Compact course notes

Physics 234, Fall 2010

Quantum Mechanics

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Contents

1 Fundamentals

1.1 The photoelectric effect

$$
\begin{aligned}
E_p &= hf \\
hf &= E_{k_{max}} + W_o \\
hf_o &= W_o \\
E_{k_{max}} &= q_e V_s \\
p &= \frac{h}{\lambda} \\
\lambda' - \lambda &= \lambda_c (1 - \cos(\theta)) \\
\Delta\lambda &= \frac{h}{m_e c}(1 - \cos(\theta)) \\
E &= \frac{p}{2m}
\end{aligned}
$$

E_p : Energy of a photon
h : Planck's constant
f : Frequency
$E_{k_{max}}$: Maximum kinetic energy
W_o : Work function
f_o : Threshold frequency
q_e : Charge of an electron
V_s : Stopping voltage
p : Momentum
λ_c : Compton wavelength
$\Delta\lambda$: Compton shift
m_e : Mass of target electron

· Classical physics cannot explain the following:

1. Stopping potential is independent of intensity. Classically, high intensity should impart more E_k to electrons.

2. Existence of cut-off frequency and independence of intensity. Classically, intensity governs energy, not frequency.

3. Additional experimental observation: Zero time lag between incident light and photoemission, regardless of low intensity. Classically, at low intensity, photoemission will not occur until electron has absorbed sufficient energy.

Bragg scattering

$$n\lambda = 2d\sin(\theta)$$

n : Order number
d : Planar separation
θ : Angle between incident and scattered electrons

1.2 Historical background

2 The Stern-Gerlach experiment

2.1 Background

$$\vec{\mu}_s = g\frac{q}{2m}\vec{s}$$

$\vec{\mu}_s$: Intrinsic magnetic dipole momentum

2.2 Bra-ket notation

State vectors in quantum mechanics, which represent arbitrary states of atoms, are denoted by $|\ \rangle$ with an appropriate label inside, depending on the state.

For $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ composed in the z-basis:

$$\begin{aligned}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = |\ \rangle &= |{\uparrow_z}\rangle\alpha + |{\downarrow_z}\rangle\beta\\ &= |{\uparrow_x}\rangle\frac{\alpha+\beta}{\sqrt{2}} + |{\downarrow_x}\rangle\frac{\alpha-\beta}{\sqrt{2}}\\ &= |{\uparrow_y}\rangle\frac{\alpha-i\beta}{\sqrt{2}} + |{\downarrow_y}\rangle\frac{\alpha+i\beta}{\sqrt{2}}\end{aligned}$$

The adjoint of bra is ket, and the adjoint of ket is bra. That is,
$|\ \rangle^\dagger = \langle\ |$ and $\langle\ |^\dagger = |\ \rangle$

2.3 Pauli operators

· The three main experiments are given by the Pauli operators:

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{with} \quad \left.\begin{matrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{matrix}\right\} = \left\{\begin{matrix}|{\uparrow_x}\rangle\langle{\uparrow_x}| + |{\downarrow_x}\rangle\langle{\downarrow_x}|\\ |{\uparrow_y}\rangle\langle{\uparrow_y}| + |{\downarrow_y}\rangle\langle{\downarrow_y}|\\ |{\uparrow_z}\rangle\langle{\uparrow_z}| + |{\downarrow_z}\rangle\langle{\downarrow_z}|\end{matrix}\right.$$

Above, while the sum of ket-bras gives the operator, each ket-bra represents the + or − operator in each direction. Any experiment may be represented by $a_o I + \vec{a}\cdot\vec{\sigma}$.

· Any beam of atoms may be passed through a flipper, which reverses the magnetic moment of the input. The operator that rotates the beam through an angle φ is given by:

$$R_y(\varphi) = \begin{pmatrix}\cos\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right)\\ -\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)\end{pmatrix} \quad \text{with} \quad R_y(\varphi_1)\cdot R_y(\varphi_2) = R_y(\varphi_1+\varphi_2)$$

The flipper turns magnetic moments clockwise, given by $R_y(\pi)$, while the antiflipper rotates the moments counter clockwise, so is represented by $R_y(-\pi)$.

