in M_{n\times n}(\mathbb{R})$ be invertible. Then there exists $\lambda > 0$ such that if $N = [\beta_{ij}] \in M_{n\times n}(\mathbb{R})$ with $|\alpha_{ij} - \beta_{ij}| < \lambda$ for all i, j, then N is invertible as well.

Proposition 16.3.$*$ Let $A \subseteq \mathbb{R}^n$ be open and $\vec{a} \in A$, with $f \in C^1(A, \mathbb{R}^n)$ such that $(Jf)(\vec{a})$ is invertible. Then there exists $r > 0$ such that $B(\vec{a}; r) \subseteq A$ and f is one-to-one on $B(\vec{a}; r)$.

17 Implicit function theorem

Definition 17.1. Let $A \subseteq \mathbb{R}^n$ be open with $\vec{a} \in A$ and $f \in C^1(A, \mathbb{R}^m)$ with $m \leqslant n$. Choose m principal directions in $\mathbb{R}^n$, that is, choose $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m \leqslant n$. The <u>partial Jacobian matrix</u> of f at $\vec{a}$ with respect to directions $j_1, \ldots, j_m$ is given by

$$\begin{bmatrix} (\partial_{j_1} f^{(1)}(\vec{a}) & \cdots & (\partial_{j_m} f^{(1)})(\vec{a}) \\ \vdots & \ddots & \vdots \\ (\partial_{j_1} f^{(m)})(\vec{a}) & \cdots & (\partial_{j_m} f^{(m)})(\vec{a}) \end{bmatrix} \in M_{m\times m}(\mathbb{R})$$

If this matrix is invertible, then $\vec{a}$ is <u>regular</u> wrt $j_1, \ldots, j_m$. Otherwise, $\vec{a}$ is <u>singular</u>.

Definition 17.2. Let $d, n \in \mathbb{N}$ with $n \geqslant d$, and $\emptyset \neq V \subseteq \mathbb{R}^d$. Let $\varphi : V \to \mathbb{R}^n$ be one-to-one and φ be a C^1-function such that $(J\varphi)(\vec{z}) \in M_{n\times d}(\mathbb{R})$ has rank d for every $\vec{z} \in V$. Then the set $S = \varphi(V) \subseteq \mathbb{R}^n$ is termed a <u>parametrized C^1 manifold of dimension d</u>.

Theorem 17.3. [Implicit function theorem]
Let $A \subseteq \mathbb{R}^n$ be nonempty with $\vec{a} \in A$ and $f \in C^1(A, \mathbb{R}^m)$ for $m < n$ and $n - m = d$. Denote $f(\vec{a}) = \vec{b} \in \mathbb{R}^m$ and consider the level set $L = \{\vec{x} \in A \,|\, f(\vec{x}) = \vec{b}\}$. Suppose that $\vec{a}$ is a regular point for f with respect to directions $1, 2, \ldots, m$. Then there exists $r > 0$ such that $L \cap B(\vec{a}; r)$ is a parametrized C^1 manifold of dimension d with parametrization obtained by solving for the first m components.

More precisely, while $\vec{a} = (\vec{p}, \vec{q})$ for $\vec{p} \in \mathbb{R}^m$, $\vec{q} \in \mathbb{R}^d$, then there exists an open set $V \subseteq \mathbb{R}^d$ such that $\vec{q} \in V$ and there exists $h \in C^1(V, \mathbb{R}^m)$ such that

- $h(\vec{q}) = \vec{p}$
- $\{(h(\vec{z}), \vec{z}) \,|\, \vec{z} \in V\} = L \cap B(\vec{a}; r)$

File II
Selected proofs

Theorem 1.1. [Cauchy-Bunyakovsky-Schwarz Inequality]
$| < \vec{x}, \vec{y} > | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$

Proof: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \langle \vec{x} - t\vec{y}, \vec{x} - t\vec{y} \rangle$ for all $t \in \mathbb{R}$.

$$\begin{aligned} &= \langle \vec{x}, \vec{x} \rangle - \langle \vec{x}, t\vec{y} \rangle - \langle t\vec{y}, \vec{x} \rangle + \langle t\vec{y}, t\vec{y} \rangle \\ &= \|\vec{x}\|^2 - 2t\langle \vec{x}, \vec{y} \rangle + t^2\|\vec{y}\|^2 \end{aligned}$$

Hence f is a quadratic function of the form $f = at^2 + bt + c$, with $a = \|\vec{y}\|^2, b = -2\langle \vec{x}, \vec{y} \rangle, c = \|\vec{x}\|^2$.
But observe that $f(t) = \|\vec{x} - t\vec{y}\|^2 \geqslant 0$
Hence the discriminant $\Delta = b^2 - 4ac \leqslant 0$

$$\begin{aligned} &= (-2\langle \vec{x}, \vec{y} \rangle)^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \\ &= 4(\langle \vec{x}, \vec{y} \rangle^2 - \|\vec{x}\|^2\|\vec{y}\|^2) \end{aligned}$$

Now since $\Delta \leqslant 0$, $\langle \vec{x}, \vec{y} \rangle^2 \leqslant \|\vec{x}\|^2\|\vec{y}\|^2$

$$|\langle \vec{x}, \vec{y} \rangle| \leqslant \|\vec{x}\|\|\vec{y}\|$$

Proposition 2.7. Let $(\vec{x}_k)_{k=1}^{\infty} \in \mathbb{R}^n$ and $\vec{a} \in \mathbb{R}^n$. Then $(\vec{x}_k \xrightarrow[k\to\infty]{} \vec{a}$ in $\mathbb{R}^n) \iff \left(\begin{array}{ll} \vec{x}_k^{(1)} \xrightarrow[k\to\infty]{} \vec{a}^{(1)} & \\ \quad\vdots & \text{in } \mathbb{R} \\ \vec{x}_k^{(n)} \xrightarrow[k\to\infty]{} \vec{a}^{(n)} & \end{array} \right)$

Proof: Suppose that $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{a}$ in $\mathbb{R}^n$.
Fix i.
Observe that for every $k \geqslant 1$, $|x^{(i)} - a^{(i)}| = |(\vec{x}_k - \vec{a})^{(i)}| \leqslant \|\vec{x}_k - \vec{a}\| \xrightarrow[k\to\infty]{} 0$
Hence $x_k^{(i)} \xrightarrow[k\to\infty]{} a^{(i)}$.

Now suppose that $x_k^{(i)} \xrightarrow[k\to\infty]{} a^{(i)}$ for every $1 \leqslant i \leqslant n$.
Hence $|x^{(i)} - a^{(i)}| \xrightarrow[k\to\infty]{} 0$ for every $1 \leqslant i \leqslant n$.
So $|x^{(1)} - a^{(1)}| + |x^{(2)} - a^{(2)}| + \cdots + |x^{(n)} - a^{(n)}| \xrightarrow[k\to\infty]{} 0$.
Hence $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{a}$.

