Compact course notes

PURE MATH 351, FALL 2012

Real Analysis

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Contents

1 Set Theory

1.1 Definitions

Definition 1.1.1. Given a set X, the power set of X is defined to be $\mathbf{P}(X) = \{A \mid A \subset X\}$.

Definition 1.1.2. Given sets A, B define the symmetric difference of them to be $A \triangle B = (A \cup B) \setminus (A \cap B) = (A \cap B^C) \cup (A^C \cap B)$

Proposition 1.1.3. [De Morgan's Laws]
Let $\{A_\alpha\}_{\alpha \in I} \subset \mathbf{P}(X)$. Then

1. $\left(\bigcup_{\alpha \in I} A_\alpha \right)^C = \bigcap_{\alpha \in I} A_\alpha^C$

2. $\left(\bigcap_{\alpha \in I} A_\alpha \right)^C = \bigcup_{\alpha \in I} A_\alpha^C$

Proof: **1.**

$$\begin{aligned} x \in \left(\bigcup_{\alpha \in I} A_\alpha \right)^C &\iff x \notin \bigcup_{\alpha \in I} A_\alpha \\ &\iff x \notin A_\alpha \ \forall \ \alpha \in I \\ &\iff x \in A_\alpha^C \ \forall \alpha \in I \\ &\iff x \in \bigcap_{\alpha \in I} A_\alpha^C \end{aligned}$$

2. Similar to above, by replacing A with A^C. ■

Definition 1.1.4. Given $A_1, \ldots, A_n \subset X$, define their product to be

$$A_1 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \ldots, a_n) \mid a_i \in A_i \ \forall \ i\}$$

Definition 1.1.5. The size of a set A, denoted by $|A|$, is the number elements A has.

If $|A_i| = m_i$, then $\left| \prod_{i=1}^{n} A_i \right| = \prod_{i=1}^{n} m_i$.

1.2 Problems arising

Proposition 1.2.1. Suppose that $I = \emptyset$. If the expression $\{A_\alpha\}_{\alpha \in I}$ is meaningful, then clearly $\bigcup_{\alpha \in I} A_\alpha = \emptyset$. But then by de Morgan's laws, $\bigcap_{\alpha \in I} A_\alpha = X$.

Axiom 1.2.2. [Axiom of choice]
If $I \neq \emptyset$ and $A_\alpha \neq \emptyset$ for all $\alpha \in I$, then $\prod_{\alpha \in I} A_\alpha \neq \emptyset$.

Axiom 1.2.3. [Equivalent to AoC]
If A is non-empty, there exists a function $f : \mathbf{P}(A) \setminus \{\emptyset\} \to A$ such that $f(A) \in A$.

1.3 Relations

Definition 1.3.1. A relation R on sets X, Y is a subset of $X \times Y$. In general, we write $xRy \iff (x, y) \in R$ for $x \in X$ and $y \in Y$. Interpreted as a set, R is termed the graph of the relation.

If $X = Y$, then R is termed a relation on X.

Definition 1.3.2. Let R be a relation on $X \ni x, y, z$. Then:
1. R is reflexive iff for all $x \in X$, xRx
2. R is symmetric iff $xRy \iff yRx$
3. R is anti-symmetric iff xRy and yRx implies $x = y$
4. R is transitive iff xRy and yRz implies xRz

Example 1.3.3.
1. Let R be a relation on $\mathbb{R}$ and $xRy \iff x \leqslant y$. This is a poset.
2. Let R be a relation on $\mathbf{P}(X)$ for X any set and $ARB \iff A \subset B$. This is a poset.
In this case we say $\subset$ orders $\mathbf{P}(X)$ by inclusion.
3. Let R be a relation on $\mathbf{P}(X)$ for X any set, and $ARB \iff A \supset B$. This is a poset.
In this case we say $\supset$ orders $\mathbf{P}(X)$ by containment.

Definition 1.3.4. A partial order on a set X is a relation $\preccurlyeq$ on X that is reflexive, anti-symmetric, and transitive. As an ordered pair, $(X, \preccurlyeq)$ is termed a poset.

X is a poset off for all $x, y \in X$, either $x \preccurlyeq y$ ar $y \preccurlyeq x$.

Definition 1.3.5. A chain is a subset of $(X, \preccurlyeq)$ that is totally ordered, i.e. that has $x \preccurlyeq y$ or $y \preccurlyeq x$ for all $x, y \in X$.

Definition 1.3.6. Let $(X, \preccurlyeq)$ be a poset with $A \subset X$. Then:
1a. We say that $\alpha \in A$ is an upper bound of A iff $x \preccurlyeq \alpha$ for all $x \in A$
1b. We say that α is the least upper bound of A iff α is an upper bound of A and for all other upper bounds β of A, $\alpha \preccurlyeq \beta$.
2a. We say that $\alpha \in A$ is an lower bound of A iff $x \succcurlyeq \alpha$ for all $x \in A$
2b. We say that α is the greatest lower bound of A iff α is a lower bound of A and for all other lower bounds β of A, $\alpha \succcurlyeq \beta$.
3. We say that A is bounded if it has a lower bound and an upper bound.

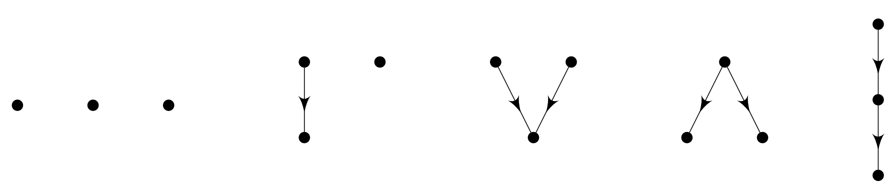
Axiom 1.3.7. [Least upper bound principle]
If $A \subset \mathbb{R}$ is bounded above and is non-empty, then there exists a least upper bound for A.

Definition 1.3.8. Let $(X, \leqslant)$ be a poset. Then $x \in X$ is termed maximal if whenever $x \leqslant y$, $x = y$.

Example 1.3.9.
1. For $\mathbb{R}$, there is no maximal element
2. For $(\mathbf{P}(X), \subset)$, X is the maximal element
3. For $(\mathbf{P}(X), \supset)$, $\emptyset$ is the maximal element

Remark 1.3.10. Note that finite posets may be represented by finite digraphs. As such, two elements are termed comparable if there is a dipath joining them. We assume that $x \lneq y$ iff there is a path from y to x.

Example 1.3.11. Let $X = \{x, y, z\}$ have distinct elements. There are 5 basic posets.



There are 2^9 relations on X, and of them, 19 are posets.

Theorem 1.3.12. If $(X, \leqslant)$ is a finite non-empty poset, then $(X, \leqslant)$ has a maximal element.

Proof: Induction on the number of elements in X. ■

Axiom 1.3.13. [Zorn's Lemma]
Let $(X, \leqslant)$ be a non-empty, partially ordered set. Assume that every chain $C \subset X$ has an upper bound. Then $(X, \leqslant)$ has a maximal element.

Zorn's lemma is logically equivalent to the axiom of choice.

Example 1.3.14. Let $(V, +)$ be a non-zero vector space. Let $\mathcal{L} = \{L \subset V \mid L \text{ is linearly independent}\}$. Then a basis for V is a maximal element of $\mathcal{L}$, given the ordering $\subset$.

Theorem 1.3.15. Every non-zero vector space has a basis.

