Morse Theory

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Seminars were organized around John Milnor's *Morse Theory*.

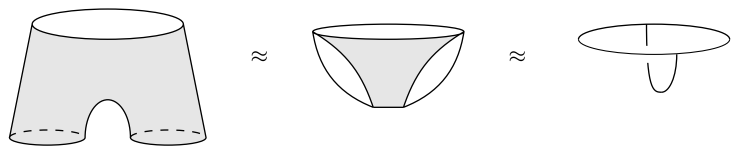
1 Background

1.1 Topology

Theorem 1.1. Let M be a compact manifold, and $f : M \to \mathbb{R}$ a smooth function with 2 critical points. Then M is homeomorphic to $\mathbb{S}^n$.

1.1.1 Homotopy

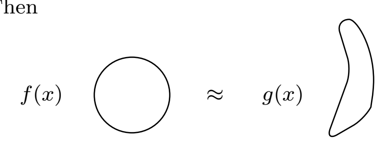
For our uses, homomorphism is too restrictive. We will use homotopy, in which case the following figures are said to be homotopic to each other.



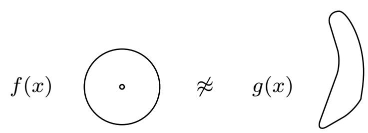
These spaces are known as, left to right, the pair of pants, the pair of underwear, and the g-string.

Definition 1.2. Let X, Y be topological spaces and $f; g : X \to Y$ continuous functions. Then the maps f and g are termed homotopic iff there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Example 1.3. Let $X = \mathbb{S}^1$ and $Y = \mathbb{R}^2$. Then



Let $X = \mathbb{S}^1$ and $Y = \mathbb{R}^2 \setminus \{(0, 0)\}$. Then as now $f(x)$ encircles a hole, and $g(x)$ does not, one may not be deformed continously in this space to get the other, so



Definition 1.4. A continuous map $f : X \to Y$ is termed a homotopy equivalence if there exists a continuos map $g : Y \to X$ such that $f \circ g \cong \mathrm{id}_Y$ and $g \circ f \cong \mathrm{id}_X$. In this case, the spaces X and Y are termed homotopic. Note that this is an equivalence relation.

Example 1.5. Let $X = \mathbb{R}^n \setminus \{0\}$ and $Y = \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \ : \ |x| = 1\}$. Fix two maps:

$$\begin{array}{rccl} f: & X & \to & Y \\ & x & \mapsto & \frac{x}{|x|} = \hat{x} \end{array} \qquad \begin{array}{rccl} g: & Y & \to & X \\ & x & \mapsto & x \end{array}$$

Now check that they are homotopy equivalent.

$$(f \circ g)(x) = f(x) = x$$
$$(g \circ f)(x) = g(\hat{x}) = \hat{x}$$

Next consider the following two homotopies.

$$\begin{array}{rccl} H_1: & X \times I & \to & X \\ & (x, t) & \mapsto & tx + (1 - t)\hat{x} \end{array} \qquad \begin{array}{rccl} H_2: & Y \times I & \to & Y \\ & (x, t) & \mapsto & x \end{array}$$

Then we have that $H_1(x, 0) = \hat{x} = (g \circ f)(x)$, and $H_1(x, 1) = x = \mathrm{id}_x$. This shows that X and Y are homotopic.

1.1.2 Retracts

Definition 1.6. Let X be a topological space with $A \subseteq X$. Then A is termed a retract of X iff there exists a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

A simple example is $X = \mathbb{S}^1$ and $A = \{y\}$ for some $y \in \mathbb{S}^1$, with $r(x) = y$.

Definition 1.7. Let X be a topological space with $A \subseteq X$. Then A is termed a deformation retract of X iff there exists a retraction $r : X \to A$ and a homotopy $H : X \times I \to X$ with:

$$\begin{array}{rcll} H(x, 0) & = & x & \\ H(x, 1) & = & r & \\ H(a, t) & = & a & \forall\, a \in A, t \in I \end{array}$$

Lemma 1.8. If A is a deformation retract of X, then $X \approx A$.

Proof: Let $r : X \to A$ be the retraction, and $\iota : A \hookrightarrow X$ be the inclusion. It is clear that $r \circ \iota = \mathrm{id}_A$ and $\iota \circ r = \mathrm{id}_X$, hence A and X are homotopic. ∎

1.1.3 CW-complexes

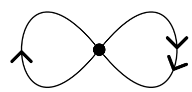
In the finite case, CW stands for *nice*.

Definition 1.9. An n-cell is e^n, most commonly given by $e^n = \{x \in \mathbb{R}^n \ : \ |x| \leqslant 1\}$, with $\partial e^n = \mathbb{S}^{n-1}$.

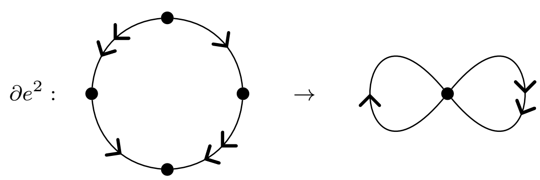
Definition 1.10. Let X be a topological space. Let $f : \partial e^n = \mathbb{S}^{n-1} \to X$ be a continuous map. Then gluing e^n to X via f results in the space $Y = X \sqcup_f e^n = (X \sqcup e^n)/\sim$, for $a \sim f(a)$ for all $a \in \partial e^n$.

Example 1.11. For the map $\varphi : \partial e^n \to \mathrm{pt}$, we have $e^n \sqcup_\varphi \{\mathrm{pt}\} = \mathbb{S}^n$.

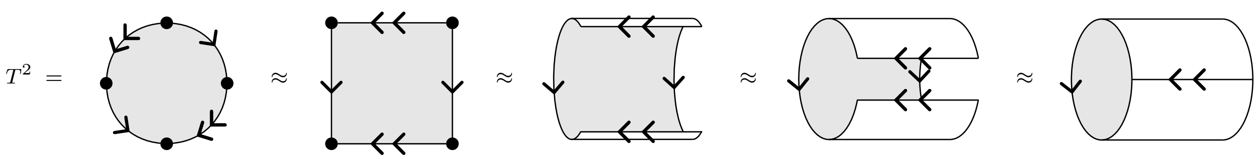
Example 1.12. The torus T^2 may be constructed by gluing cells together. Start with a point and attach two 1-cells to the point by their boundaries as below.



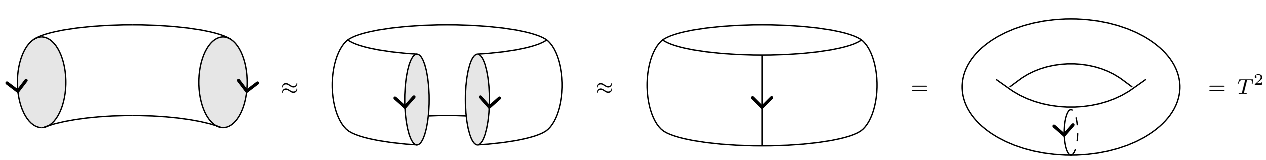
Attach a 2-cell to the shape above, by the identification of ∂e^2 as below.



The resulting shape will be T^2, which materializes by identifying in the same direction pairs of commonly labeled edges. Start with the pair labelled by double arrows:



This produces a cylinder, for which we now identify the other pair of arrows:



Definition 1.13. A CW-complex may be constructed by starting with a finite number of discrete points, then attaching a finite number of cells of varying dimension.