Proposition 3.6. Let $\emptyset \neq C \in \mathbb{R}^n$. Then C is closed $\iff$ C has the cannot escape property.

Proof: Since C is closed, $\mathbb{R}^n \setminus C$ is open.
Take $(\vec{x}_k)_{k=1}^{\infty} \in C$, with $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{b} \in \mathbb{R}^n$.
Assume by contradiction that $\vec{b} \notin C$ and $\vec{b} \in \mathbb{R}^n \setminus C$.
Since $\mathbb{R}^n \setminus C$ is open, we can find $r > 0$ such that $B(\vec{b}; r) \subseteq \mathbb{R}^n \setminus C$.
But then since $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{b}$, we can find $k_o \in \mathbb{N}$ such that $\vec{x}_k \in B(\vec{b}; r)$ for all $k \geqslant k_o$.
In particular, $\vec{x}_{k_o} \in B(\vec{b}; r) \subseteq \mathbb{R}^n \setminus C$.
Contradiction.

Now suppose that C has the cannot escape property.
Assume by contradiction that there does not exist any such r such that for $\vec{b} \in \mathbb{R}^n \setminus C$ we have $B(\vec{b}; r) \subseteq \mathbb{R}^n \setminus C$.
So in particular, $B(\vec{b}; r) \not\subseteq \mathbb{R}^n \setminus C \Longrightarrow$ there exists $\vec{x}_1 \in B(\vec{b}; 1)$ such that $\vec{x}_1 \in C$.
In general for every $k \geqslant 1$, we have $B(\vec{b}; \frac{1}{k}) \subseteq \mathbb{R}^n \setminus C$, hence there exists $\vec{x}_k \in B(\vec{b}; \frac{1}{k})$ such that $\vec{x}_k \in C$.
In this way we get a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in C.
Observe that for every $k \geqslant 1$, we have $\vec{x}_k \in B(\vec{b}; \frac{1}{k}) \Longrightarrow \|\vec{x}_k - \vec{b}\| < \frac{1}{k}$.
So we have $\|\vec{x}_k - \vec{b}\| \xrightarrow[k\to\infty]{} 0$, and we obtain that $\vec{x}_k \xrightarrow[k\to\infty]{} \vec{b}$.
But $\vec{x}_k \notin C$ and $\vec{b} \in C$, so we have a contradiction.

Proposition 4.3. For $\emptyset \neq A \in \mathbb{R}^n$, a function $f : A \to \mathbb{R}^n$, and $\vec{a} \in A$, we have that

$$\begin{pmatrix} f \text{ respects sequences in} \\ A \text{ which converge to } \vec{a} \end{pmatrix} \iff \begin{pmatrix} f \text{ is continuous} \\ \text{at } \vec{a} \end{pmatrix}.$$

Proof: Suppose f respects sequences in A which converge to A.

Fix $\epsilon > 0$.

Suppose there exists no such $\delta > 0$ such that $\|\vec{x} - \vec{a}\| < \delta$ and $\|f(\vec{x}) - f(\vec{a})\| < \epsilon$.

In the process of trying different δ, we have a sequence $(\vec{x}_k)_{k=1}^{\infty} \in A$ with $\|\vec{x}_k - \vec{a}\| < \frac{1}{k}$ for all $k \geqslant 1$ and $\|f(\vec{x}) = f(\vec{a})\| \geqslant \epsilon$ for all $k \geqslant 1$.

Since $\|\vec{x}_k - \vec{a}\| < \frac{1}{k}$, we have $\|\vec{x}_k - \vec{a}\| \xrightarrow[k\to\infty]{} 0$.

Also we do not have that $\|f(\vec{x}) = f(\vec{a})\| \xrightarrow[k\to\infty]{} 0$.

So f does not respect $(\vec{x}_k)_{k=1}^{\infty}$, which is a contradiction.

Suppose that f is continuous at $\vec{a}$, using the $\epsilon - \delta$ definition.

Let $(\vec{a}_k)_{k=1}^{\infty}$ be some sequence of points converging to $\vec{a}$.

Let $\epsilon > 0$.

Then we can find $\delta > 0$ such that whenever $\|\vec{a} - \vec{x}\| < \delta$ for $\vec{x} \in A$, we have $\|f(\vec{x}) - f(\vec{a})\| < \epsilon$.

Since $(\vec{a}_k)_{k=1}^{\infty}$ converges to $\vec{a}$, we can find $k_o \geqslant 0$ such that $\|\vec{a} - \vec{a}_k\| < \delta$ whenever $k \geqslant k_o$.

Let $k \geqslant k_o$, and we have that $\|\vec{a} - \vec{a}_k\| < \delta$, so $\|f(\vec{a}) - f(\vec{a}_k)\| < \epsilon$.

Hence $f(\vec{a}_k)$ converges to $f(\vec{a})$.

Theorem 5.3. Suppose $\emptyset \neq A \in \mathbb{R}^n$ is compact. If $f : A \to \mathbb{R}^n$ is continuous, then f is uniformly continuous on A.

Proof: Suppose that no δ exists such that for any $\epsilon > 0$, $\|\vec{x} - \vec{a}\| < \delta \Longrightarrow \|f(\vec{x}) - f(\vec{a})\| < \epsilon$ for $\vec{a}, \vec{x} \in A$.

Let $\delta = \frac{1}{k}$ for $k \in \mathbb{N}$.

Then there exist $\vec{x}_k, \vec{a}_k \in A$ such that $\|\vec{x}_k - \vec{a}_k\| < \frac{1}{k} \Longrightarrow \|f(\vec{x}_k) - f(\vec{a}_k)\| \geqslant \epsilon$.

This generates two sequences, $(\vec{x}_k)_{k=1}^{\infty}$ and $(\vec{a}_k)_{k=1}^{\infty}$ in A.

Since A is compact, we can find $1 \leqslant k(1) \leqslant k(2) \leqslant \cdots \leqslant k(p) \leqslant \cdots$ such that $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ converges to some $\vec{x}_o \in A$.

But then also we have that $(\vec{a}_k)_{k=1}^{\infty}$ converges to the same limit:

$$\begin{aligned}
\|\vec{a}_{k(p)} - \vec{x}_o\| &\leqslant \|\vec{a}_{k(p)} - \vec{x}_{k(p)}\| + \|\vec{x}_{k(p)} - \vec{x}_o\| \\
&< \frac{1}{k(p)} + \|\vec{x}_{k(p)} - \vec{x}_o\| \\
&= 0 \qquad\qquad \text{as } p \to \infty
\end{aligned}$$

Since f is continuous at $\vec{x}_o \in A$, it respects convergence of $\vec{x}_{k(p)} \xrightarrow[p\to\infty]{} \vec{x}_o$ and $\vec{a}_{k(p)} \xrightarrow[p\to\infty]{} \vec{x}_o$.

Therefore $\|f(\vec{x}_{k(p)}) - f(\vec{a}_{k(p)})\| \xrightarrow[p\to\infty]{} 0$.

Contradiction, so such a δ exists.