Proof: Let $\mathcal{L} = \{L \subset V \mid L \text{ is linearly independent}\} \subset \mathbf{P}(V)$.
Then $\mathcal{L} \neq \emptyset$, as for $v \in V$ nonzero, $\{v\} \in \mathcal{L}$.
Let $L^* = \bigcup_{\alpha \in I} L_\alpha$.
We claim that L^* is linearly independent, so $L^* \in \mathcal{L}$ and L^* is an upper bound.
Let $\{v_1, \ldots, v_n\}$ be distinct elements of L^* with $a_1 v1 + \cdots + a_n v_n = 0$.
For each $i = 1, 2, \ldots, n$, $v_i \in L_{\alpha_i}$ for some $\alpha_i \in I$, and we may assume that $L_{\alpha_1} \subset L_{\alpha_2} \subset \cdots \subset L_{\alpha_n}$.
Hence $\{v_1, \ldots, v_n\} \subset L_{\alpha_n}$ so that $a_1 = a_2 = \cdots = a_n = 0$.
Since every chain has an upper bound, Zorn's lemma gives us a maximal element. ■

Definition 1.3.16. A poset $(X, \leqslant)$ is termed well-ordered if every non-empty subset has a least element.

Well-ordered sets are totally ordered.

Example 1.3.17.

1. $\mathbb{N}$ with the usual order is well-ordered
2. $(\mathbb{Q}, \leqslant)$ is not well-ordered, as $\{r \in \mathbb{Q} \mid r > \sqrt{2}\}$ has no least element

Proposition 1.3.18. The set $\mathbb{Q}$ can be injected into the set $\mathbb{N}$. Consider:

$$\varphi : \quad \mathbb{Q} = \left\{ \tfrac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{N}, \gcd(n, m) = 1 \right\} \to \mathbb{N}$$

$$\varphi\left(\tfrac{n}{m}\right) = \begin{cases} 1 & \text{if } n = 0 \\ 2^n 3^m & \text{if } \frac{n}{m} > 0 \\ 2^{-n} 5^m & \text{if } \frac{n}{m} < 0 \end{cases}$$

The fundamental theorem of arithmetic gives us that φ is injective.

Proposition 1.3.19. The set $\mathbb{Q}$ is well-ordered.

Proof: Using the above function and the relation $r \preccurlyeq q$ iff $\varphi(r) \leqslant \varphi(q)$ in the usual order on $\mathbb{N}$. ■

Axiom 1.3.20. [Well-ordering principle]
Every non-empty set can be well-ordered.

Theorem 1.3.21. The following axioms are logically equivalent:

1. The axiom of choice
2. Zorn's lemma
3. The well-ordering principle

1.4 Equivalence relations and cardinaltiy

Definition 1.4.1. A relation $\sim$ on a set X is termed an equivalence relation iff it is:

1. reflexive
2. symmetric
3. transitive

Definition 1.4.2. Given an equivalence relation $\sim$ on X, the equivalence class of an element $x \in X$ is defined as

$$[x] = \{y \in X \mid x \sim y\}$$

The following properties hold for all $x, y \in X$:

1. $x \in [x]$
2. either $[x] = [y]$ or $[x] \cap [y] = \emptyset$

Definition 1.4.3. Given a non-empty set X, a partition on X is a collection $\{A_\alpha\}_{\alpha \in I}$ of pairwise disjoint nonempty subsets of X such that

$$X = \dot{\bigcup_{\alpha \in I}} A_\alpha$$

Remark 1.4.4.

1. Any equivalence relation $\sim$ partitions X
2. Any partition $\{A_\alpha\}_{\alpha \in I}$ of X defines an equivalence relation on X.

Example 1.4.5. Given a set X, let $\sim$ be an equivalence relation on $\mathbf{P}(X)$ by $A \sim B$ iff there exists a bijection $f : A \to B$. Then A is equivalent to B, or $A = B$ iff $|A| = |B|$. Heuristically, $A = B$ iff both have the same number of elements.

Definition 1.4.6. A set X is termed finite if either $X = \emptyset$ or $X \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. If $X = \emptyset$, then X is said to have cardinality 0. If $X \sim \{1, 2, \ldots, n\}$, then X is said to have cardinality n. If X is not finite, then it is termed infinite.

Theorem 1.4.7. If X is finite, then X cannot be equivalent to a proper subset of itself.

Proof: This is clearly false for $X = \emptyset$, so we will not consider that case.
Assume that $X = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
Let P_n be the statement "the set $\{1, 2, \ldots, n\}$ is not equivalent to a proper subset of itself".
Base case: The case P_1 clearly holds.
Inductive step: Suppose that P_k holds for $k \in \mathbb{N}$.
Also suppose that there exists a bijective function $f : \{1, 2, \ldots, k, k+1\} \to S$ for $S \subsetneq \{1, 2, \ldots, k, k+1\}$.

Case 1: $k + 1 \notin S$
Let $S' = S \setminus \{f(k + 1)\} \subsetneq \{1, 2, \ldots, k\}$.
Then $f|_{\{1,2,\ldots,k\}}$ is bijective from $\{1, 2, \ldots, k\}$ to $S' \subsetneq \{1, 2, \ldots, k\}$.
This contradicts P_k.

Case 2: $k + 1 \in S$ and $f(k + 1) = k + 1$
Then $f|_{\{1,2,\ldots,k\}}$ has range $S' = S \setminus \{k + 1\} \subsetneq \{1, 2, \ldots, k\}$.
Since f is bijective on $\{1, 2, \ldots, k\}$, we have that $\{1, 2, \ldots, k\} \nsim S$.

Case 3: $k + 1 \in S$ and $f(k + 1) \neq k + 1$
Then $f(j_0) = k + 1$ for some $j_0 \in \{1, 2, \ldots, k\}$.
Let $g : \{1, 2, \ldots, k + 1\} \to S$ be defined by

$$g(j) = \begin{cases} k+1 & \text{if } j = k+1 \\ f(k+1) & \text{if } j = j_0 \\ f(j) & \text{if } j \in \{1, 2, \ldots, k\} \text{ with } j \neq j_0 \end{cases}$$

Then g is a bijection on S, which by the above case, is impossible.

Now suppose that $X \sim \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ and $X \sim S$ for S a proper subset of X.
Then there exists a bijective function $f : X \to \{1, 2, \dots, n\}$.
Then $S \sim f(S) \subsetneq \{1, 2, \dots, n\}$.
But then $\{1, 2, \dots, n\} \sim X \sim S \sim f(S)$. ■

Proposition 1.4.8. If X is infinite, then there exists a subset $X \subset X$ with $S \sim \mathbb{N}$.

<u>*Proof:*</u> Since X is non-empty, there is a choice function f on $\mathbf{P}(X) \setminus \{\emptyset\}$.
Let $x_1 = f(X)$, $X_2 = f(X \setminus \{x_1\})$, and proceed recursively with $x_{n+1} = f(X \setminus \{x_1, \dots, x_n\})$.
This gives $S = \{x_1, \dots, x_n, \dots\}$. ■

Theorem 1.4.9. A set X is infinite if and only if it is equivalent to one of its proper subsets.

<u>*Proof:*</u> We know that if X is finite, then it is not equivalent to any one of its proper subsets.
Then suppose that X is infinite.
Choose $S = \{x_1, x_2, \dots, x_n\}$ as in the previous proposition.
Define $f : X \to X \setminus \{x_1\}$ by

$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \in S \\ x & \text{if } x \notin X \setminus S \end{cases}$$

This proves the theorem. ■

Definition 1.4.10. A set X is termed <u>countable</u> iff it is either finite or $X \sim \mathbb{N}$.