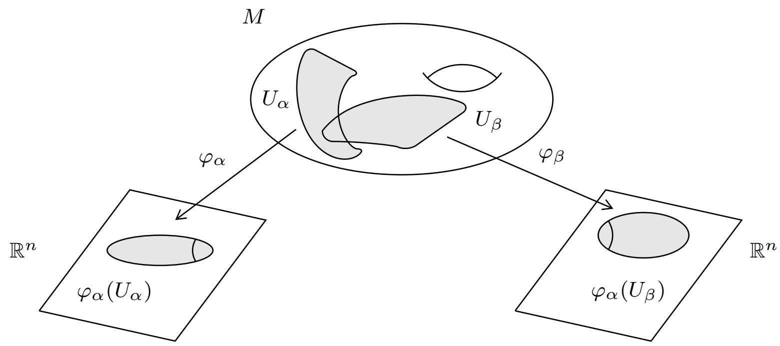
1.2 Differential geometry

1.2.1 Smooth manifolds

We wish to give our topological spaces some more structure.

Definition 1.14. A smooth n-manifold is a topological space M such that:

1. $\{U_\alpha\}$ covers M such that for each α, there exists a homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ open
2. Where $U_\alpha \cap U_\beta \neq 0$, the composition $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism

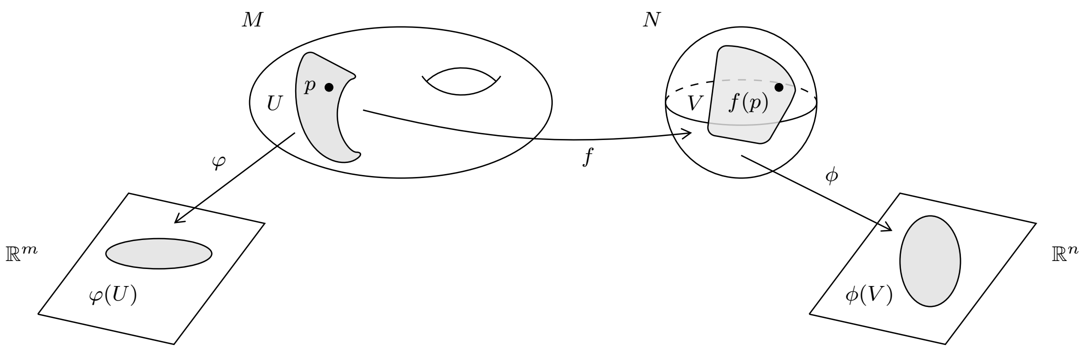


Around each $p \in M$, we may pick a pair $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$, and thus give p local coordinates, as $\varphi_\alpha = (x_1, \ldots, x_n)$. The pair $(U_\alpha, \varphi_\alpha)$ is termed a coordinate chart.

1.2.2 Smooth maps

Let M be a smooth m-manifold, and N a smooth n-manifold. We already know what it means for $F : \mathbb{R}^n \to \mathbb{R}^m$ to be smooth.

Definition 1.15. A map $f : M \to N$ is termed smooth if it is smooth locally everywhere, i.e. given $p \in N$, there exist charts $(U \ni p, \varphi)$ and $(V \ni f(p), \phi)$ with $f(U) \subseteq f(V)$, such that $\phi \circ f \circ \varphi^{-1} : \varphi(U) \to \phi(V)$ is smooth.



1.2.3 Tangent spaces and pushforwards

Definition 1.16. *(Geometric)* Let (U, φ) be a coordinate chart around p. A tangent vector is as equivalence class of curves through p, where a curve is a smooth map $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$, given $c_1 \sim c_2$ iff $(\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$.

The tangent space T_pM at p is the set of all tangent vectors at p.

2 Foundations

2.1 Basic tools

2.1.1 The tangent space

Definition 2.1. Let $M \supseteq U \ni p$ be a smooth manifold. For a chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$, define A_pM by

$$A_pM = \left\{ [(-\epsilon, \epsilon) \xrightarrow{c} M] \ : \ c \text{ is smooth, } c(0) = p, \ \mathrm{Im}(c) \subseteq U \right\}$$

This space is restricted by the relation $[c] = [\tilde{c}]$ iff $(\varphi \circ c)'(0) = (\varphi \circ \tilde{c})'(0)$. Further, the choice of chart is not important, as for any other chart $\psi : W \to \tilde{W} \subseteq \mathbb{R}$ around p, we set $[c] = \psi[\tilde{c}]$ iff $[c] = \varphi[\tilde{c}]$.

Proposition 2.2. Let $f, g \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and $p \in \mathbb{R}^2$. Then

$$\left.\frac{\partial}{\partial x}\right|_p (fg) = \left(\left.\frac{\partial}{\partial x}\right|_p (f) \right) g(p) + \left(\left.\frac{\partial}{\partial x}\right|_p (g) \right) f(p)$$

This is termed the product rule.

Proposition 2.3. Suppose that $\varphi : C^\infty(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}$ is $\mathbb{R}$-linear, and satisfies, for some $p \in \mathbb{R}$,

$$\varphi(fg) = \varphi(f)g(p) + \varphi(g)f(p) \ \forall \ f, g$$

Then $\varphi \in \operatorname{span}\left\{ \frac{\partial}{\partial x}\Big|_p, \frac{\partial}{\partial y}\Big|_p \right\}$.

Definition 2.4. Let $M \supseteq U \ni p$ be a smooth manifold. Define B_pM by

$$B_pM = \{\varphi : C^\infty(M, \mathbb{R}) \to \mathbb{R} \ : \ \varphi \text{ is } R\text{-linear, } \varphi(fg) = \varphi(f)g(p) + \varphi(g)f(p)\}$$

Note that $\dim(A_pM) = \dim(B_pM) = \dim(M)$.

Definition 2.5. For A_pM and B_pM as above, define $\mu : A_pM \to B_pM$ by

$$\begin{array}{rccc} \mu([c]) : & C^\infty(M,\mathbb{R}) & \to & \mathbb{R} \\ & f & \mapsto & (f \circ c)'(0) \end{array}$$

This map is an isomorphism.

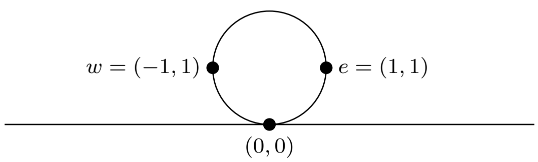
Definition 2.6. The space T_pM, which is both A_pM and B_pM, is termed the tagent space of M at p.

Next consider a map $f : M \to N$, with $p \in M$. The map and the point p induce a map $(f_*)_p : T_pM \to T_{f(p)}N$ between tangent spaces on different manifolds. The definition is straightforward - if we have a path $c : [-\epsilon, \epsilon] \to M$, then this becomes $f \circ c : [-\epsilon, \epsilon] \to N$, a path on N.

Definition 2.7. Let M, N be smooth manifolds and $f : M \to N$ a smooth map. If $p \in M$ satisfies $(f_*)_p = 0$, then p is termed a critical point.

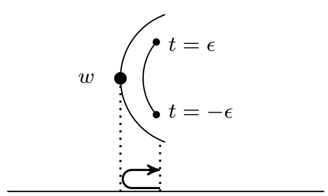
2.1.2 Some motivating examples

Example 2.8. Consider the circle sitting on the real line.