Proposition 5.4. Suppose $\emptyset \neq A \in \mathbb{R}^n$ is compact. Let $f : A \to \mathbb{R}^n$ be continuous on A. Then the image set $B = f(A) \subseteq \mathbb{R}^n$, or $B = \{\vec{y} \in \mathbb{R}^n \mid \text{there exists } \vec{x} \in A \text{ such that } f(\vec{x}) = \vec{y}\}$ is a compact subset of $\mathbb{R}^n$.

Proof: Fix $\vec{b} \in cl(B)$.

From definition of $cl(B)$, there exists $(\vec{y}_k)_{k=1}^{\infty} \in B$ with $\vec{y}_k \xrightarrow[k\to\infty]{} \vec{b}$.

For every $k \geqslant 1$, we have $\vec{y}_k \in B = f(A)$, hence there exists $\vec{x}_k \in A$ such that $f(\vec{x}_k) = \vec{y}_k$.

Since A is compact and $(\vec{x}_k)_{k=1}^{\infty} \in A$, there exists a subsequence $\left(\vec{x}_{k(p)}\right)_{p=1}^{\infty}$ of the original sequence with $\vec{x}_{k(p)} \xrightarrow[p\to\infty]{} \vec{x}_o \in A$ for some $\vec{x}_o$.

Since f is continuous, $f(\vec{x}_{k(p)}) \xrightarrow[p\to\infty]{} f(\vec{x}_o)$.

Therefore $\vec{y}_{k(p)} \xrightarrow[p\to\infty]{} f(\vec{x}_o)$ while $\vec{y}_k \xrightarrow[k\to\infty]{} \vec{b}$, which implies that $\vec{y}_{k(p)} \xrightarrow[p\to\infty]{} \vec{b}$.

Since limits are unique, $f(\vec{x}_o) = \vec{b}$.
Hence B is closed.
Suppose B is not bounded.
Then there exists a sequence $(\vec{y}_k)_{k=1}^{\infty} \in B$ such that $\|\vec{y}_k\| > k$ for all $k \geqslant 1$.
For every $k \geqslant 1$, pick $\vec{x}_k \in A$ such that $f(\vec{x}_k) = \vec{y}_k$.
Then select a convergent subsequence $\left(\vec{y}_{k(p)}\right)_{p=1}^{\infty}$ for $(\vec{y}_k)_{k=1}^{\infty}$.
Then we reach a contradiction by the same process as above.

Theorem 5.5. [Extreme Value Theorem]
Suppose $\emptyset \neq A \in \mathbb{R}^n$ is compact. Let $f : A \to \mathbb{R}^n$ be continuous. Then f has a minimum and maximum on A. That is, there exist $\vec{\gamma}_1, \vec{\gamma}_2 \in A$ such that $f(\vec{\gamma}_1) \leqslant f(\vec{x}) \leqslant f(\vec{\gamma}_2)$ for all $\vec{x} \in A$.

Proof: Denote $f(A) = K \subseteq \mathbb{R}$.
From a previous proposition, K is compact.
Denote $\inf(A) = \alpha, \sup(A) = \beta$.
Then $\alpha, \beta \in A$.
Then there exist $\vec{\gamma}_1, \vec{\gamma}_2 \in A$ such that $f(\vec{\gamma}_1) = \alpha, f(\vec{\gamma}_2) = \beta$.
Then for every $\vec{x} \in A$, we have $f(\vec{x}) \in K \Longrightarrow \alpha \leqslant f(\vec{x}) \leqslant \beta$, or $f(\vec{\gamma}_1) \leqslant f(\vec{x}) \leqslant f(\vec{\gamma}_2)$.

Lemma 6.11. Suppose P is a closed rectangle on $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ is bounded. Let Γ, Δ be divisions of P such that $\Gamma \prec \Delta$. Then $U(f, \Gamma) \leqslant U(f, \Delta)$ and $L(f, \Gamma) \geqslant L(f, \Delta)$.

Proof: This proof will show inequality for U; the procedure for L is analogous.
Write $\Delta = \{P_1, P_2, \ldots, P_k\}$ and $\Gamma = \{Q_{1_1}, \ldots, Q_{1_{m_1}}, Q_{2_1}, \ldots, Q_{2_{m_2}}, \ldots\ldots, Q_{k_1}, \ldots, Q_{k_{m_k}}\}$
Then we have $P_i = \displaystyle\bigcup_{j=1}^{m_i} Q_{i_j}$ for all $i \in [1, k]$.
Then $U(f, \Gamma) = \displaystyle\sum_{j=1}^{k} vol(Q_{ij}) \sup_{Q_{ij}}\{f\}$
Observe that for every $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m_k$, we have $\sup\{f(\vec{x}) \,|\, \vec{x} \in Q_{ij}\} \leqslant \sup\{f(\vec{x}) \,|\, \vec{x} \in P_i\} = \sup_{P_i}\{f\}$
Then we have

$$
\begin{aligned}
U(f, \Gamma) &\leqslant \sum_{i=1}^{k} \left(\sum_{j=1}^{m_i} vol(Q_{ij}) \sup_{P_i}\{f\} \right) \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{m_i} vol(Q_{ij}) \right) \sup_{P_i}\{f\} \\
&= \sum_{i=1}^{k} vol(P_i) \sup_{P_i}\{f\} \\
&= U(f, \Delta)
\end{aligned}
$$

Proposition 6.15. Let P be a closed rectangle in $\mathbb{R}^n$ and $F : P \to \mathbb{R}$ be bounded. Then

$$\big(\, f \text{ is integrable } \big) \iff \left(\begin{array}{c} \text{for every } \epsilon > 0 \text{, there exists a division } \Delta \text{ of } P \\ \text{such that } U(f, \Delta) - L(f, \Delta) < \epsilon \end{array} \right)$$

Proof: We know that $\sup\{L(f, \Delta) \,|\, \Delta \text{ is a partition of } P\} = \underline{\int}_P f = \overline{\int}_P f = \inf\{U(f, \Delta) \,|\, \Delta \text{ is a partition of } P\}$.
Let $\epsilon > 0$.

Then we can find a division Δ_1 of P such that $L(f\Delta_1) > \underline{\int}_P f - \frac{\epsilon}{2}$ by the definition of sup.

Similarly, we can find a division Δ_2 of P such that $U(f\Delta_2) < \overline{\int}_P f + \frac{\epsilon}{2}$.

From this we have that for $\Delta \prec \Delta_1, \Delta_2$, $U(f,\Delta) \leqslant U(f,\Delta_2) < L(f,\Delta_1) + \epsilon \leqslant L(f,\Delta) + \epsilon$.

Now, for every $k \in \mathbb{N}$, apply the hypothesis with $\epsilon = \frac{1}{k}$.

This gives divisions Δ_k of P such that $U(f,\Delta_k) < L(f,\Delta_k) + \frac{1}{k}$.

But then $\overline{\int}_P f \leqslant U(f,\Delta_k) < L(f,\Delta_k) + \frac{1}{k} \leqslant \underline{\int}_P f + \frac{1}{k}$.