If $X \sim \mathbb{N}$, then $|X| = \aleph_0$.

Theorem 1.4.11. [Cantor, Shroeder, Bernstein]
Let $A_2 \subset A_1 \subset A_0$. If $A_2 \sim A_0$, then $A_1 \sim A_0$.

<u>*Proof:*</u> Note that there exists a bijection $f : A_0 \to A_2$, so $f(A_0) = A_2$.
Let $A_3 = f(A_1)$, $A_4 = f(A_2), \dots, A_n = f(A_{n-2}), \dots$
Then $A_{n+2} \sim A_n$ via f, as well as $A_{n+2} \setminus A_n \sim A_{n+2} \setminus A_{n+3}$ also via f.
We may decompose A_0 and A_1 as follows:

$$A_0 = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \cdots \cup \bigcap_{i=0}^{\infty} A_i$$

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup (A_3 \setminus A_4) \cup \cdots \cup \bigcap_{i=1}^{\infty} A_i$$

Identification between sets is made if they are equal and otherwise through $g : A_0 \to A_1$:

$$g(x) = \begin{cases} f(x) & \text{if } x \in (A_{2k} \setminus A_{2k+1}) \\ x & \text{if } x \in (A_{2k+1} \setminus A_{2k+2}) \\ x & \text{if } x \in \bigcap_{i=0}^{\infty} A_i \end{cases}$$

Since g is a bijection, $A_1 \sim A_0$. ■

Corollary 1.4.12. If $A_1 \subset A_0$ and $B_1 \subset B_0$ with $B_1 \sim A_0$ and $A_1 \sim B_0$, then $A_0 \sim B_0$.

Proof: Let $f : A_0 \to B_1$ and $g : B_0 \to A_1$ be bijective.
Define $A_2 \subset A_1 \subset A_0$ by $A_2 = g \circ f(A_0) = g(B_1)$.
Therefore $A_2 \sim A_0$.
By CSB, we have that $A_1 \sim A_0$ and so $A_0 \sim B_0$. ■

Example 1.4.13. These are some examples of equivalent sets.

· $\mathbb{Q} \sim \mathbb{N}$

· $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

This is given by two injective functions, $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

$$\begin{aligned} f(n) &= (n, n) \\ g((n, m)) &= 2^n 3^m \end{aligned}$$

Since both are injective, CSB says that the sets are equivalent.

· $\prod_{i=1}^{n} \mathbb{N} \sim \mathbb{N}$ for $n \in \mathbb{N}$

Theorem 1.4.14. The product of finitely many countable sets is countable.

Theorem 1.4.15. Let $\{X_n\}_{n=1}^{\infty}$ be a countable collection of countable sets. Then $X = \bigcup_{n=1}^{\infty} X_n$ is countable.

Proof: Recall that if S is countable with $T \subset S$, then T is also countable by CSB.
Let

$$\begin{aligned} E_1 &= X_1 \\ E_2 &= X_2 \setminus X_1 \\ E_3 &= X_3 \setminus (X_1 \cup X_2) \\ E_4 &= X_4 \setminus (X_1 \cup X_2 \cup X_3) \\ &\vdots \\ E_n &= X_n \setminus \bigcup_{i=1}^{n-1} X_i \end{aligned}$$

Then $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} X_n$ and $\{E_1, E_2, \dots, E_n\}$ is a pairwise disjoint sequence of countable sets.
For each E_n, write $E_n = \{x_{n,1}, x_{n,2}, \dots\}$ possibly terminating.
Let $f : \bigcup_{n=1}^{\infty} E_n \to \mathbb{N}$ by $f(x_{i,j}) = 2^i 3^j$.
Since f is injective, the theorem is proven. ■

Definition 1.4.16. A set is termed <u>uncountable</u> if it is not countable.

Proposition 1.4.17. The set $(0, 1) \subset \mathbb{R}$ is not countable.

Proof: Suppose that $(0, 1)$ is countable.
Then $(0, 1) = \{\alpha_1, \alpha_2, \dots\}$ for each $\alpha_j = 0.b_{j1}b_{j2}\cdots = \sum_{i=1}^{\infty} \frac{b_{ji}}{10^i}$ for each $b_{ji} \in \{0, 1, 2, \dots, 9\}$.

Consider the following expansion:

$$\begin{aligned}
\alpha_1 &= 0.b_{11}b_{12}b_{13}\ldots \\
\alpha_2 &= 0.b_{21}b_{22}b_{23}\ldots \\
\alpha_3 &= 0.b_{31}b_{32}b_{33}\ldots \\
&\vdots \\
\alpha_n &= 0.b_{n1}b_{n2}b_{n3}\ldots \\
&\vdots
\end{aligned}$$

Now define an element $\alpha = 0.b_1b_2b_3\ldots$ by

$$b_n = \begin{cases} 1 & \text{if } b_{nn} \neq 0 \\ 2 & \text{else} \end{cases}$$

Clearly $\alpha \in (0,1)$, but there is also clearly no $i \in \mathbb{N}$ such that $\alpha = \alpha_i$.
Therefore α is not in our enumeration, and so $(0,1)$ is not countable. ∎

Remark 1.4.18.
1. For any $a < b \in \mathbb{R}$, we have that $(0,1) \sim (a,b) \sim \mathbb{R}$, and $(0,1) \sim \mathbb{R}$ via $f(x) = \arctan\left(\pi x - \frac{\pi}{2}\right)$.
2. $|\mathbb{R}| = c$, which is the first uncountable ordinal.

Axiom 1.4.19. [Continuum hypothesis]
For X any set, if $\aleph_0 \preccurlyeq |X| \preccurlyeq c$, then either $|X| = c$ or $|X| = \aleph_0$.

Definition 1.4.20. For sets W, V, let $h : W \to V$ be a function. Denote the pullback of h by $h^{-1} : \mathbf{P}(V) \to \mathbf{P}(W)$, with $h^{-1}(B) = \{w \in W \mid h(w) \in B\}$ for any $B \subset V$.

Proposition 1.4.21. Assume that there exists a surjective function $g : Y \to X$. Then there exists an injective function $f : X \to Y$.

Proof: Let $g : Y \to X$ be surjective.
For each $x_0 \in X$, $g^{-1}(\{x_0\}) \neq \emptyset$, as g is surjective.
By the axiom of choice, there if a choice function h on $\mathbf{P}(Y) \setminus \{\emptyset\}$.
Define $f(x_0) = h(g^{-1}(\{x_0\})) = y_0 \in Y$.
Since g is a function, $f : X \to Y$ is injective. ∎

Corollary 1.4.22. Given nonempty sets X, Y, the following are equivalent:
1. $|X| \preccurlyeq |Y|$
2. There exists an injective function $f : X \to Y$
3. There exists a surjective function $g : Y \to X$

Theorem 1.4.23. [Computability theorem]
Given any sets X, Y, either $|X| \preccurlyeq |Y|$ or $|Y| \preccurlyeq |X|$.

Proof: We may assume that X, Y are nonempty.
Define $S = \{(A, B, f) \mid A \subset X, B \subset Y, f : A \to B \text{ is bijective}\}$.
We may order S by $\preccurlyeq$, with $(A_1, B_1, f_1) \preccurlyeq (A_2, B_2, f_2)$ iff $A_1 \subset A_2$, $B_1 \subset B_2$, and $f_a|_{A_1} = f_1$.
Let $C = \{(A_\alpha, B_\alpha, f_\alpha)\}_{\alpha \in I}$ be a chain in S.
Let $A = \bigcup_{\alpha \in I} A_\alpha$, $B = \bigcup_{\alpha \in I}$, and $f : A \to B$ given by $f(x) = f_\alpha(x)$ if $x \in A_\alpha$.