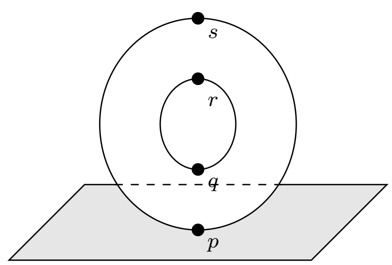


$$\begin{array}{rccc} \mu : & \mathbb{S}^1 & \to & \mathbb{R} \\ & e^{i\theta} & \mapsto & \cos(\theta) \end{array} \qquad \operatorname{rank}((\mu_*)_p) = \begin{cases} 0 & \text{if } p \in \{w, e\} \\ 0 & \text{else} \end{cases}$$

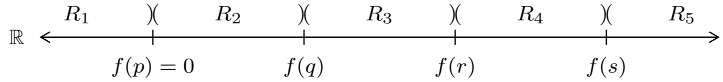
The map $(\mu_*)_w : T_w : \mathbb{S}^1 \to T_{-1}\mathbb{R}$ is given in the following diagram:



Example 2.9. Place the torus T^2 on the plane as shown below. Let $f : \mathbb{T}^2 \to \mathbb{R}$ be defined by $f(x) =$ (the height of x above the plane). Further, for $a \subseteq T^2$, let $T^2_a = \{x \in T^2 \ : \ f(x) \leqslant a\}$. Mark the points p, q, r, s on the torus as shown.



It is clear that the marked points are exactly the critical points of f. Based on these points, define several regions on $\mathbb{R}$ as follows:

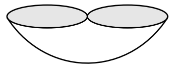


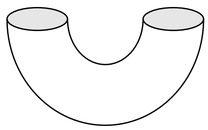
Let $r_i \in R_i$ be any element, and define $T_a^2 = \{x \in T^2 \;:\; f(x) \leqslant a\}$ for $a \in \mathbb{R}$. Then T^2 may be described at the all the ciritical points and defined regions as follows:

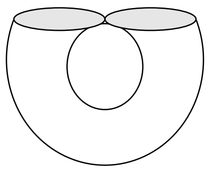
$T^2_{r_1} \quad = \quad \emptyset$

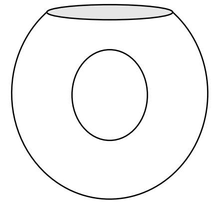
$T^2_{f(p)} \quad = \quad \{p\}$

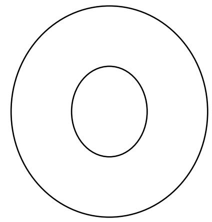
$T^2_{r_2} \quad =$ 

$T^2_{f(q)} \quad =$ 

$T^2_{r_3} \quad =$ 

$T^2_{f(r)} \quad =$ 

$T^2_{r_4} \quad =$ 

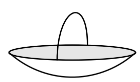
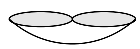
$T^2_{f(s)} = T^2_{r_5} \quad =$ 

Next we consider how, in terms of cells, to go from one stage to the next.

$T^2_{r_1} \to T^2_{f(p)}$: Glue on a 0-cell

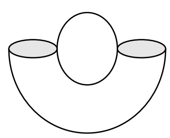
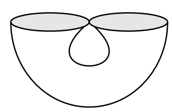
$T^2_{f(p)} \to T^2_{r_2}$: Nothing, as $T^2_{r_2}$ deformation retracts to $T^2_{f(p)}$

$T^2_{r_2} \to T^2_{f(q)}$: Glue on a 1-cell

$T^2_{r_2} \sqcup e^1 \quad =$  $\xrightarrow{\text{def. ret}}$  $= \; T^2_{f(q)}$

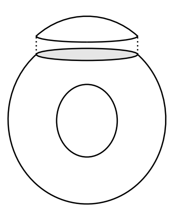
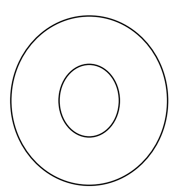
$T^2_{f(q)} \to T^2_{r_3}$: Nothing, as $T^2_{r_3}$ deformation retracts to $T^2_{f(q)}$

$T^2_{r_3} \to T^2_{f(r)}$: Glue on a 1-cell

$T^2_{r_3} \sqcup e^1 \quad =$  $\xrightarrow{\text{def. ret}}$  $= \; T^2_{f(r)}$

$T^2_{f(r)} \to T^2_{r_4}$: Nothing, as $T^2_{r_4}$ deformation retracts to $T^2_{f(r)}$

$T^2_{r_4} \to T^2_{f(s)}$: Glue on a 2-cell

$T^2_{r_4} \sqcup e^2 \quad =$  $=$  $= \; T^2_{f(s)}$

The behavior of the manifold around the four critical points is interesting to note. If F describes the manifold, and positive and negative directions are given canonically, then

$$\left.\begin{array}{lcc} \text{around } p & F \approx x^2 + y^2 & 0 \\ \text{around } q & F \approx -x^2 + y^2 & 1 \\ \text{around } r & F \approx x^2 - y^2 & 1 \\ \text{around } s & F \approx -x^2 - y^2 & 2 \end{array}\right\} \text{number of negative signs}$$

Finally, observe that if around a point F has n negative signs, then to get to that point most recently an n-cell was glued on.

2.2 Morse's lemma

2.2.1 Preliminaries

Definition 2.10. For M, N smooth manifolds, $f : M \to N$ a smooth map, and $p \in M$, the map $f_* : T_pM \to T_{f(p)}N$ is termed the pushforward of f. Given $v \in T_pM$, the map is given by

$$f_*(v)(g) = v(g \circ f) \in \mathbb{R}$$

where g is a smooth map on M passing through p in the direction of v at argument zero, or equivalently, $g'(0) = v$.

Definition 2.11. Recall that for $f : M \to \mathbb{R}$ with $p \in M$, the point p is a critical point of f_* iff $f_* : T_pM \to T_{f(p)}\mathbb{R}$ is zero. A critical point is termed non-degenerate iff given a local coordinate system $(x_1, \dots, x_n)$, the matrix representing the second derivative of f at p is non-singular, or

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right| \neq 0 \ \forall\ 1 \leqslant i, j \leqslant n$$

Further, given a critical point p of f, define the Hessian of f at p to be the bilinear form on T_pM given by

$$f_{**}(v, w) = v(\bar{w}(f))$$

where $\bar{w}$ is an extension of w to a vector field. That is, $\bar{w} : M \to \bigsqcup_{q \in M} T_qM$ such that we have a map

$$\begin{array}{rccc} \bar{w}(f) : & M & \to & \mathbb{R} \\ & p & \mapsto & w_p(f) \end{array}$$

2.2.2 Lemmae to the lemma

Lemma 2.12. The map $f{**}$ is symmetric and does not depend on the choice of $\bar{w}$. In general, given vector fields X, Y, we define $[X, Y]$ to be the vector field given by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$$

Proof: To check symmetry, let $\bar{v}$ be an extension of v to a vector field, for which

$$\begin{aligned} f_{**}(v, w) - f_{**}(w, v) &= v(\bar{w}(f)) - w(\bar{v}(f)) \\ &= \bar{v}_p(\bar{w}(f)) - \bar{w}_p(\bar{v}(f)) \\ &= [\bar{v}, \bar{w}]_p(f) \\ &= [\bar{v}, \bar{w}]_p(\mathrm{id}_{\mathbb{R}} \circ f) \\ &= f_*([\bar{v}, \bar{w}]_p)(\mathrm{id}_{\mathbb{R}}) \\ &= 0 \end{aligned}$$

To check that this is wel-defined, let $\hat{w}$ be another extension of w to a vector field, so

$$\begin{aligned} v(\bar{w}(f)) &= w(\bar{v}(f)) \\ &= \hat{w}_p(\bar{v}(f)) \\ &= v(\hat{w}(f)) && \text{(by symmetry)} \end{aligned}$$

■

Next we take a local coordinate system $(x_1, \dots, x_n)$ on M. One can check that in terms of the basis of T_pM given by $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$, the matrix representation of the bilinear form f_{**} is given as before, by $\partial^2 f / \partial x_i \partial x_j(p)$. It follows that f_{**} as a bilinear form is non-degenerate iff p is non-degenerate.