So $\overline{\int}_P f \leqslant \underline{\int}_P f + \frac{1}{k}$ for all $k \in \mathbb{N}$.

Let $k \to \infty$, so that $\underline{\int}_P f \leqslant \overline{\int}_P f$ becomes $\underline{\int}_P f = \overline{\int}_P f$.

Hence f is integrable.

Proposition 6.16. Let P be a closed rectangle in $\mathbb{R}^n$ and $f : P \to \mathbb{R}$ be bounded. Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be a sequence of divisions of P such that $\lim_{k\to\infty}[U(f,\Delta_k) - L(f,\Delta_k)] = 0$, and f be integrable, then

$$\lim_{k\to\infty}[U(f,\Delta_k)] = \int_P f = \lim_{k\to\infty}[L(f,\Delta_k)]$$

Proof: Let $\epsilon > 0$.

We know that there exists k_o such that $U(f,\Delta_k) - L(f,\Delta_k) < \epsilon$ for all $k \geqslant k_o$.

So then for $k \geqslant k_o$, $\overline{\int}_P f \leqslant U(f,\Delta_k) < L(f,\Delta_k) + \epsilon \leqslant \underline{\int}_P f + \epsilon = \int_P f + \epsilon$

This implies that $|U(f,\Delta_k) - \int_P f| < \epsilon$ for all $k \geqslant k_o$.

To prove that lower sums converge to the same integral, write

$$L(f,\Delta_k) = U(f,\Delta_k) - (U(f,\Delta_k) - L(f,\Delta_k)) \xrightarrow[k\to\infty]{} \int_P f - 0 = \int_P f$$

Proposition 7.2. Let P be a closed rectangle in $\mathbb{R}^n$ and $f, g : P \to \mathbb{R}$ be bounded and integrable on P. Then $f+g$ is integrable also with $\int_P (f+g) = \int_P f + \int_P g$.

Proof: Find sequences of divisions of P, $(\Delta'_k)_{k=1}^{\infty}$ such that $U(f,\Delta'_k) - L(f,\Delta'_k) \xrightarrow[k\to\infty]{} 0$ and $(\Delta''_k)_{k=1}^{\infty}$ such that $U(g,\Delta''_k) - L(g,\Delta''_k) \xrightarrow[k\to\infty]{} 0$.

For every $k \geqslant 1$, let Δ_k be a common refinement for Δ'_k and Δ''_k.

Then $0 \leqslant U(f,\Delta_k) - L(f,\Delta_k) \leqslant U(f,\Delta'_k) - L(f,\Delta'_k) \xrightarrow[k\to\infty]{} 0$

Hence $U(f,\Delta_k) - L(f,\Delta_k) \xrightarrow[k\to\infty]{} 0$ and similarly $U(g,\Delta_k) - L(g,\Delta_k) \xrightarrow[k\to\infty]{} 0$.

Also for every $k \geqslant 1$, we have

$$\begin{aligned} U(f+g,\Delta_k) - L(f+g,\Delta_k) &\leqslant (U(f,\Delta_k) + U(g,\Delta_k)) - (L(f,\Delta_k) + L(g,\Delta_k)) \\ &= (U(f,\Delta_k) - L(f,\Delta_k)) + (U(g,\Delta_k) - L(g,\Delta_k)) \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

Then apply a previous proposition to get that $f+g$ is integrable.

Moreover, the above gives us that $\displaystyle\int_P f + g \leqslant U(f+g,\Delta_k) \leqslant U(f,\Delta_k) + U(g,\Delta_k)$ for all $k \geqslant 1$.

Let $k \to \infty$, and we have that $\displaystyle\int_P f + g \leqslant \lim_{k\to\infty}[U(f,\Delta_k) + U(g,\Delta_k)] = \int_P f + \int_P g$

In a similar fashion with the lower sums we get that $\displaystyle\int_P f + g \geqslant \int_P f + \int_P g$.

Therefore $\displaystyle\int_P f + g = \int_P f + \int_P g$.

Theorem 8.2. Let $P \subseteq \mathbb{R}^n$ and $f : P \to \mathbb{R}$ be bounded. Suppose we find subsets $B, G \subseteq P$ such that $B \cup G = P$ and such that

1. f is continuous at every $\vec{x} \in G$

2. B is a null set

Then f is integrable over P.

Proof: Let $\epsilon > 0$.

We want to find a division Δ of P such that $U(f,\Delta) - L(f,\Delta) < \epsilon$.

Fix some bounds α, β for values of f. That is, have $\alpha \leqslant f(\vec{x}) \leqslant \beta$ for all $\vec{x} \in P$.

Consider $\epsilon_o = \frac{\epsilon}{(\beta-\alpha)+vol(P)+1}$

Now, B is a null set, that is, we can find closed rectangles $Q_1, \dots, Q_m$ in $\mathbb{R}^n$ with $Q_1, \dots, Q_m \subseteq P$ and $Q_1 \cup \dots \cup Q_m \supset P$ with $\sum_{i=1}^m Q_i < \epsilon_o$.

By enlarging $Q_1, \dots, Q_m$ as necessary, assume that $int(Q_1) \cup \dots \cup int(Q_m) \supset B$

Let us denote $int(Q_1) \cup \dots \cup int(Q_m) = D$. Then D is an open set.

Let us denote $P \setminus D = K$. Since $K = P \cap (\mathbb{R}^n \setminus D)$ it is a closed set.

K is also bounded, so K is compact.

Observe that $K \subseteq G$, since $D \supset B$ by **1.**

Since $K \subseteq G$, f is continuous at every $\vec{x} \in K$.

Hence f is uniformly continuous on K.

Then there exists $\delta > 0$ such that for $\vec{x}, \vec{y} \in K$ with $\|\vec{x} - \vec{y}\| < \delta$, it follows that $|f(\vec{x}) - f(\vec{y})| < \epsilon$.

So we can make a division Δ of P such that every rectangle of Δ is either counted in $Q_1 \cup \dots \cup Q_m$ or contained in K.

Then write $\Delta = \{\underbrace{P_1', \dots, P_q'}_{\text{in } Q_1 \cup \dots \cup Q_m}, \underbrace{P_1'', \dots, P_r''}_{\text{in } K}\}$

By further refinement, we may arrange the divisions such that $\text{diam}(P_j'') < \delta$ for all j.

Note that for every j, we have $\vec{x}, \vec{y} \in P_j''$ which implies that $\|\vec{x} - \vec{y}\| < \delta$.

This implies that $|f(\vec{x}) - f(\vec{y})| < \epsilon_o$.