First it must be shown that f is well defined.
Assume that $x \in A_\alpha$, $x \in A_\beta$.
WLOG we may assume $A_\alpha \subset A_\beta$.
Then $f(x) = f_\alpha(x) = f_\beta(x)$.

Thus f is well-defined.

Now we must show that f is injective.
Let $x_1 \neq x_2 \in A_\alpha \subset A_\beta$ so $x_1, x_2 \in A_\beta$.
Now $f_\alpha(x_1) = f_\beta(x_1) \neq f_\beta(x_2) = f(x_2)$.

Finally it must be shown that f is surjective.
Let $w \in B$.
Then $w \in B_\alpha$ for some α.
So here exists $x \in A_\alpha$ with $f_\alpha(x) = w$.
Then $x \in A$ and $f(x) = w$.
Therefore (A, B, f) is an upper bound of C.
By Zorn's lemma, S has a maximal element (A_0, B_0, f_0).
If $A_0 = X$, then $|X| \preccurlyeq |Y|$.
Assume $A_0 \neq X$.
If $B_0 = Y$, then $|Y| \preccurlyeq |X|$.
If $B_0 \neq Y$, then choose $x_0 \in X \setminus A_0$ with $y_0 \in Y \setminus B_0$.
Define $f_1 : A_0 \cup \{x_0\} \to B_0 \cup \{y_0\}$ by

$$f_1(x) = \begin{cases} f_0(x) & \text{if } x \in A_0 \\ f(x_0) = y_0 & \text{if } x = x_0 \end{cases}$$

Then $(A_0, B_0, f_0) \prec (A_1, B_1, f_1)$.
This is a contradiction, and hence the last situation cannot hold. ■

1.5 Cardinal arithmetic

Definition 1.5.1. Given two sets X, Y with $X \cap Y = \emptyset$, define $|X| + |Y| := |X \cup Y|$.

Example 1.5.2. Consider $\mathbb{N} = \{1, 3, 5, \dots\} \cup \{2, 4, 6, \dots\}$, and so $|\mathbb{N}| = |\mathbb{N}| + |\mathbb{N}| = \aleph_0 + \aleph_0$.

Theorem 1.5.3. Given two sets X, Y with X infinite,
1. $|X| + |X| = |X|$
2. $|X| + |Y| = \max\{|X|, |Y|\}$

Definition 1.5.4. Given two nonempty sets X, Y, define $|X||Y| := |X \times Y|$.

This means that $\aleph_0 \cdot \aleph_0 = \aleph_0$ and $c \cdot c = c$.

Theorem 1.5.5. Given two nonempty sets X, Y with X infinite,
1. $|X||X| = |X|$
2. $|X||Y| = \max\{|X|, |Y|\}$

Definition 1.5.6. Given two nonempty sets X, Y, define $|Y|^{|X|} := |Y^X| = |\prod_{x \in X} Y| = |\{f : X \to Y\}|$.

Proposition 1.5.7. For any set X, $|\mathbf{P}(X)| = 2^{|X|}$.

Proof: Given any $A \subset X$, define $\chi_A : X \to \{0, 1\}$ by $\chi_A(x) = \left\{ \begin{smallmatrix} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{smallmatrix} \right.$.
Then $\mathbf{P}(X) \sim \{f : X \to \{0, 1\}\}$ via $A \iff \chi_A$. ■

Theorem 1.5.8. [Russell]
For any set X, $|X| \prec 2^{|X|}$.

Proof: Let $f : X \to \mathbf{P}(X)$ be injective.
Suppose that f is onto.
Let $A \subset X$ be defined by $A = \{x \in X \mid x \notin f(x)\}$.

Then there exists x_0 with $f(x_0) = A$.
But if $x_0 \in A$, then $x_0 \notin f(x_0) = A$.
And if $x_0 \in A$, then $x_0 \in f(x_0) = A$.
This is a contradiction.
Hence no such f injective exists. ■

Remark 1.5.9. Given a set A, the number of relations on A is equal to $|\mathbf{P}(A \times A)|$.

The number of equivalence relations on A is equal to the number of partitions of A.

2 Metric spaces

Definition 2.0.1. Given a set X, a function $d : X \times X \to \mathbb{R}$ is termed a metric iff for all $x, y, z \in X$:

1. $d(x, y) \geqslant 0$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) + d(y, z) \geqslant d(x, z)$

Example 2.0.2. These are some examples of metrics.

1. $X = \mathbb{R}$ and $d(x, y) = |x - y|$
2. $X =$ any set and $d(x, y) = \left\{ \begin{smallmatrix} 0 & \text{if } x=y \\ 1 & \text{else} \end{smallmatrix} \right.$, the discrete metric

Definition 2.0.3. Given a set X and a metric d on X, the pair (X, d) is termed a metric space.

2.1 Normed linear spaces

Definition 2.1.1. Let V be a vector space. A function $\|\cdot\| : V \to \mathbb{R}$ is termed a norm iff for all $v, w \in V$ and $\alpha \in \mathbb{R}$:

1. $\|v\| \geqslant 0$ and $\|v\| = 0 \iff v = 0$
2. $|\alpha v\| = |\alpha|\|v\|$
3. $\|v + w\| \leqslant \|v\| + \|w\|$

Given a vector space V and a norm $\|\cdot\|$ on V, the pair $(V, \|\cdot\|)$ is termed a normed linear space.

Definition 2.1.2. Let $(V, \|\cdot\|)$ be a normed linear space. If $d(x, y) = \|x - y\|$, then d is a metric on V, and d is termed the metric induced by $\|\cdot\|$.

Example 2.1.3. These are some examples of norms.

1. the standard norm: $\|(x_1, \ldots, x_n)\|_1 = |x_1| + \cdots + |x_n|$
2. the Euclidean norm: $\|(x_1, \ldots, x_n)\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$
3. the p-norm: $\|(x_1, \ldots, x_n)\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ for $1 < p < \infty$
4. the sup norm: $\|(x_1, \ldots, x_n)\|_\infty = \max_i \{|x_i|\}$

Then we have that $\|x\|_\infty \leqslant \|x\|_p \leqslant \|x\|_1 \leqslant n\|x\|_\infty$ for $x \in \mathbb{R}^n$ and for $1 < p < \infty$.

Lemma 2.1.4. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (or $q(p - 1) = p$), where p, q is a conjugate pair. Then for any $\alpha, \beta > 0$,

$$\alpha\beta \leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Theorem 2.1.5. [Holder's inequality]
Let $a, b \in \mathbb{R}^n$ with $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$. Then

$$\|ab\|_1 \leqslant \|a\|_p \|b\|_q$$

Proof: We may assume that a, b are nonzero.

Note that the result holds iff it holds for αa and βb for nonzero scalars α, β.

Then we may assume that $\left(\sum_{i=1}^n |a_i|^p\right)^{1/p} = 1$ and $\left(\sum_{i=1}^n |b_i|^q\right)^{1/q} = 1$.