Definition 2.13. Let H be a non-degenerate bilinear form. Then the index of H is the number of negative eigenvalues in any matrix representation of H. The independence of matrix representations comes from a theorem of Sylvester.

Lemma 2.14. Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be in a convex neighborhood of $0 \in \mathbb{R}^n$, with $f(0) = 0$. Then

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n)$$

for some smooth g defined in this same neighborhood with $g(0) = \frac{\partial f}{\partial x}(0)$.

Proof: This is merely a straightforward calculation.

$$\begin{aligned} f(x_1,\dots,x_n) &= \int_0^1 \frac{df(tx_1,\dots,tx_n)}{dt}\ dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1,\dots,tx_n)\ dt \\ &= \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx_1,\dots,tx_n)\ dt}_{g_i} \end{aligned}$$

■

2.2.3 The real lemma

Lemma 2.15. [Morse]
Let p be a non-degenerate critical point for f. Then there is a local coordinate system $(y_1,\dots,y_n)$ on a neighborhood U of p with $y_i(p)=0$ for all i, and

$$f = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2$$

holds throughout U, where λ is the index of f at p.

Proof: First we show that if there is any such expression, then λ is indeed the index of p. Suppose that f is as given above. Then the matrix of f_{**} with respect to the basis mentioned above is

$$f_{**} = \begin{bmatrix} -2 & & & & & \\ & \ddots & & & 0 & \\ & & -2 & & & \\ & & & 2 & & \\ & 0 & & & \ddots & \\ & & & & & 2 \end{bmatrix}$$

As the entries -2 extend up to the λth row of the matrix, λ is indeed the index of f_{**}.

Next, assume that $f(p)=0$. Let $(x_1,\dots,x_n)$ be a local coordinate system on the manifold, so $x_1(p)=\cdots=x_n(p)=0$. By the previous lemma, $f(x_1,\dots,x_n)=\sum_{i=1}^n x_j g_j(x-1,\dots,x_n)$ where $g_j(0)=\partial f/\partial x_j(0)$. Since p is a critical point, $\partial f/\partial x_j(0)=0$, and so we may apply the lemma again, to get that

$$g_j(x_1,\dots,x_n) = \sum_{i=1}^n x_i h_{ij}(x_1,\dots,x_n) \quad\Longrightarrow\quad f(x_1,\dots,x_n) = \sum_{i,j=1}^n x_i x_j h_{ij}(x-1,\dots,x_n)$$

Let $\bar{h}_{ij} = \frac{1}{2}(h_{ij}+h_{ji})$, so $\bar{h}_{ij}=\bar{h}_{ji}$, and $f=\sum x_i x_j \bar{h}_{ij}$. Further,

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right) = \bar{h}_{ij}(0) \quad\Longrightarrow\quad |\bar{h}_{ij}(0)| \neq 0$$

That is, $\bar{h}_{ij}$ is non-singular, which shows the base case of the following claim:

For each $0 \leqslant r \leqslant n$, there are coordinates $(y_1(p),\dots,y_r(p))=0$ such that

$$f = \pm y_1^2 \pm \cdots \pm y_r^2 + \sum_{i,j \geqslant r+1} y_i y_j H_{ij}(y_1,\dots,y_n)$$

where $H_{ij}=H_{ji}$ and $H_{ij}(0)$ is non-singular.

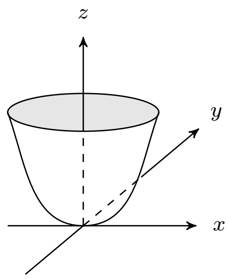
The induction step is much more involved, and will not be covered in this lecture. ■

Corollary 2.16. Non-degenerate critical points are isolated.

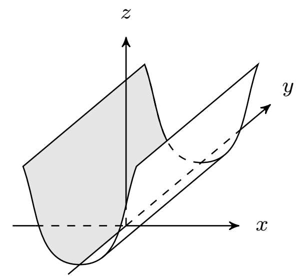
subsectionExamples from the previous lecture
The previous lecture concluded with the corallary that non-degenerate points are isolated. Let us consider a few manifolds with different types of critical points.

Example 2.17. For the function F, the only critical point, at $(0,0,0)$, is non-degenerate, and is indeed isolated. The function G, however, has critical points that are degenerate, all along the y-axis, and they are not isolated.

$F(x,y) = x^2 + y^2$:



$G(x,y) = x^2$:



3 Morse's theorem

An important theorem about compact sets needs to be proven before we can mave to the full proof of Morse's theorem.

3.1 Diffeomorphism groups

Definition 3.1. A 1-parameter group of diffeomorphisms of a manifold M is a smooth map $\varphi : \mathbb{R} \times M \to M$ such that

1. $\varphi(t,q) = \varphi_t(q) : M \to M$ is a diffeomorphism
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$

Note that group properties are indeed satisfied - we have associativity from the composition of maps, the identity from the identity isomorphism q_0, and inverses from φ_{-t} for φ_t.

Next, for a given smooth map $f : M \to \mathbb{R}$, we define a 1-dimensional vector field X:

$$X_q(t) = \frac{\partial}{\partial t} f(\varphi_t(q))|_{t=0}$$

So we associate X to a 1-parameter group of diffeomorphisms, and say that X generates φ, implying that we may recover the group only having the vector field. The succeeding lemma will show that we may do this in a unique manner.

Lemma 3.2. A compactly-supported vector field on a smooth manifold M generates a unique 1-parameter group of diffeomorphisms of M.

Proof: Let φ be a 1-parameter group of diffeomorphisms generated by X. From the definition of this vector field, we have a differential equation with a boundary condition:

$$\frac{\partial}{\partial t}\varphi_t(q) = X_{\varphi_t(q)}$$

The boundary condition is the given vector field $X_{\varphi_0(q)}$.

By the uniqueness of solutions for diffirential equations, for each $q \in M$ there exists a neighborhood $U_q \subseteq M$ containing q, and $\epsilon_q > 0$ such that the differential equation has a unique solution for all $|t| < \epsilon_q$ and $p \in U_q$. Further, as the vector field is supported on some $K \subseteq M$, we can find elements $q_i \in M$ such that

$$\bigcup_{i=1}^{\ell} U_{q_i} \supset K$$

Setting $\varphi_t(q) = q$ for all $q \notin K$, we have a unique solution on M for all t such that $|t| < \epsilon_0 = \min\{\epsilon_{q_i} \ : \ 1 \leqslant i \leqslant \ell\}$. And for $|t| \geqslant \epsilon_0$, we set

$$\varphi_t(q) = (\underbrace{\varphi_{\epsilon_0/2} \circ \cdots \circ \varphi_{\epsilon_0/2}}_{s} \circ \varphi_r)(q)$$

Here, $t = s \cdot \epsilon_0/2 + r$, and $0 \leqslant r < \epsilon_0/2$. This completes the proof. ∎

3.2 Homotopy type

Recall that for a smooth function $f : M \to \mathbb{R}$, we had $M^a = \{p \in M \ : \ f(p) \leqslant a\}$. Further, recall that a Riemannien metric $T_pM \times T_pM \to \mathbb{R}$ is a smooth, bilinear, symmetric, positive definite inner product on a tangent space.