This implies that $\sup\limits_{P_j''}\{f\} - \inf\limits_{P_j''}\{f\} \leqslant \epsilon_o$ for each j

Then we have

$$\begin{aligned}
U(f,\Delta) - L(f,\Delta) &= \left(\sum_{i=1}^q vol(P_i') \sup_{P_i'}\{f\} + \sum_{j=1}^r vol(P_j'') \sup_{P_j''}\{f\}\right) - \left(\sum_{i=1}^q vol(P_i') \inf_{P_i'}\{f\} + \sum_{j=1}^r vol(P_j'') \inf_{P_j''}\{f\}\right) \\
&= \sum_{i=1}^q vol(P_i') \left(\sup_{P_i'}\{f\} - \inf_{P_i'}\{f\}\right) + \sum_{j=1}^r vol(P_j'') \left(\sup_{P_j''}\{f\} - \inf_{P_j''}\{f\}\right) \\
&\leqslant \sum_{i=1}^q vol(P_i') \, (\beta - \alpha) + \sum_{j=1}^r vol(P_j'') \epsilon_o \\
&= (\beta - \alpha) \sum_{i=1}^q vol(P_i') + \epsilon_o \sum_{j=1}^r vol(P_j'') \\
&\leqslant (\beta - \alpha)\epsilon_o + \epsilon_o vol(P) \\
&\leqslant \frac{\epsilon(\beta - \alpha + vol(P))}{(\beta - \alpha) + vol(P) + 1} \\
&< \epsilon
\end{aligned}$$

Corollary 8.4. Let $A \subseteq \mathbb{R}^n$ be compact such that $bd(A)$ is a null set. Let P be a closed rectangle with $A \subseteq P$, and define $f : P \to \mathbb{R}$ by $f(\vec{x}) = \begin{cases} 1 & \vec{x} \in A \\ 0 & \text{else} \end{cases}$ Then f is integrable over P.

Proof: Take $B = bd(A)$, which is a null set by the hypothesis, and $G = P \setminus bd(A)$.
Note that $bd(A) = cl(A) \setminus int(A)$.
Therefore $G = int(A) \cup (P \setminus A)$
For every $\vec{a} \in int(A)$ we can find $r > 0$ such that $f(\vec{x}) = 1$ for all $\vec{x} \in B(\vec{a}; r)$
This implies that f is continuous at $\vec{a}$.
For every $\vec{v} \in P \setminus A$, we can find $r > 0$ such that $f(\vec{x}) = 0$ for all $\vec{x} \in B(\vec{v}; r) \cap P$
This implies that f is continuous on G.
Hence f is integrable on G, and by the above theorem, over P.

Theorem 9.3. [THEOREM OF STOLZ-FUBINI]
Let $P = M \times N$ as above, and let $f : P \to \mathbb{R}$ as above. Suppose that
i. f is integrable on P
ii. For every $\vec{v} \in M$, the function $f_{\vec{v}} : N \to \mathbb{R}$ is integrable over N.
Define $F : M \to \mathbb{R}$ by putting $F(\vec{v}) = \int_N f_{\vec{v}}$ for all $\vec{v} \in M$. Then F is integrable on M, and

$$\int_M F = \int_P f$$

Proof: Given f is integrable, there exists a sequence $(\Delta_k)_{k=1}^{\infty}$ of divisions of P with $U(f, \Delta_k) - L(f, \Delta_k) \xrightarrow[k\to\infty]{} 0$.
Refine every Δ_k to a division $\Delta'_k = \Phi_k \times \Psi_k$ where Φ_k is a division of M and Ψ_k is a division of N.
For every $k \geqslant 1$, we have $U(F, \Phi_k) \leqslant U(f, \Delta'_k) \leqslant U(f, \Delta_k)$ and $L(F, \Phi_k) \geqslant L(f, \Delta'_k) \geqslant L(f, \Delta_k)$.
Hence we have $U(F, \Phi_k) - L(F, \Phi_k) \leqslant U(f, \Delta_k) - L(f, \Delta_k) \xrightarrow[k\to\infty]{} 0$.
Therefore F is integrable.
From a previous proposition, we have $\displaystyle\int_M F = \lim_{k\to\infty} [U(F, \Phi_k)]$
And we also have $U(f, \Delta_k) \geqslant U(F, \Phi_k) \geqslant L(F, \Phi_k) \geqslant L(f, \Delta_k)$ so as $k \to \infty$, $\int_P f \geqslant \int_M F \geqslant \int_P f$
Therefore $\int_M F = \int_P f$.

Proposition 10.8. Let $\emptyset \neq P \subset \mathbb{R}^n$ be bounded, and let $f : P \to \mathbb{R}$ be integrable, with $f(\vec{x}) \geqslant 0$ for all $\vec{x} \in P$. Let $S \subset \mathbb{R}^{n+1}$ be the subgraph of f. Then S has volume, and $vol(S) = \int_P f$.

Proof: Pick a sufficiently large upper bound for for f, $c = \sup_P \{f\} + 2$.
Then clearly $S \subset P \times [0, c]$, which is a rectangle in $\mathbb{R}^{n+1}$.
Define $\chi : P \times [0, c] \to \mathbb{R}$ by $\chi((\vec{x}, t)) = \begin{cases} 1 & \text{if } (\vec{x}, t) \in S \\ 0 & \text{else} \end{cases}$
It will be shown that χ is integrable with $\displaystyle\int_{P\times[0,c]} \chi = \int_P f$.
Since f is integrable, there exists a sequence $(\Delta_k)_{k=1}^{\infty}$ of divisions of P such that $U(f, \Delta_k) - L(f, \Delta_k) \xrightarrow[k\to\infty]{} \infty$.
Moreover, we have that $\lim_{k\to\infty} [U(f, \Delta_k)] = \lim_{k\to\infty} [L(f, \Delta_k)]$.
Suppose that $\Delta_k = \{P_1, P_2, \ldots, P_r\}$.
For every $1 \leqslant i \leqslant r$, denote $\alpha_i = \inf_{P_i} f$ and $\beta_i = \sup_{P_i} f$.
Construct $\widetilde{\Delta}_k = \{P_i \times [0, \alpha_i] \,\big|\, 1 \leqslant i \leqslant r\} \cup \{P_i \times [\alpha_i, \beta_i + \frac{1}{k}] \,\big|\, 1 \leqslant i \leqslant r\} \cup \{P_i \times [\beta_i + \frac{1}{k}, c] \,\big|\, 1 \leqslant i \leqslant r\}$.
Note that $\begin{array}{ccccc} \sup\limits_{P\times[0,\alpha_i]} \chi & = & \inf\limits_{P\times[0,\alpha_i]} \chi & = & 1 \\ \sup\limits_{P\times[\beta_i+\frac{1}{k},c]} \chi & = & \inf\limits_{P\times[\beta_i+\frac{1}{k},c]} \chi & = & 0 \end{array}$ and $\begin{array}{ccc} \sup\limits_{P_i\times[\alpha_i,\beta_i+\frac{1}{k}]} \chi & = & 1 \\ \inf\limits_{P_i\times[\alpha_i,\beta_i+\frac{1}{k}]} \chi & = & 0 \end{array}$ hence we have

$$
\begin{aligned}
U(\chi, \widetilde{\Delta}_k) &= \sum_{i=1}^{r} vol(P_i \times [0, \alpha_i]) \cdot 1 + \sum_{i=1}^{r} vol(P_i \times [\alpha_i, \beta_i + 1/k]) \cdot 1 + \sum_{i=1}^{r} vol(P_i \times [\beta_i + 1/k, c]) \cdot 0 \\
&= \sum_{i=1}^{r} vol(P_i)\alpha_i \cdot 1 + \sum_{i=1}^{r} vol(P_i)(\beta_i + 1/k - \alpha_i) \cdot 1 \\
&= \sum_{i=1}^{r} vol(P_i)(\beta_i + 1/k) \\
&= \sum_{i=1}^{r} vol(P_i)\beta_i + \sum_{i=1}^{r} vol(P_i)1/k \\
&= U(f, \Delta_k) + vol(P)/k
\end{aligned}
$$