Now $|a_i b_i| \leqslant \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q}$ for all $i = 1, \ldots, n$, so

$$\sum_{i=1}^n |a_i b_i| \leqslant \frac{\sum_{i=1}^n |a_i|^p}{p} + \frac{\sum_{i=1}^n |b_i|^q}{q} = 1$$

Replacing 1 with the norms gives the result. ■

Theorem 2.1.6. [Minkowski's inequality]
Let $a, b \in \mathbb{R}^n$ with $1 < p < \infty$. Then

$$\|a + b\|_p \leqslant \|a\|_p + \|b\|_p$$

Proof: Let p, q be a conjugate pair.

Note that

$$\sum_{i=1}^n |a_i + b_i|^p = \sum_{i=1}^n |a_i||a_i + b_i|^{p-1} + \sum_{i=1}^n |b_i||a_i + b_i|^{p-1}$$

Then by Holder, we have that

$$\begin{aligned}\sum_{i=1}^n |a_i||a_i + b_i|^{p-1} &\leqslant \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^{(p-1)q}\right)^{1/q} \\ &= \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/q} \\ \sum_{i=1}^n |b_i||a_i + b_i|^{p-1} &\leqslant \left(\sum_{i=1}^n |b_i|^p\right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/q}\end{aligned}$$

The original equation then becomes

$$\|a + b\|_p = \left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1-1/p-1/q} \leqslant \|a\|_p + \|b\|_q$$

This completes the proof. ■

Definition 2.1.7. The following are all spaces of infinite sequences.

1. $\ell_1(\mathbb{N}) = \ell_1 = \{\{x_n\} \mid x_n \in \mathbb{R}, \sum_{i=1}^\infty |x_n| < \infty\}$
2. $\ell_p(\mathbb{N}) = \ell_p = \{\{x_n\} \mid x_n \in \mathbb{R}, \sum_{i=1}^\infty |x_n|^p < \infty\}$
3. $\ell_\infty(\mathbb{N}) = \ell_\infty = \{\{x_n\} \mid x_n \in \mathbb{R}, \max_i\{|x_i|\} < \infty\}$

By checking that $\|\cdot\|_p$ for each respective p is a norm, it may be shown that $(\ell_p, \|\cdot\|_p)$ is a normed linear space, for $a \leqslant p \leqslant \infty$.

Remark 2.1.8. We have the following sequence of inclusions, for all $1 < p_2 < p_2 < \infty$:

$$\ell_1 \subsetneq \ell_{p_1} \subsetneq \ell_{p_2} \subsetneq \ell_\infty$$

Proposition 2.1.9. Let $\{x_n\} \in \ell_p$ and $\{y_n\} \in \ell_q$ with p, q a conjugate pair. Then $\sum_{n=1}^\infty x_n y_n$ converges absolutely with $\|\{x_n y_n\}\|_1 \leqslant \|\{x_n\}\|_p + \|\{y_n\}\|_q$.

Example 2.1.10. Let $X = C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$. Then for

$$\|f\|_\infty = \sup_{x\in[a,b]} \{|f(x)|\} = \max_{x\in[a,b]} \{|f(x)|\}$$
$$|f+g| \leqslant |f| + |g| \leqslant \|f\|_\infty + \|g\|_\infty$$

the space $(C[a,b], \|\cdot\|_\infty)$ is a normed linear space. We may define other norms on $C[a,b]$ by:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$$

so then $(C[a,b], \|\cdot\|_p)$ will be a norm for all $1 \leqslant p < \infty$.

Example 2.1.11. Given normed linear spaces $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$, let

$$\mathcal{L}(X,Y) = \{T : X \to Y \mid T \text{ is linear}\}$$
$$\|T\|_\infty = \sup\{\|Tx\|_Y \mid \|x\|_X \leqslant 1 \ \forall\ x \in X\}$$
$$B(X,Y) = \{T \in \mathcal{L}(X,Y) \mid T \text{ is bounded}\}$$

Then the space $(B(X,Y), \|\cdot\|_\infty)$ is a normed linear space.

2.2 The topology of metric spaces

Definition 2.2.1. Let (X,d) be a metric space with $x \in X$ and $\epsilon > 0$. Define
· the open ball of radius ϵ centered at x: $B(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$
· the closed ball of radius ϵ centered at x: $B[x,\epsilon] = \{y \in X \mid d(x,y) \leqslant \epsilon\}$
· an open set $U \subset X$ has for all $y \in U$ some $\epsilon_y > 0$ such that $B(y, \epsilon_y) \subset U$
· a closed set $V \subset X$ has $X \setminus V$ open

Theorem 2.2.2. Let (X,d) be a metric space. Then
1. $X, \emptyset$ are open
2. if $\{U_\alpha\}_{\alpha\in I}$ is a collection of open sets in X, then $\bigcup_{\alpha\in I} U_\alpha$ is open in X
3. if $\{U_1, \ldots, U_n\}$ is a finite collection of open sets in X, then $\bigcap_{i=1}^n U_i$ is open in X

Proof: **1.** This is clear.

2. Let $x \in \bigcup_{\alpha\in I} U_\alpha$.
Then there exists $\alpha_0 \in I$ with $x \in U_{\alpha_0}$, so there is $\epsilon > 0$ with $B(x,\epsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha\in I} U_\alpha$.

3. Let $x_i n \bigcap_{i=1}^n U_i$.
For each i, $x_0 \in U_i$, so there is $\epsilon = \min_i\{\epsilon_i\}$, for $B(x_0, \epsilon_i) \subset U_i$ for all i.
Hence $B(x_0, \epsilon) \subset \bigcap_{i=1}^n U_i$. ■

Theorem 2.2.3. Let (X,d) be a metric space. Then
1. $X, \emptyset$ are closed
2. if $\{F_\alpha\}_{\alpha\in I}$ is a collection of open sets in X, then $\bigcap_{\alpha\in I} F_\alpha$ is closed in X
3. if $\{F_1, \ldots, F_n\}$ is a finite collection of open sets in X, then $\bigcup_{i=1}^n F_i$ is closed in X

Definition 2.2.4. Given a set X, a topology on X is a set $\tau \subset \mathbf{P}(X)$ such that
1. $X, \emptyset \in \tau$
2. if $\{U_\alpha\}_{\alpha\in I} \subset \tau$, then $\bigcup_{\alpha\in I} U_\alpha \in \tau$
3. if $\{U_1, \ldots, U_n\} \subset \tau$, then $\bigcap_{i=1}^n U_i \in \tau$

The pair (X, τ) is termed a topological space, with elements of τ termed τ-open, or simply open sets.

Proposition 2.2.5. Let $X \ni x$ be a space with $\epsilon > 0$. Then

1. The open ball $B(x, \epsilon)$ is open.
2. $U \subset X$ is open iff it is the union of open balls.
3. The closed ball $B[x, \epsilon]$ is closed.
4. The set $\{x_0\}$ is closed.

Definition 2.2.6. Let $A \subset (X, d)$. Define a metric $d_A : A \times A \to \mathbb{R}$ by $d_A(x, y) = d(x, y)$ iff $x, y \in A$.

Definition 2.2.7. Given $A \subset (X, d)$, define a topology τ_A on A by $W \in \tau_A$ iff $W = A \cap U$ for some $U \in \tau_d$. Then τ_A is termed the relative topology on A induced by τ_d.

Proposition 2.2.8. $\tau_A = \tau_{d_A}$

Proof: Let $W \in \tau_{d_A}$, so for each $x \in W$ there exists $\epsilon_x > 0$ so that $W = \bigcup_{x \in W} B_{d_A}(x, \epsilon_x)$.
Then for $U = \bigcup_{x \in W} B_d(x, \epsilon_x)$, we have that U is open in X and $W = U \cap A$.
Hence $W \in \tau_A$.