Theorem 3.3. Let $f : M \to \mathbb{R}$ be smooth, and $a < b \in \mathbb{R}$ with no critical points of f in $[a,b]$. If $f^{-1}([a,b])$ is compact, then M^a is diffeomorphic to M^b. Moreover, M^a is a deformation retract of M^b, and hence induces a homotopy equivalence between M^a and M^b.

Proof: Fix a Riemannien metric $\langle\cdot\rangle$ on M, and let the gradient of f be the vector field grad(f) on M given by $\langle X, \mathrm{grad}(f)\rangle = X(f)$ for any vector field X on M. Heuristically, grad(f) is the directional derivative of f along X. Define a map $\rho : M \to \mathbb{R}$ by

$$\begin{array}{rcl} q \in f^{-1}([a,b]) & \mapsto & \langle \mathrm{grad}(f)_q, \mathrm{grad}(f)_q\rangle^{-1} \\ q \in K & \mapsto & 0 \end{array}$$

Here, $K \supset f^{-1}([a,b])$ is a compact neighborhood such that in $K \setminus f^{-1}([a,b])$ we may interpolate ρ to get a smooth function. Next, define a vector field X on M by $X_q = \rho(q)\mathrm{grad}(f)_q$, which satisfies the conditions of the previous lemma, hence generates a unique 1-parameter group of diffeomorphisms $\varphi_t : M \to M$. Then, for $\varphi_t(q) \in f^{-1}([a,b])$, we have

$$\begin{aligned} \frac{\partial}{\partial t} f(\varphi_t(q)) &= \langle \frac{\partial}{\partial t}\varphi_t(q), \mathrm{grad}(f)\rangle \\ &= \langle X, \mathrm{grad}(f)\rangle \\ &= X(f) \\ &= 1 \end{aligned}$$

Hence the map $t \mapsto \varphi_t(q)$ is linear for $\varphi_t(a) \in f^{-1}([a,b])$, and so $\varphi_{b-a} : M \to M$ is a dffeomorphism from M^a to M^b.

For the deformation retraction map, let it be given by

$$\begin{array}{rccl} r: & M^b \times I & \to & M^a \\ & (q,t) & \mapsto & \begin{cases} \varphi_{t(a-f(q))}(q) & f(q) \in [a,b] \\ q & f(q) \leqslant a \end{cases} \end{array}$$

Then $r_0 = \mathrm{id}_{M^b}$ and $\mathrm{Im}(r_1) = M^a$, so r is indeed a retraction from M^b to M^a. ■

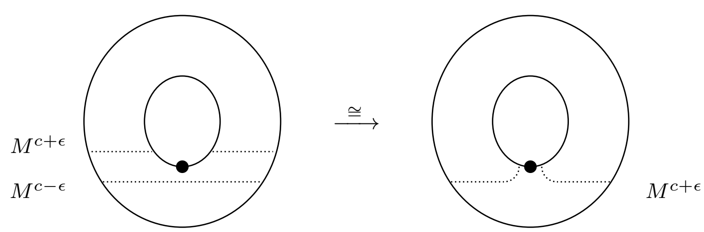
3.3 The theorem

Theorem 3.4. [Morse]
Let $f : M \to \mathbb{R}$ be smooth, and $p \in M$ a non-degenerate critical point. Suppose that $f(p) = c$, and for some $\epsilon > 0$,

1. $f^{-1}([c-\epsilon, c+\epsilon])$ is compact
2. $f^{-1}([c-\epsilon, c+\epsilon])$ contains no critical points of f except p

Then $M^{c-\epsilon} \cup e^\lambda \cong M^{c+\epsilon}$, where λ is the index of f at p.

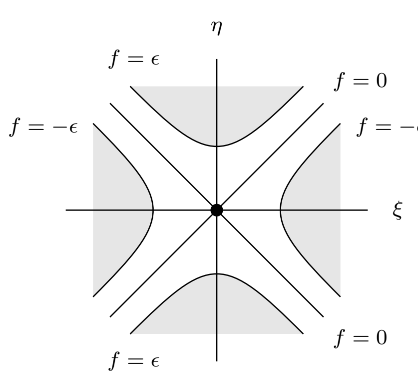
Proof: The proof will proceed by pushing $M^{c+\epsilon}$ down to $M^{c-\epsilon}$ except at p:



Without loss of generality, we assume that $f(p) = 0$. Choose local coordinates $u^1, \ldots, u^n$ in a small neighborhood U of p such that

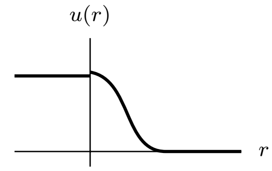
$$f(x) = \underbrace{-(u^1)^2 - \cdots - (u^\lambda)^2}_{-\xi} + \underbrace{(u^{\lambda+1})^2 + \cdots + (u^n)^2}_{\eta}$$

Choose an $\epsilon > 0$ such that conditions 1 and 2 above hold, and so that $B(0, \sqrt{2}\epsilon) \subseteq U$. Let $e^\lambda = \{(u^1, \ldots, u^n) \in U \ : \ \xi \leqslant \epsilon, \eta = 0\}$. This gives the following situation:



Consider the following function:

$$u(r) = \begin{cases} 0 & r \geqslant 2\epsilon \\ \epsilon\left(\frac{1}{1000000}+1\right)\exp\left(\frac{1}{2\epsilon}\right)\exp\left(\frac{-1}{2\epsilon - r}\right) & r \in [0, 2\epsilon) \\ \epsilon & \text{else} \end{cases} \quad =$$



Note that $u(0) > \epsilon$, and that u vanishes on $[2\pi, \infty)$. On $[0, 2\pi]$, we also have that $u' \in (-1, 0)$. Next, we define a function $F : M \to \mathbb{R}$ given by $F = f - u(\xi + 2\eta)$. We claim that $F^{-1}((-\infty, \epsilon]) = M^\epsilon$. To see this, note that outside the hyperbola $f = \epsilon$ described above, $F = f$. Inside the hyperbola,

$$F \leqslant f = -\xi + \eta \leqslant \frac{\epsilon}{2} + \eta \leqslant \epsilon$$

This follows as $\xi + 2\eta \leqslant \epsilon$, and proves the claim. Next we claim that F and f have the same critical points. To see this, observe that outside $B(0, \sqrt{2}\epsilon)$, $F = f$. Inside $B(0, \sqrt{2}\epsilon) \subseteq U$, we have that

$$F = \xi + \eta - u(\xi + 2\eta) \quad \Longrightarrow \quad dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta$$