Similarly, we find $L(\chi, \widetilde{\Delta}_k) = L(f, \Delta_k)$. Then we have

$$U(\chi, \widetilde{\Delta}_k) - L(\chi, \widetilde{\Delta}_k) = U(f, \Delta_k) - L(f, \Delta_k) + vol(P)\tfrac{1}{k} \xrightarrow[k \to \infty]{} 0$$

Hence χ is integrable. Moreover,

$$\int_{P \times [0,c]} \chi = \lim_{k \to \infty} \left[L(\chi, \widetilde{\Delta}_k) \right] = \lim_{k \to \infty} \left[U(\chi, \widetilde{\Delta}_k) \right] = \lim_{k \to \infty} [L(f, \Delta_k)] = \lim_{k \to \infty} [U(f, \Delta_k)] = \int_P f$$

Theorem 12.3. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. Let $f \in C^1(A, \mathbb{R})$. Then for every $\vec{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)}) \in \mathbb{R}^n$, the directional derivative $\partial_{\vec{v}} f$ exists, and we have $\partial_{\vec{v}} f = v^{(1)}(\partial_1 f) + \cdots + v^{(n)}(\partial_n f)$.

Proof: Fix $\vec{a} \in A$ and let $\sum_{i=1}^{n} v^{(i)}(\partial_i f)(\vec{a}) = L$.

It will be shown that $(\partial_{\vec{v}} f)(\vec{a})$ for all $\vec{a} \in A$ exists and is equal to L.

Given $\epsilon > 0$, use the lemma with $\frac{\epsilon}{1+\|\vec{v}\|}$ to get $\delta_o > 0$ such that $B(\vec{a}; \delta_o) \subseteq A$.

So then for every $\vec{x} \neq \vec{a} \in B(\vec{a}; \delta_o)$, we have $\left| \dfrac{f(\vec{x}) - f(\vec{a}) - \sum_{i=1}^{n}(x^{(i)} - a^{(i)}) \cdot (\partial_i f)(\vec{a})}{\|\vec{x} - \vec{a}\|} \right| < \dfrac{\epsilon}{1 + \|\vec{v}\|}$

Let $\delta = \frac{\delta_o}{1+\|\vec{v}\|}$.

It is claimed that for every $t \neq 0$ such that $|t| < \delta$, we have $\left| \frac{f(\vec{a}+t\vec{v}) - f(\vec{a})}{t} - L \right| < \epsilon$, as follows.

Let $t \neq 0$ be such that $|t| < \delta$.

Let $\vec{x} = \vec{a} + t\vec{v}$.

Then $\|\vec{x} - \vec{a}\| = \|t\vec{v}\| < \delta\|\vec{v}\| < \delta_o$

Also $x^{(i)} - a^{(i)} = (a^{(i)} + tv^{(i)}) - a^{(i)} = tv^{(i)}$ for $1 \leqslant i \leqslant n$.

Hence $\displaystyle\sum_{i=1}^{n}(x^{(i)} - a^{(i)})(\partial_i f)(\vec{a}) = \sum_{i=1}^{n} tv^{(i)}(\partial_i f)(\vec{a}) = tL$.

Finally we have

$$
\begin{aligned}
\left| \frac{f(\vec{x}) - f(\vec{a}) - \sum_{i=1}^{n}(x^{(i)} - a^{(i)}) \cdot (\partial_i f)(\vec{a})}{\|\vec{x} - \vec{a}\|} \right| = \left| \frac{f(\vec{a} + t\vec{v}) - f(\vec{a}) - tL}{|t|\|\vec{v}\|} \right| &< \frac{\epsilon}{1 + \|\vec{v}\|} \\
\left| \frac{f(\vec{a} + t\vec{v}) - f(\vec{a}) - tL}{|t|} \right| &< \frac{\epsilon\|\vec{v}\|}{1 + \|\vec{v}\|} \\
\left| \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} - L \right| &< \epsilon
\end{aligned}
$$

Theorem 12.4. Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be an open set. Let $\vec{a} \in A$ and $r > 0$ be such that $B(\vec{a}; r) \subseteq A$. Fix $i \in \{1, \dots, n\}$ and let $\vec{x}, \vec{y} \in B(\vec{a}; r)$ be such that they only differ in component i. Let $f \in C^1(A, \mathbb{R})$. Then there exists $\vec{b} \in B(\vec{a}; r)$ such that $f(\vec{x}) - f(\vec{y}) = (y^{(i)} - x^{(i)})(\partial_i f)(\vec{b})$.

Proof: The case $x^{(i)} = y^{(i)}$ is trivial, so assume $x^{(i)} < y^{(i)}$.

Note that $(x^{(1)}, \dots, x^{(i-1)}, s, x^{(i+1)}, \dots, x^{(n)}) \in B(\vec{a}; r)$ for all $s \in [x^{(i)}, y^{(i)}]$.

Define $\varphi : [x^{(i)}, y^{(i)}] \to \mathbb{R}$ by $\varphi(s) = (x^{(1)}, \dots, x^{(i-1)}, s, x^{(i+1)}, \dots, x^{(n)})$ for all $s \in [x^{(i)}, y^{(i)}]$.

Then φ is continuous on $[x^{(i)}, y^{(i)}]$.

Then φ is differentiable on $(x^{(i)}, y^{(i)})$, and for every $s \in (x^{(i)}, y^{(i)})$, we have $\varphi'(s) = (\partial_i f)(\vec{b})$ for $\vec{b} = \varphi(s)$.

Apply the mean value theorem to φ, so there exists $s \in (x^{(i)}, y^{(i)})$ such that

$$\begin{aligned}
\frac{\varphi(y^{(i)}) - \varphi(x^{(i)})}{y^{(i)} - x^{(i)}} &= \varphi'(s) \\
\frac{f(\vec{y}) - f(\vec{y})}{y^{(i)} - x^{(i)}} &= (\partial_i f)(\vec{b}) \\
f(\vec{y}) - f(\vec{y}) &= (y^{(i)} - x^{(i)})(\partial_i f)(\vec{b})
\end{aligned}$$

Theorem 13.4. [Chain rule 1]
Let $\emptyset \neq A \subseteq \mathbb{R}^n$ be open with $f \in C^1(A, \mathbb{R})$. Suppose $\gamma : I \to \mathbb{R}^n$ is a differentiable path such that $\gamma(t) \in A$ for all $t \in I$. Consider the composed function $\varphi = f \circ \gamma$, with $\varphi : I \to \mathbb{R}$ given by $\varphi(t) = f(\gamma(t))$ for all $t \in I$. Then φ is differentiable and we have $\varphi'(t) = \sum_{i=1}^{n} (\partial_i f)(\gamma(t))(\gamma^{(i)})'(t)$ for all $t \in I$.