2.4 Expectation values

If a source emits a state $|\ \rangle$ and the SGE is calibrated in $\vec{e}$ direction, then
probability for atoms to be deflected up: $p_+ = |\langle{\uparrow_e}|\ \rangle|^2$
probability for atoms to be deflected down: $p_- = |\langle{\downarrow_e}|\ \rangle|^2$

For an arbitrary operator Λ, its expectation value is given by $\langle\Lambda\rangle$.
For the three σ operators, we define this to be $\langle\sigma_i\rangle = (+1)P(|{\uparrow_i}\rangle) + (-1)P(|{\downarrow_i}\rangle)$

2.5 Statistical operator

Given a source of atoms that is not of a pure state (i.e. mixed atom states), employ the statistical operator to find the outcome with such a beam. The operator is given by:

$$\rho = \tfrac{1}{2}(I + \vec{s}\cdot\vec{\sigma}) = \sum_i p_i|i\rangle\langle i| \quad \text{with} \quad \sum_i p_i = 1 \quad \text{and} \quad \vec{s} = \begin{pmatrix}\langle\sigma_x\rangle\\ \langle\sigma_y\rangle\\ \langle\sigma_z\rangle\end{pmatrix}$$

The above described vector $\vec{s}$ is termed the Bloch vector, and has the property that $||\vec{s}|| \leqslant 1$.

In the generalized version, if given a state characterized by ρ and a Stern-Gerlach experiment in the direction $\vec{e}$, represented by $\vec{e}\cdot\vec{\sigma}$, the expectation value is then given by

$$\langle \vec{e}\cdot\vec{\sigma}\rangle = Tr\{\vec{e}\cdot\vec{\sigma}\rho\} = \vec{e}\cdot\vec{s}$$

With respect to the above situation, $\vec{e} = \frac{\vec{s}}{|\vec{s}|}$

3 Linear algebra

3.1 Fundamentals

Definition 3.1.1. A vector space over $\mathbb{C}$ consists of vectors $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \ldots$ and scalars $a, b, c, \ldots, \in \mathbb{C}$.

Definition 3.1.2. The dual vector to $c|\alpha\rangle$ is $c^*|\alpha\rangle$, where c^* represents the complex conjugate of c.
Note that $c^*c = |c|^2$

Definition 3.1.3. The inner product of $|\alpha\rangle$ and $|\beta\rangle$ is $\langle\alpha|\beta\rangle \in \mathbb{C}$.

Theorem 3.1.4. [Properties of the Inner Product]

1. $\langle\alpha|\beta\rangle = \langle\alpha|\beta\rangle^*$
2. $\langle\alpha|\alpha\rangle \geqslant$ with equality $\iff$ $\langle\alpha| = |\alpha\rangle = 0$
3. $\langle\alpha|\Big(b|\beta\rangle + c|\gamma\rangle\Big) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle$

3.2 Orthogonality

$+z$ atoms: $\begin{pmatrix}1\\0\end{pmatrix}$	$+x$ atoms: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$+y$ atoms: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$
$-z$ atoms: $\begin{pmatrix}0\\1\end{pmatrix}$	$-x$ atoms: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$-y$ atoms: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

· Note that $+e$ and $-e$ atoms for any e are mutually exclusive. Therefore orthogonality exists between $+e$ and $-e$ atoms. However, there is not orthogonality between $+e_i$ and $+e_j$ atoms for $i \neq j$.

3.3 Notation

· The *transpose:* $\widetilde{\begin{pmatrix}\alpha\\\beta\end{pmatrix}} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}^T = \begin{pmatrix}\alpha & \beta\end{pmatrix}$

· The *(complex) conjugate:* $\begin{pmatrix}\alpha\\\beta\end{pmatrix}^* = \begin{pmatrix}\alpha^*\\\beta^*\end{pmatrix}$

· The *adjoint* or *(complex) conjugate transpose:* $\begin{pmatrix}\alpha\\\beta\end{pmatrix}^\dagger = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$

· The *commutator:* $[A, B] = AB - BA$

· The *anticommutator:* $\{A, B\} = AB + BA$

3.4 Identities

$$\begin{aligned}
e^{ix} + e^{ix} &= 2\cos(x) + 2i\sin(x)\\
e^{ix} + e^{-ix} &= 2\cos(x)\\
e^{ix} - e^{ix} &= 0\\
e^{ix} - e^{-ix} &= 2i\sin(x)
\end{aligned}$$