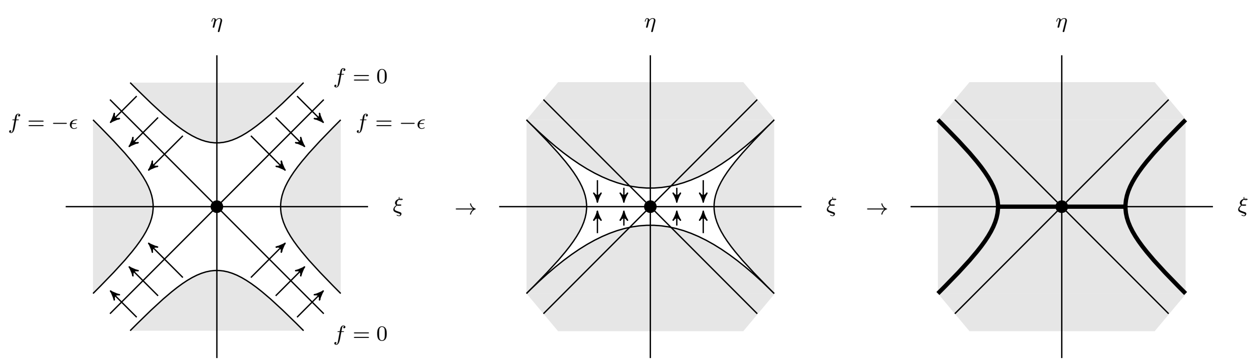
Observe that $d\xi$ and $d\eta$ are of the form $\sum_i c_i u^i du^i$, for some constants c_i. Hence $d\xi$ and $d\eta$ only vanish at the origin (that is, at p). Further observe that

$$\frac{\partial F}{\partial \xi} = -i - u'(\xi + 2\eta) < 0 \qquad\qquad \frac{\partial F}{\partial \eta} = 1 - 2u'(\xi + 2\eta) \geqslant 1$$

Hence dF only vanishes at 0, and the claim is proven. The final claim to be proven is that $F^{-1}((-\infty, -\epsilon])$ is a deform retract of M^ϵ. First we note that as $F \leqslant f$, if $F(x) \geqslant -\epsilon$, then $f(x) \geqslant F(x) \geqslant -\epsilon$, so $F^{-1}([-\epsilon, \infty)) \subseteq f^{-1}([-\epsilon, \infty))$. By the first claim above, $F^{-1}([-\epsilon, \epsilon]) \subseteq f^{-1}([-\epsilon, \epsilon])$, and as $f^{-1}([-\epsilon, \epsilon])$ is compact and $F^{-1}([-\epsilon, \epsilon])$ is closed, it follows that $F^{-1}([-\epsilon, \epsilon])$ is compact.

Since $f^{-1}([-\epsilon, \epsilon])$ has no critical points, $F^{-1}([-\epsilon, \epsilon])$ has no critical points by the claim above, except possibly at 0. And at p, $F(p) = f(p) - u(0) = -u(0) < \epsilon$, so $p \notin F^{-1}([-\epsilon, \epsilon])$. Now apply the previous theorem to get a deformation retract from M^ϵ to $F^{-1}((-\infty, -\epsilon))$. This completes the final claim.

We proceed as in the pictures below, by pushing along the indicated edges.



Although these pictures are heuristic, it is possible to construct formal retractions as above, by dividing up the disappearing region into two cases, and formulating two appropriate functions. This completes the proof. ■

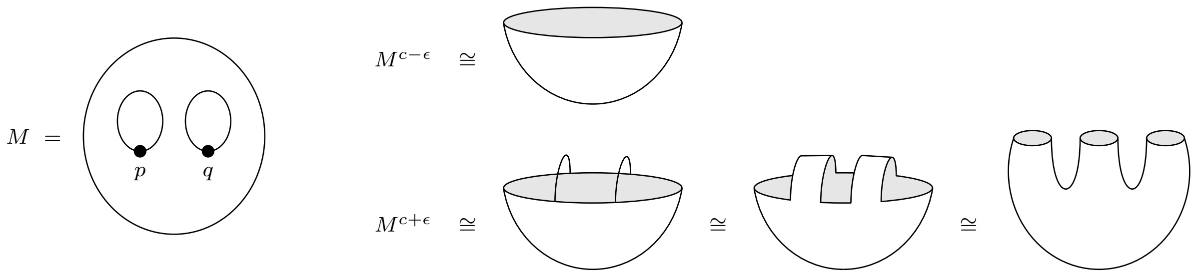
4 The homotopy type of a manifold

4.1 Motivation

Previously we saw that for $f : M \to \mathbb{R}$ smooth and $p \in M$ a critical point with $f(p) = c$, if $f^{-1}([c - \epsilon, c + \epsilon])$ is compact and continuous with no other critical points except p, then $M^{c+\epsilon} \cong M^{c-\epsilon} \cup e^\lambda$, for λ the index of p.

Proposition 4.1. Suppose $p_1, \ldots, p_k \in f^{-1}(c)$ are all non-degenerate critical points. Then $M^{c+\epsilon} \cong M^{c-\epsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$, where λ_k is the index of p_k.

For example, we may have the following situation:



4.2 Preliminary lemmae

Lemma 4.2. Let $\varphi_0 \cong \varphi_1 : \mathbb{S}^{\lambda-1} \to X$ via $H(x,t) = \varphi_t(x)$. Then the identity on X extends to a homotopy equivalence

$$k : X \bigcup_{\varphi_0} e^\lambda \to X \bigcup_{\varphi_1} e^\lambda$$

Proof: Define the maps k and its inverse ℓ as follows:

$$\begin{aligned} k(x) &= x \quad \forall\, x \in X & \qquad \ell(x) &= x \quad \forall\, x \in X \\ k(tu) &= \begin{cases} 2tu & t \in [0,1/2] \\ \varphi_{2-2t}(u) & t \in [1/2,1] \end{cases} & \qquad \ell(tu) &= \begin{cases} 2tu & t \in [0,1/2] \\ \varphi_{2t-1}(u) & t \in [1/2,1] \end{cases} \end{aligned}$$

Here $u \in \mathbb{S}^{\lambda-1}$. The effect, heuristically, is simply stretching parts of e^λ along X to where φ_1 glues e^λ. ∎

Lemma 4.3. Let $\varphi : \mathbb{S}^{\lambda-1} \to X$ be a gluing map. Let $\delta : X \to Y$ be a homotopy equivalence. Then f extends to a homotopy equivalence

$$F : X \bigcup_{\varphi} e^\lambda \to Y \bigcup_{\varphi} e^\lambda$$

Proof: Let the maps $f : X \to Y$ and $g : Y \to X$ define the homotopy equivalence between X and Y, so $gf = \mathrm{id}_X$ via h_t. Define the following two maps:

$$\begin{aligned} F : X \bigcup_{\varphi} e^\lambda \to Y \bigcup_{f\varphi} e^\lambda \qquad &\text{by} \qquad F|_X = f, \quad F|_{e^\lambda} = \mathrm{id} \\ G : X \bigcup_{f\varphi} e^\lambda \to Y \bigcup_{gf\varphi} e^\lambda \qquad &\text{by} \qquad G|_Y = g, \quad G|_{e^\lambda} = \mathrm{id} \end{aligned}$$

First note that $gf\varphi : \mathbb{S}^{\lambda-1} \to X$, which follows if we let $H : \mathbb{S}^{\lambda-1} \times I \to X$ be given by $H(y,t) = h_t(\varphi(y))$. By the above lemma, there exists a function $k : X \cup_{gf\varphi} e^\lambda \to X \cup_\varphi e^\lambda$. We next claim that if F has a left and right homotopy inverse (call them L and R), then F is a homotopy equivalence, and L (or R) is a 2-sided inverse.