Proof: Fix $t_o \in I$.

Denote $\vec{a} := \gamma(t_o) = (\gamma^{(1)} t_o, \dots, \gamma^{(n)}(t_o))$
$\vec{v} := \gamma'(t_o) = ((\gamma^{(1)})' t_o, \dots, (\gamma^{(n)})'(t_o))$

It will be proven that $\displaystyle\lim_{\substack{h \to 0 \\ h \neq 0}} \left[\frac{\varphi(t_o + h) - \varphi(t_o)}{h} \right] = \varphi'(t_o) = (\partial_{\vec{v}} f)(\vec{a})$.

Choose $r > 0$ such that $B(\vec{a}; r) \subseteq A$

Fix $c > 0$ such that if $\vec{x}, \vec{y} \in \overline{B}(\vec{a}; \frac{r}{2})$, then $|f(\vec{x}) - f(\vec{y})| \leqslant c \|\vec{x} - \vec{y}\|$.

Fix $\ell > 0$ such that $(t_o - \ell, t_o + \ell) \subseteq I$ and such that $|t - t_o| < \ell \Longrightarrow \|\gamma(t) - \gamma(t_o)\| < \frac{r}{2}$.

Use $h \neq 0$ such that $|h| < \ell$ and $|h| < \frac{r}{2(1 - \|\vec{v}\|)} \Longrightarrow \|(\vec{a} + h\vec{v}) - \vec{a}\| < \frac{r}{2}$.

Denote $\ell_o = \max\{\ell, \frac{r}{2(1 - \|\vec{v}\|)}\}$.

For $0 < |h| \leqslant \ell_o$ write

$$\begin{aligned}
\frac{\varphi(t_o + h) - \varphi(t_o)}{h} &= \frac{f(\gamma(t_o + h)) - f(\gamma(t_o))}{h} \\
&= \frac{f(\gamma(t_o + h)) - f(\gamma(t_o) + h\vec{v})}{h} + \frac{f(\gamma(t_o) + h\vec{v}) - f(\gamma(t_o))}{h} \\
&= \frac{f(\gamma(t_o + h)) - f(\vec{a} + h\vec{v})}{h} + \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}
\end{aligned}$$

Now the limit of the first term as $h \to 0$ will go to zero by the following:

$$\begin{aligned}
\left| \frac{f(\gamma(t_o + h)) - f(\vec{a} + h\vec{v})}{h} \right| &\leqslant c \left\| \frac{\gamma(t_o + h) - \vec{a} - h\vec{v}}{h} \right\| \\
&= c \left\| \frac{\gamma(t_o + h) - \gamma(t_o)}{h} - \gamma'(t_o) \right\| \\
&\leqslant c \sum_{i=1}^{n} \left| \frac{\gamma^{(i)}(t_o + h) - \gamma^{(i)}(t_o)}{h} - (\gamma^{(i)})'(t_o) \right|
\end{aligned}$$

Taking limits of the above, since the last line is a sum of n zeros, we have

$$\lim_{h\to 0}\left[\left|\frac{f(\gamma(t_o+h))-f(\vec{a}+h\vec{v})}{h}\right|\right]=0$$

Clearly, we also have $\lim_{h\to 0}\left[\frac{f(\vec{a}+h\vec{v})-f(\vec{a})}{h}\right]=(\partial_{\vec{v}}f)(\vec{a})$, so the theorem is proved.

Theorem 14.15. Let $\emptyset\neq A\subseteq\mathbb{R}^n$ be open and $f\in C^2(A,\mathbb{R})$ with $\vec{a}\in A$ a stationary point for f. If the Hessian matrix $(Hf)(\vec{a})$ is positive definite, then $\vec{a}$ is a local minimum for f.

Proof: Pick $r>0$ as in first lemma above.

Then for every $\vec{b}\neq\vec{a}$ in $B(\vec{a};r/2)$ we have $\vec{c}\in\operatorname{Co}(\vec{a},\vec{b})\subseteq B(\vec{a};r/2)$.

Therefore $f(\vec{b})-f(\vec{a})=t_o\langle T_{(Hf)(\vec{c})}(\vec{v}),\vec{v}\rangle>0$ since $(Hf)(\vec{c})$ is positive definite.

Hence $f(\vec{b})>f(\vec{a})$.

Proposition 15.3. [Chain rule 2]
Let $A\subseteq\mathbb{R}^n, B\subseteq\mathbb{R}^m$ be open nonempty sets. Let $f\in C^1(A,\mathbb{R}^m)$ such that $f(\vec{x})\in B$ for all $\vec{x}\in A$. Suppose also $g\in C^1(B,\mathbb{R}^p)$. Consider $h=g\circ f:A\to\mathbb{R}^p$ defined by $h(\vec{x})=g(f(\vec{x}))$ for $\vec{x}\in A$. Then $(Jh)(\vec{x})=(Jg)(f(\vec{x}))\cdot(Jf)(\vec{x})$ for all $\vec{x}\in A$.

Proof: This will be reduced to the chain rule as proved above.

Fix $\vec{x}\in A$, $1\leqslant k\leqslant p$, $1\leqslant j\leqslant n$.

It will be shown that $\lim\limits_{\substack{t\to 0\\ t\neq 0}}\left[\frac{h^{(k)}(\vec{x}+t\vec{e}_j)-h^{(k)}(\vec{x})}{t}\right]=(\partial_j h^{(k)})(\vec{x})=\sum\limits_{i=1}^{m}(\partial_i g^{(k)})(f(\vec{x}))\cdot(\partial_j f^{(i)})(\vec{x})$.

For $\vec{x}\in A$, there exists $r>0$ such that $B(\vec{x};r)\subseteq A$, in particular $\vec{x}+t\vec{e}_j\in A$ for all $t\in(-r,r)$.

Define $\gamma:(-r,r)\to B\subseteq\mathbb{R}^m$ by $\gamma(t)=f(\vec{x}+t\vec{e}_j)$ for $t\in(-r,r)$.

Apply the chain rule above to γ and to $g^{(k)}\in C^1(B,\mathbb{R}^m)$.

Define $\varphi:(-r,r)\to\mathbb{R}^p$ by $\varphi(t)=g^{(k)}(\gamma(t))$ for $t\in(-r,r)$.

Observe that for every $t\in(-r,r)$, we have $\varphi(t)=g^{(k)}(\gamma(t))=g^{(k)}(f(\vec{x}+t\vec{e}_j))=h^{(k)}(\vec{x}+t\vec{e}_j)$.