Let $W \in \tau_A$ and $x \in W$.
Then there exists $U \subset X$ so that $W = A \cap U$.
Then as $x \in U$, there exists $\epsilon > 0$ with

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\} \subset U$$

Then we also have that

$$B_{d_A}(x, \epsilon) = \{y \in A \mid d_A(x, y) < \epsilon\} \subset W$$

Therefore $W \in \tau_{d_A}$.

The result follows. ■

2.3 Closures, interiors, and boundaries

Definition 2.3.1. Let $A \subset (X, d)$. Define

· closure of A: $\overline{A} = \bigcap\{F \subset X \mid A \subset F, F \text{ is closed}\}$
· interior of A: $\mathsf{int}(A) = A^\circ = \bigcup\{U \subset X \mid U \subset A, U \text{ is open}\}$
· neighborhood of x: N with $x \in N^\circ$

Note that $\bar{A}$ is the smallest closed set containing A and A° is the largest open set contained in A.

Remark 2.3.2.

· $A^\circ \subset A \subset \overline{A}$
· A is closed iff $A = \overline{A}$
· A is open iff $A = A^\circ$

Definition 2.3.3. Given $A \subset (X, d)$, a point $x \in A$ is termed a boundary point of A iff every neighborhood N of x is such that $N \cap A \neq \emptyset$ and $N \cap A^c \neq \emptyset$. Equivalently, $x \in A$ is a boundary point iff

$$B(x, \epsilon) \cap A \neq \emptyset, \quad B(x, \epsilon) \cap A^c \neq \emptyset \quad \forall\ \epsilon > 0$$

A point $x \in A$ is termed a limit point (or cluster point) of A iff for all $\epsilon > 0$ $B(x, \epsilon) \cap A$ contains a point different from x.

The set of all boundary points of A is denoted $\mathsf{bdy}(A)$.
The set of all limit points of A is denoted $\mathsf{Lim}(A)$.

Proposition 2.3.4. Let (X, d) be a metric space and $A \subset X$. Then

1. $\overline{A} = A \cup \mathsf{bdy}(A)$
2. A is closed iff $\mathsf{bdy}(A) \subset A$

Proposition 2.3.5. Let (X, d) be a metric space and $A \subset X$. Then

1. $\overline{A} = A \cup \mathsf{Lim}(A)$
2. A is closed iff $\mathsf{Lim}(A) \subset A$

Definition 2.3.6. Let (X, d) be a metric space and $A \subset X$. Then A is termed dense in X iff $\overline{A} = X$. In general, if $A \subset B \subset X$, then A is termed dense in B iff $B \subset \overline{A}$.

Another way to characterize denseness is to say $A \subset X$ is dense in X iff every open ball $B(z, \epsilon) \subset X$ intersects A.

Example 2.3.7. $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$ are dense in $\mathbb{R}$.

Proposition 2.3.8.

1. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
2. $\mathsf{int}(A \cap B) = \mathsf{int}(A) \cap \mathsf{int}(B)$

Proposition 2.3.9.

1. $(\overline{A})^c = \mathsf{int}(A^c)$
2. $\mathsf{bdy}(A) = \overline{A} \setminus \mathsf{int}(A)$

Definition 2.3.10. Given a metric space (X, d), the space is termed separable iff X has a countable dense set. Otherwise the space is termed non-separable.

Example 2.3.11.

· $\mathbb{R}$ is separable
· $\mathbb{R}^n$ is separable
· $\mathbb{R}^\infty$ is not separable

· $(\ell_1, \|\cdot\|_1)$ is separable
· $(\ell_\infty, \|\cdot\|_\infty)$ is not separable

It is a direct consequence of the definition of a separable metric space that any separable metric space has cardinality at most $\mathfrak{c}$.

2.4 Sequences in metric spaces

Definition 2.4.1. For (X, d) a metric space, $\{x_n\} \subset X$ converges to $x_0 \in X$ iff for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n \geqslant N_0$, $d(x_0, x) < \epsilon$. This relationship is expressed as $\lim_{n\to\infty}[x_n] = x_0$ or $x_n \to x_0$. If such an x_0 does not exist, then $\{x_n\}$ is said to diverge.

Proposition 2.4.2. Given a sequence $\{x_n\}$ in a metric space (X, d),

$$\lim_{n\to\infty}[x_n] = x_0 \text{ and } \lim_{n\to\infty}[x_n] = y_0 \implies x_0 = y_0$$

Proof: Suppose that $x_0 \neq y_0$, or equivalently, that $d(x_0, y_0) = \epsilon > 0$.

Then we can find $N_0 \in \mathbb{N}$ such that if $n \geqslant N_0$, then $d(x_n, x_0) < \frac{\epsilon}{2}$ and $d(x_n, y_0) < \frac{\epsilon}{2}$.

This implies that

$$d(x_0, y_0) \leqslant d(x_0, x_n) + d(y_0, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

As ϵ was arbitrary, $x_0 = y_0$. ■

Remark 2.4.3. A sequence $x_n \to x_0$ iff $y_n \to x_0$ for all subsequences $\{y_n\}$ of $\{x_n\}$.

Definition 2.4.4. Given a sequence $\{x_n\}$, a point x_0 is termed a limit point of $\{x_n\}$ iff there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with $x_{n_k} \xrightarrow{k\to\infty} x_0$.

Thes set of all limit points of a sequence x_n is denoted by $\mathsf{lim}^*(\{x_n\})$.

Remark 2.4.5. Note that $\mathsf{lim}^*(\{x_n\}) \neq \mathsf{Lim}(\{x_n\})$. For example, for $x_n = (-1)^{n-1}$, we have $\mathsf{lim}^*(\{x_n\}) = \{-1, 1\}$ and $\mathsf{Lim}(\{x_n\}) = \emptyset$.

Theorem 2.4.6. Let (X, d) be a metric space and $A \subset X$. Then

1. $x_0 \in \mathsf{bdy}(A)$ iff there exists $\{x_n\} \subset A$ and $\{y_n\} \subset A^c$ with $x_n, y_n \to x_0$
2. $x_0 \in \mathsf{Lim}(A)$ iff there exists $\{x_n\} \subset A \setminus \{x_0\}$ with $x_n \to x_0$
3. A is closed iff $\{x_n\} \subset A$ and $x_n \to x_0$ implies $x_0 \in A$

Proof: **1.** Suppose that $x_0 \in \mathsf{bdy}(A)$.
For each $n \in \mathbb{N}$, we can choose $x_n \in B(x_0, \frac{1}{n}) \cap A$ and $y_n \in B(x_0, \frac{1}{n}) \cap A^c$.
This gives us $\{x_n\} \subset A$ and $\{y_n\} \subset A^c$ with $x_n, y_n \to x_0$.

Suppose that there exist $\{x_n\} \subset A$ and $\{y_n\} \subset A^c$ with $x_n, y_n \to x_0$.
Let $\epsilon > 0$ so we can find $N_0 \in \mathbb{N}$ so that $x_{N_0}, y_{N_0} \in B(x_0, \epsilon)$.
Hence $x_0 \in \mathsf{bdy}(A)$.

2. Suppose that $x_0 \in \mathsf{Lim}(A)$.
For any $n \in \mathbb{N}$, there exists $x_n \in B(x_0, \frac{1}{n}) \cap (A \setminus \{x_0\})$.
Hence $\{x_n\}$ is such that $x_n \neq x_0$, but $x_n \to x_0$.