To prove the claim, note that $LF \approx \mathrm{id}$ and $FR \approx \mathrm{id}$. Hence

$$L \approx L(FR) = (LF)R \approx \mathbb{R} \quad \Longrightarrow \quad LF = RF$$

Hence with this map k, we have $kGF : X \cup_\varphi e^\lambda \to X \cup_\varphi e^\lambda \approx \mathrm{id}$. Hence F is a homotopy equivalence. To see that G has a left homotopy inverse, observe that

$$kGF \approx \mathrm{id} \quad \Longrightarrow \quad GFk \approx \mathrm{id} \quad \Longrightarrow \quad FkG \approx \mathrm{id}$$

Hence F has a right homotopy inverse, and the proof is complete. ∎

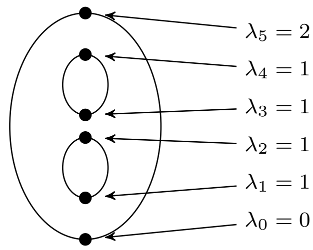
4.3 Implications of the lemmae

Theorem 4.4. Let $f : M \to \mathbb{R}$ be smooth with no degenerate critical points. If each M^a is compact, then M has the homotopy type of a CW complex with one cell of dimension λ for each index λ of a critical point.

Proof: This follows by applying the second lemma as many times as there are critical points at each a. ∎

Example 4.5. This fact extends to the Euler characteristic of a space X. For example,

$X =$



$$\chi(X) = \sum_i (-1)^{\lambda_i} = 1 - 1 - 1 - 1 - 1 + 1 = -2$$

4.4 Cellular approximation

Definition 4.6. Let X be a CW complex. The <u>n-skeleton</u> of X, denoted X^n, is the union of all cells of X of dimension at most n.

Definition 4.7. A map $f : X \to Y$ between CW complexes is termed <u>cellular</u> if $f(X^n) \subseteq Y^n$ for all n.

Theorem 4.8. [Cellular approximation theorem]
Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map.

Theorem 4.9. Let M be a compact n-manifold and $f : M \to \mathbb{R}$ smooth with exactly 2 critical points, both non-degenerate. Then M is homeomorphic to $\mathbb{S}^n$.

<u>Proof:</u> The compact points must be the maximum and minimum of f. Without loss of generality, we may assume that $f(p) = 0$ and $f(q) = 1$, for p, q the critical points. Then for small enough ϵ, the sets $f^{-1}([0, \epsilon])$ and $f^{-1}([1-\epsilon, 1])$ are closed n-cells. The first set M^ϵ is homeomorphic to $M^{1-\epsilon} = f^{-1}([0, 1-\epsilon])$, and so

$$M = f^{-1}([0, 1-\epsilon]) \cup f^{-1}([1-\epsilon, 1]) \approx e^n \underset{\substack{\text{glue on}\\ \text{boundary}}}{\bigsqcup} e^n = \mathbb{S}^n$$

■

Note that the above theorem holds also if the critical points are degenerate. Moreover, we must have homeomorphism, not diffeomorphism, as Milnor, in 1956, constructed an exotic 7-sphere that was homeomorphic, but not diffeomorphic to $\mathbb{S}^7$.

Example 4.10. Let $M = \mathbb{C}P^n = \{[z_0, \ldots, z_n]\}$, and pick distinct $c_0, c_1, \ldots, c_n \in \mathbb{R}$. Define $f : M \to \mathbb{R}$ by

$$[z_0, z_1, \ldots, z_n] \mapsto \frac{\sum c_i |z_i|^2}{\sum |z_i|^2} = \sum c_i |z_i|^2$$

This follows as we may assume that $\sum |z_i|^2 = 1$ by normalisation. Let $U_0 = \{z_0 \neq 0\}$, with $|z_0| \frac{z_j}{z_0} = x_j + iy_j$. Here we use $x_1, y_1, \ldots, x_n, y_n$ as local coordinates around $[1, 0, \ldots, 0]$. Note that U_0 is diffeomorphic to a closed $2n$-ball in $\mathbb{R}^{2n}$, and

$$\sum_{j=1}^n |x_j|^2 + |y_j|^2 = \sum_{j=1}^n |z_j|^2 < \sum_{j=0}^n |z_j|^2 = 1$$

Hence all the points are within the unit ball. Furthermore,

$$\begin{aligned}
f &= c_0 |z_0|^2 + \sum_{j=1}^n c_j |z_j|^2 \\
&= c_0(1 - |z_1|^2 - \cdots - |z_n|^2) + \sum_{j=1}^n c_j |z_j|^2 \\
&= c_0 + \sum_{j=1}^n (c_j - c_0)(x_j^2 + y_j^2)
\end{aligned}$$

And this holds on all of U_0. Next we take the partials, and note that they vanish when $x_j = 0$ or $y_j = 0$, respectively.

$$\frac{\partial f}{\partial x_j} = 2x_j(c_j - c_0) \qquad\qquad \frac{\partial f}{\partial y_j} = 2y_j(c_j - c_0)$$

Note the only critical point is $x_1 = y_1 = \cdots = x_n = y_n = 0$, which corresponds to $p_0 = [1, 0, \ldots, 0]$. Then

$$\text{Hessian}(f)_{p_0} = \begin{bmatrix} 2c_1 - 2c_0 & 0 & & & & & \\ 0 & 2c_1 - 2c_0 & & & & & \\ & & 2c_2 - 2c_0 & 0 & & & \\ & & 0 & 2c_2 - 2c_0 & & & \\ & & & & \ddots & & \\ & & & & & 2c_n - 2c_0 & 0 \\ & & & & & 0 & 2c_n - 2c_0 \end{bmatrix}$$

Zeroes are in the cells not filled in. Next,

$$\text{index}(f, p_0) = (\#\text{ of eigenvalues of Hessian}(f)_{p_0}) = 2|\{c_k \ : \ c_k < c_0\}|$$

Similarly, $p_1 = [0, 1, 0, \ldots, 0]$, and p_i for all $i = 1, \ldots, n$ are all the other critical points. Iterating $\text{index}(f, p_i)$ over $i = 0, \ldots, n$, we will get every even index between 0 and $2n$ exactly once, and hence

$$M \approx e^0 \sqcup e^2 \sqcup e^4 \sqcup \cdots \sqcup e^{2n}$$

As, for example, $\mathbb{C}P^2 = \mathbb{C}^2 \sqcup \mathbb{C}P^1 = \mathbb{C}2 \sqcup \mathbb{C}^0 = e^4 \sqcup e^2 \sqcup e^0$, this is a CW decomposition of $\mathbb{C}P^n$. ■

Example 4.11. Suppose that $f : M \to \mathbb{R}$ has exactly 3 non-degenerate critical points and is orientable. Then the indeces of cells are 0, n, $n/2$, and M has homotopy type of $e^{n/2} \sqcup e^n$.

5 The nature of Morse functions

So far we have been studying Morse functions, but the question arises, do Morse functions always exist? The answer is yes, as for $M \hookrightarrow \mathbb{R}^n$, with M $n/2 = k$-dimensional, we fix a $p \in \mathbb{R}^n$, and define

$$\begin{array}{rccc} L_p : & M & \to & \mathbb{R} \\ & q & \mapsto & \|p - q\| \end{array}$$

This may not work directly; we must put some conditions on p. The function L_p will be Morse iff p is not a focal point of M, a trem which we will define later.