Then we have $\lim\limits_{\substack{t\to 0\\ t\neq 0}}\left[\frac{h^{(k)}(\vec{x}+t\vec{e}_j)-h^{(k)}(\vec{x})}{t}\right]=\lim\limits_{\substack{t\to 0\\ t\neq 0}}\left[\frac{\varphi(t)-\varphi(0)}{t}\right]$

Proving the necessary limit then amounts to proving that φ is differentiable at 0 and that $\varphi'(0)$ is the desired limit.

Now it will be shown that γ is a differentiable path with a formula for $\gamma'(t)$.

Then we have

$$\begin{aligned}(\gamma^{(i)})'(t)=\lim_{\substack{s\to 0\\ s\neq 0}}\left[\frac{\gamma^{(i)}(t+s)-\gamma^{(i)}(t)}{s}\right]&=\lim_{\substack{s\to 0\\ s\neq 0}}\left[\frac{f^{(i)}(\vec{x}+(t+s)\vec{e}_j)-f^{(i)}(\vec{x}+t\vec{e}_j)}{s}\right]\\ &=\lim_{\substack{s\to 0\\ s\neq 0}}\left[\frac{f^{(i)}(\vec{x}+t\vec{e}_j+s\vec{e}_j)-f^{(i)}(\vec{x}+t\vec{e}_j)}{s}\right]\\ &=(\partial_j f^{(i)})(\vec{x}+t\vec{e}_j)\end{aligned}$$

Now apply chain rule 1 to $\varphi=g^{(k)}\circ\gamma$ to get

$$\begin{aligned}\varphi'(0)&=\sum_{i=1}^{m}(\partial_i g^{(k)})(\gamma(0))(\gamma^{(i)})'(0)\\ &=\sum_{i=1}^{m}(\partial_i g^{(k)})(f(\vec{x}))(\partial_j f^{(i)})(\vec{x})\end{aligned}$$

And this is the desired limit.

Lemma 16.2. Let $M = [\alpha_{ij}] \in M_{n\times n}(\mathbb{R})$ be invertible. Then there exists $\lambda > 0$ such that if $N = [\beta_{ij}] \in M_{n\times n}(\mathbb{R})$ with $|\alpha_{ij} - \beta_{ij}| < \lambda$ for all i, j, then N is invertible as well.

Proof: Denote $|\det(M)| = \epsilon > 0$, since M is invertible.
Recall that the determinant of an $n \times n$ matrix T is a polynomial P_n of the matrix with $P_n(T) = \det(T)$.
So we have continuity of P_n at $(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{nn})$ for $\epsilon/2$.
Hence there exists $\delta > 0$ such that $\|(\beta_{11}, \beta_{12}, \ldots, \beta_{nn}) - (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{nn})\| < \delta$.
This implies that $|P_n((\beta_{11}, \beta_{12}, \ldots, \beta_{nn})) - P_n((\alpha_{11}, \alpha_{12}, \ldots, \alpha_{nn}))| < \epsilon/2$.
Let $\lambda = \delta/n$.
If $N = [\beta_{ij}]$ has $|\beta_{ij} - \alpha_{ij}| < \lambda$ for all i, j, then we get

$$\begin{aligned}\|(\beta_{11}, \beta_{12}, \ldots, \beta_{nn}) - (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{nn})\| &= \sqrt{\sum_{i,j} (\beta_{ij} - \alpha_{ij})^2} \\ &< \sqrt{n^2\lambda^2} \\ &= n\lambda \\ &= \delta\end{aligned}$$

Then also

$$\begin{aligned}\epsilon &= |\det(M)| \\ &\leqslant |\det(N)| + |\det(M) - \det(N)| \\ &< |\det(N)| + \epsilon/2 \\ |\det(N)| &> \epsilon/2\end{aligned}$$

Therefore N is invertible.

Proposition 16.3. Let $A \subseteq \mathbb{R}^n$ be open and $\vec{a} \in A$, with $f \in C^1(A, \mathbb{R}^n)$ such that $(Jf)(\vec{a})$ is invertible. Then there exists $r > 0$ such that $B(\vec{a}; r) \subseteq A$ and f is one-to-one on $B(\vec{a}; r)$.

Proof: Denote $(Jf)(\vec{a}) = M = [\alpha_{ij}]$, which is invertible, with $\alpha_{ij} = (\partial_j f^{(i)})(\vec{a})$.
Lemma gives a $\lambda > 0$ such that if $N = [\beta_{ij}]$ with $|\beta_{ij} - \alpha_{ij}| < \lambda$ for all $1 \leqslant i, j \leqslant n$, then N is invertible.
Due to continuity of $\partial_j f^{(i)}$at $\vec{a}$, we can find $r > 0$ such that $B(\vec{a}; r) \subseteq A$ and such that $|(\partial_j f^{(i)})(\vec{x}) - (\partial_j f^{(i)})(\vec{a})| < \lambda$ for all $\vec{x} \in B(\vec{a}; r)$.
Assume that f is not one-to-one on $B(\vec{a}; r)$.
Hence there exists $\vec{p}, \vec{q} \in B(\vec{a}; r)$ such that $f(\vec{p}) = f(\vec{q})$.
For every $1 \leqslant i \leqslant n$, apply the mean value theorem to $f^{(i)} \in C^1(A, \mathbb{R})$ between $\vec{p}$ and $\vec{q}$.
The mean value theorem gives a point $\vec{c}_i \in \mathrm{Co}(\vec{a}, \vec{b})$ such that $0 = f^{(i)}(\vec{p}) - f^{(i)}(\vec{q}) = \langle(\nabla f^{(i)})(\vec{c}_i), \vec{p} - \vec{q}\rangle$
Consider the matrix $N = \begin{bmatrix}(\nabla f^{(1)})(\vec{c}_1) \\ (\nabla f^{(2)})(\vec{c}_2) \\ \vdots \\ (\nabla f^{(n)})(\vec{c}_n)\end{bmatrix}$ where $N = [\beta_{ij}]$ with $\beta_{ij} = (\partial_j f^{(i)})(\vec{c}_i)$ for all $1 \leqslant i, j \leqslant n$.
By choice of r and because $\vec{c}_i \in B(\vec{a}; r)$, we get $|\beta_{ij} - \alpha_{ij}| = |(\partial_j f^{(i)})(\vec{c}_i) - (\partial_j f^{(i)})(\vec{a})| < \lambda$ for all $1 \leqslant i, j \leqslant n$.
Hence N is invertible.
Since $\langle(\nabla f^{(i)})(\vec{c}_i), \vec{p} - \vec{q}\rangle = 0$ for all $1 \leqslant i \leqslant n$, we have that $\vec{p} - \vec{q} \in \mathrm{Null}(N)$.
This is a contradiction, since N is invertible, we have $\mathrm{Null}(N) = \{0\}$.
Therefore f is one-to-one on $B(\vec{a}; r)$.