Suppose there exists $\{x_n\} \subset (A \setminus \{x_0\})$ with $x_n \to x_0$.
Let $\epsilon > 0$ so for some $n \in \mathbb{N}$, $x_n \in B(x_0, \epsilon)$, and as $x_n \neq x_0$, x_0 is a limit point.

3. Suppose that A is closed, and let $\{x_n\} \subset A$ with $x_n \to x_0$.
If $x_0 \in A^c$, then there exists $\epsilon_0 > 0$ with $B(x_0, \epsilon_0) \cap A \neq \emptyset$.
This is impossible, as $\{x_n\} \subset A$ and $x_n \to x_0$, and so $x_n \in B(x_0, \epsilon_0)$ for all n large enough.

Suppose that A is not closed.
Then there exists $x_0 \in \mathsf{Lim}(A)$ with $x_0 \notin A$.
Then there exists (by **2.**) a sequence $\{x_n\} \subset A$ with $x_n \to x_0$, contradicting the assumption. ■

3 Completeness

3.1 Continuity

Definition 3.1.1. Given metric spaces $(X, d_X), (Y, d_Y)$ with $f : X \to Y$, the function f is termed continuous at $x_0 \in X$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$. Otherwise, x_0 is termed a point of discontinuity of f.

The function f is termed continuous on X iff it is continuous at every $x_0 \in X$.

Theorem 3.1.2. Let $(X, d_X), (Y, d_Y)$ be metric spaces with $x_0 \in X$ and $f : X \to Y$. Then the following are equivalent:

1. $f(x)$ is continuous at x_0
2. If $W \subset Y$ is a neighborhood of $y_0 = f(x_0)$, then $f^{-1}(W)$ is a neighborhood of x_0
3. If $\{x_n\} \subset X$ with $x_n \to x_0$, then $f(x_n) \to f(x_0)$

Proof: (**1.** $\implies$ **2.**) Let W be a neighborhood of $y_0 = f(x_0)$.
Then there exists $\epsilon_0 > 0$ such that $B(y_0, \epsilon_0) \subset W$.
Then there exists $\delta > 0$ such that if $x \in B(x_0, \delta)$, then $d_Y(f(x), f(x_0)) < \epsilon$, so $f(x) \in B(y_0, \epsilon_0) \subset W$.
Hence $B(x_0, \delta) \subset f^{-1}(W)$, and so $x_0 \in \mathsf{int}(f^{-1}(W))$.

(**2.** $\implies$ **3.**) Let $\{x_n\} \subset X$ with $x_n \to x_0$ and $y_0 \in f(x_0)$.
For any $\epsilon > 0$ we have that $B(y_0, \epsilon)$ is a neighborhood of y_0.
Hence $V = f^{-1}(B(y_0, \epsilon))$ is a neighborhood of x_0.

Hence there exists $\delta > 0$ with $B(x_0, \delta) \subset V$.
Then as $x_n \to x_0$, we can find $N_0 \in \mathbb{N}$ such that if $n \geqslant N_0$, then $x_n \in B(x_0, \delta)$.
Then $f(x_n) \in B(f(x_0) = y_0, \epsilon)$, and so $f(x_n) \to f(x_0)$.

(**3.** $\implies$ **1.**) Suppose that $f(x)$ is not continuous at x_0.
Then there is $\epsilon_0 > 0$ such that for $\delta > 0$, we can find x_δ with $d_X(x_\delta, x_0) < \delta$, but $d_Y(f(x_\delta), f(x_0)) \geqslant \epsilon_0$.
Let $\delta = \frac{1}{n}$ and $x_\delta = x_n$.
Then $x_n \to x_0$, but $f(x_n) \notin B(f(x_0), \epsilon_0)$ for any n.
Hence $f(x_n) \not\to f(x_0)$. ■

Theorem 3.1.3. Let $(X, d_X), (Y, d_Y)$ be metric spaces with $f : X \to Y$. Then the following are equivalent:
1. f is continuous on X
2. $f^{-1}(W)$ is open for every open $W \subset Y$
3. If $x_n \to x_0$, then $f(x_n) \to f(x_0)$

Definition 3.1.4. Given a metric space (X, d) with $A \subset X$, a function $f : X \to Y$ is termed continuous on A iff $f|_A$ is continuous on (A, d_A), where d_A is the metric on A induced by d.

3.2 Complete metric spaces

Definition 3.2.1. A metric space (X, d) is termed complete iff every Cauchy sequence in (X, d) converges.

Definition 3.2.2. Given a metric space (X, d) with $A \subset X$, the set A is termed bounded iff there exists $x_0 \in X$ and $M > 0$ such that $A \subset B[x_0, M]$.

Proposition 3.2.3. Given a metric space (X, d), if a sequence $\{x_n\} \subset X$ is Cauchy, then it is bounded.

Proposition 3.2.4. Given a metric space (X, d), if a sequence $\{x_n\} \subset X$ is Cauchy and a subsequence $\{x_{n_k}\}$ converges to x_0, then $x_n \to x_0$.

Theorem 3.2.5. [Bolzano, Weierstrass]
Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Corollary 3.2.6. The metric space $(\mathbb{R}, |\cdot|)$ is complete.

Theorem 3.2.7. $(\mathbb{R}^n, \|\cdot\|_2)$ is complete.

Proof: Let $\{\vec{x}_k\} \subset \mathbb{R}^n$ be Cauchy.
Then for all $i = 1, 2, \ldots, n$, we have $|x_{k,i} - x_{m,i}| \leqslant \|\vec{x}_k - \vec{x}_m\|_2$, hence $\{x_{k_n}\}$ is Cauchy and thus convergent.
Therefore $\vec{x}_n \to \vec{x}_0$, where $x_{0,i} = \lim_{n\to\infty}[x_{k,i}]$. ■

Theorem 3.2.8. Let $1 \leqslant p \leqslant \infty$. Then $(\ell_p, \|\cdot\|_p)$ is complete.

Proof: The cases done here are only for $p \in \{1, \infty\}$. For other p, the proof follows similarly.

Case 1: $p = \infty$
Let $\{\vec{x}_k\}_{k=1}^\infty \in \ell_\infty$ be Cauchy, with $\vec{x}_k = \{x_{k,i}\}_{i=1}^\infty$.
Note that for any $i \in \mathbb{N}$, $|x_{n,i} - x_{m,i}| \leqslant \|\vec{x}_n - \vec{x}_w\|_\infty$ for all $m, n \in \mathbb{N}$.
Hence $\{x_{k,i}\}$ is Cauchy in $\mathbb{R}$ for all i, and so it is convergent, as $\mathbb{R}$ is complete.
Let $x_{0,i} = \lim_{k\to\infty}[x_{k,i}]$ for each $i \in \mathbb{N}$, and $\vec{x}_0 = \{x_{k,i}\}_{i=1}^\infty$.
We claim that $\vec{x}_0 \in \ell_\infty$ and $\vec{x}_k \to \vec{x}_0$ in $\|\cdot\|_\infty$.