Remark 5.1. We need some structures for the following analysis. First, let $M \subseteq \mathbb{R}^n$ be a manifold of dimension $k < n$. Define an n-dimensional manifold

$$N = \{(q, v) \ : \ q \in M, v \perp M \text{ at } q\} \subseteq M \times \mathbb{R}^n$$

Next, define a function $E : N \to \mathbb{R}^n$ by $E(q, v) = q + v$, where addition is vector addition.

5.1 Focal points

Definition 5.2. A point $e \in \mathbb{R}^n$ is termed a focal point of (M, q) with multiplicity μ if $e = q + v$ for $(q, v) \in N$, and $\text{Jac}(E)_{(q,v)}$ has nullity $\mu > 0$. Then we say that e is a focal point of a manifold M if it is a focal point of (M, q) for some $q \in M$.

Heuristically, focal points are where nearby normals intersect.

Theorem 5.3. [Sard]
If M_1 and M_2 are n-dimensional manifolds and $f : M_1 \to M_2$ is C^1, then the image of the set of critical points of f has Lebesgue measure zero in M_2.

If p is a critical point, then equivalently $\det(\text{Jac}(f)_p) = 0$.

Corollary 5.4. For almost all $p \in \mathbb{R}^n$, p is not a focal point.

For the next part, we let $u_1, \ldots, u_k$ be local coordinates for $M \subseteq \mathbb{R}^n$ around q. The inclusion map induces smooth functions $X_1(u_1, \ldots, u_k), \ldots, X_n(u_1, \ldots, u_k)$. We let $\vec{x} = (x_1, \ldots, x_n)$.

Definition 5.5. The first fundamental form is the matrix $(g_{ij}) = \left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j}\right)$. Without loss of generality, we may choose coordinates such that $(g_{ij}) = I_k$.

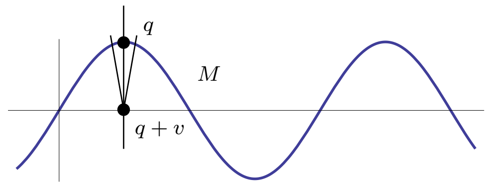
Definition 5.6. The second fundamental form is the matrix (ℓ_{ij}) defined through $\left(\frac{\partial^2 \vec{x}}{\partial u_i u_j}\right)$, which may be expressed as an element of TM plus a perpendicular vector (ℓ_{ij}). That is,

$$\left(\frac{\partial^2 \vec{x}}{\partial u_i u_j}\right) = \underbrace{A_{ij}}_{\text{tan.}} + \underbrace{\ell_{ij}}_{\text{perp.}}$$

Let $\vec{v}$ be any unit vector normal to M at q. Then $(\vec{v} \cdot \ell_{ij})$ is a matrix of real-valued functions, and the eigenvalues $k_1, \ldots, k_k$ of $(\vec{v} \cdot \ell_{ij})$ are termed the principal curvatures of M at q.

Example 5.7. Consider the simple manifold $M = (x, \sin(x))$, for which

$$\begin{aligned} TM &= c(1, \cos(x)) \quad c \in \mathbb{R} \\ NM &= ((x, \sin(x), \lambda(-\cos(x), 1)) \end{aligned}$$



Here, TM is the tangent bundle and NM is the normal bundle. We may also make the following calculations:

$$\mathrm{Im}(E) = (x - \lambda\cos(x), \sin(x) + \lambda)$$

$$\begin{aligned} \mathrm{Jac}(E) &= \begin{bmatrix} 1+\lambda\sin(x) & -\cos(x) \\ \cos(x) & 1 \end{bmatrix} \\ \det(\mathrm{Jac}(E)) &= 0 \implies \lambda = \tfrac{\cos^2(x)+1}{-\sin(x)} \end{aligned}$$

Note that at $(\pi/2, 1)$ (this is p) we have a focal point $(\pi/2, 0)$ (this is $q + v$). We may also calculate the fundamental forms:

$$\begin{aligned} (g_{ij}) &= ((1, \cos(x)) \cdot (1, \cos(x))) = (1 + \cos^2(x)) \\ (v\ell_{ij}) &= -\sin(x)/(1 + \cos^2(x)) \end{aligned}$$

5.2 Implications on a manifold

Lemma 5.8. The focal points of (M, q) along $\ell = \vec{q} + t\vec{v}$ are $\vec{q} + k_i^{-1}\vec{v}$, for $1 \leqslant i \leqslant k$ and $k \neq 0$.Hence the are at most $k(= \dim(M))$ focal points along ℓ.

Proof: We may choose coordinates so that at $\vec{q} + t\vec{v}$,

$$\mathrm{rank}\,(\mathrm{Jac}(E)_{q,tv}) = \mathrm{rank}\left(\begin{bmatrix} g_{ij} - t\vec{v}\ell_{ij} & * \\ 0 & I_{kn} \end{bmatrix}\right)$$

Hence $\vec{q} + t\vec{v}$ is a focal point with multiplicity μ if and only if $(g_{ij} - t\vec{v}\ell_{ij})$ is singular with nullity μ. But if $g_{ij} = I$, then $I_k - t\vec{v}\ell_{ij}$ is singuler if and only if $1/t$ is an eigenvalue of $(\vec{v}\ell_{ij})$. Hence $1/t = k_i$, and $t = 1/k_i$ if $k_i \neq 0$. ∎

Next we fix $p \in \mathbb{R}^n$, and let $L_p : M \to \mathbb{R}$ be given as $L_p(\vec{x}) = \|\vec{x} - \vec{p}\|^2 = \vec{x}\vec{x} - 2\vec{x}\vec{p} + \vec{p}\vec{p}$. Then due to the following calculations, L_p has a critical point at q if and only if $\vec{q} - \vec{p}$ is normal to M at q (say $\vec{p} = \vec{x} \pm t\vec{v}$).

$$\begin{aligned} \frac{\partial L_p}{\partial u_i} &= 2\frac{\partial \vec{x}}{\partial u_i}(\vec{x} - \vec{p}) \\ \frac{\partial^2 L_p}{\partial u_i u_j} &= 2\left(\frac{\partial \vec{x}}{\partial u_i} \cdot \frac{\partial \vec{x}}{\partial u_j} + \frac{\partial^2 \vec{x}}{\partial u_i u_j}(\vec{x} - \vec{p})\right) = 2(g_{ij} - t\vec{v}\ell_{ij}) \end{aligned}$$

This implies the following lemma.

Lemma 5.9. A point $\vec{q} \in M$ is a degenerate critical point of L_p if and only if p is a focal point of M_q with nullity$(\vec{q})$ = multiplicity$(\vec{p})$.

Theorem 5.10. For almost all $p \in \mathbb{R}^n$, $L_p : M \to \mathbb{R}$ has no degenerate critical points.

Proof: Follows from the lemma and the corollary to Sard's theorem. ∎

Theorem 5.11. On a manifold M, there exists a differentiable function with no degenerate critical points such that M^a is compact for all a.

Proof: We apply L_p, and note that

$$M^a = L_p^{-1}((-\infty, a]) = L_p^{-1}([0, a]) = \{a \in M \;:\; \|p - q\| < a\}$$

∎

Corollary 5.12. Any differentiable manifold M has the homotopy type of a CW complex.

Theorem 5.13. The index of L_p at a non-degenerate critical point is equal to the number of focal points of (M, q) which lie on the line segment from q to p.

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