Let $\epsilon > 0$.
Since $\{\vec{x}_i\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that if $k, m \geqslant N$, then $\|\vec{x}_k - \vec{x}_m\|_\infty < \frac{\epsilon}{2}$.
Let $n \geqslant N$, so if $m \geqslant N$, then $|x_{n,i} - x_{m,i}| < \frac{\epsilon}{2}$ for all i, so we have that

$$|x_{n,i} - x_{0,i}| = \lim_{m\to\infty}[|x_{n,i} - x_{m,i}|] \leqslant \frac{\epsilon}{2} < \epsilon$$

Therefore $\{x_{n,i} - x_{0,i}\}_{i=1}^{\infty} \in \ell_\infty$, and so $\{x_{0,i}\}_{i=1}^{\infty} \in \ell_\infty$, and to prove the claim, note that

$$\|\vec{x}_n - \vec{x}_0\|_\infty = \sup_i \{|x_{n,i} - x_{0,i}|\} \leqslant \frac{\epsilon}{2} < \epsilon$$

Case 2: $p = 1$
Let $\{\vec{x}_k\}_{k=1}^{\infty} \in \ell_1$, with $\vec{x}_k$ Cauchy.
Then $|x_{k,i} - x_{m,i}| \leqslant \|\vec{x}_k - \vec{x}_m\|_1$, implying $\{x_{k,i}\}_{i=1}^{\infty}$ is Cauchy for all $k \in \mathbb{N}$.
Let $x_{0,i} = \lim_{n\to\infty} [x_{k,i}]$ for all $i \in \mathbb{N}$.
Let $\epsilon > 0$.
Then we can find $N \in \mathbb{N}$ such that if $k, m \geqslant N$, then $\|\vec{x}_k - \vec{x}_m\|_1 < \frac{\epsilon}{2}$.
Let $n \geqslant N$, and so if $m \geqslant N$, then for all $j \in \mathbb{N}$,

$$\sum_{i=1}^{j} |x_{n,i} - x_{m,i}| \leqslant \|\vec{x}_n - \vec{x}_m\|_1 \leqslant \frac{\epsilon}{2}$$

This directly implies that, for all $i \in \mathbb{N}$,

$$\sum_{i=1}^{j} |x_{n,i} - x_{0,i}| = \lim_{m\to\infty} \left[\sum_{i=1}^{j} |x_{n,i} - x_{m,i}|\right] \leqslant \frac{\epsilon}{2} < \epsilon$$

Letting $j \to \infty$, we find that

$$\sum_{i=1}^{i} nfty|x_{n,i} - x_{0,i}| = \lim_{j\to\infty} \left[\sum_{i=1}^{j} |x_{n,i} - x_{0,i}|\right] \leqslant \frac{\epsilon}{2} < \epsilon$$

Therefore $\{x_{n,i} - x_{0,i}\}_{i=1}^{\infty} \in \ell_1$, and $\{x_{0,i}\} \in \ell_1$.
Hence $\|\vec{x}_n - \vec{x}_0\| \leqslant \frac{\epsilon}{2} < \epsilon$. ■

3.3 Completeness of $C_b(X)$

The space $C_b(X)$ is the space of all continuous bounded functions on x.

Definition 3.3.1. Let $f_n : X \to \mathbb{R}$ be a sequence of functions. Then we say that $\{f_n\}$ converges pointwise on X to some $f_0 : X \to \mathbb{R}$ iff for all $x_0 \in X$, $f_n(x_0) \xrightarrow{n\to\infty} f_0(x_0)$

Example 3.3.2. Let $X = [0,1]$ and $f_n = x^n$, with $f_0(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$
Then $f_n \to f_0$ pointwise, and every f_n is continuous, but f_0 is not.

Definition 3.3.3. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\{f_n : X \to Y\}$ a sequence of functions with $f_0 : X \to Y$ fixed. Then $\{f_n\}$ converges uniformly to f_0 on X iff for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that if $n \geqslant N_0$, then $d_Y(f_n(x), f_0(x)) < \epsilon$ for all $x \in X$.

Theorem 3.3.4. If $\{f_n : X \to Y\}$ is such that $\{f_n\}$ converges uniformly on X and if each f_n is continuous at each $x_0 \in X$, then f_0 is continuous at $x_0 \in X$. In particular, if each f_n is continuous, then so is f_0.

Proof: Let $\epsilon > 0$ and choose $N_0 \in \mathbb{N}$ such that if $n \geqslant N_0$, then $d_Y(f_n(x), f_0(x)) < \frac{\epsilon}{3}$.
As f_{N_0} is continuous at x_0, there exists $\delta > 0$ such that if $d_X(x, x_0) < \delta$, then $d_Y(f_{N_0}(x), f_{N_0}(x_0)) < \frac{\epsilon}{3}$.
Now let $d_X(x, x_0) < \delta$, so then

$$\begin{aligned} d_Y(f_0(x), f_0(x_0)) &\leqslant d_Y(f_0(x), f_{N_0}(x)) + d_Y(f_{N_0}(x), f_{N_0}(x_0)) + d_Y(f_{N_0}(x_0), f_0(x_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Therefore f_0 is continuous at x_0. ■

Theorem 3.3.5. Let (X,d) be a metric space. Let $C_b(X) = \{f : X \to \mathbb{R} \mid f(x) \text{ is bounded and continuous on } \mathbb{R}\}$. Let $\|\cdot\|_\infty = \sup\{|f(x)| \mid x \in X\}$. Then $(C_b(X), \|\cdot\|_\infty)$ is a normed linear space.

Theorem 3.3.6. $C_b(X)$ is complete.

Proof: Let $\{f_n\} \subset C_b(X)$ be Cauchy.
If $x \in X$, then $|f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_\infty$, so $\{f_n(x)\}_{n=1}^\infty$ is Cauchy for all $x \in X$.
Let $f_0(x) = \lim_{n\to\infty} [f_n(x)]$ for all $x \in X$.

Claim: $f_0 \in C_b(X)$ and $f_n \xrightarrow{n\to\infty} f_0$
Let $\epsilon > 0$.
Then there exists $N_0 \in \mathbb{N}$ such that if $n, m \geqslant N_0$, then $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ for all $x \in X$.
Let $n \geqslant N_0$, so

$$|f_n(x) - f_0(x)| = \lim_{m\to\infty} [|f_n(x) - f_m(x)|] \leqslant \frac{\epsilon}{2} < \epsilon$$

This proves that $f_n \to f_0$ uniformly on X, which implies that f_0 is continuous on X.
Since $f_n(x) \in C_b(X)$ is bounded, there exists $M \geqslant 0$ such that $\|f_n\|_\infty < M$ for all $n \in \mathbb{N}$.
Then for any $x \in X$,

$$|f_0(x)| \leqslant |f_0(x) - f_{N_0}(x)| + |f_{N_0}(x)|$$

This proves that f_0 is bounded on X.
Applying the previous result, for all $n \geqslant N_0$

$$\begin{aligned} |f_n(x) - f_0(x)| &< \frac{\epsilon}{2} \text{ for all } x \in X \\ &\implies \|f_n - f_0\|_\infty \leqslant \frac{\epsilon}{2} < \epsilon \\ &\implies f_n \to f_0 \text{ in } \|\cdot\|_\infty \end{aligned}$$

This proves the claim and completes the proof. ■

Example 3.3.7.
1. Convergence in $C_b(X)$ is exactly uniform convergence.
2. For $X = \mathbb{N}$, $C_b(X) = \ell_\infty$

Proposition 3.3.8. Let (X,d) be a complete metric space with $A \subset X$. Then (A, d_A) is complete iff A is closed in (X,d).

Proof: $(\Leftarrow)$ Suppose that A is closed in (X,d).
Let $\{x_n\} \subset A$ be Cauchy, so $\{x_n\}$ is Cauchy in X.
Therefore $x_n \to x_0 \in X$, but as A is closed, $x_0 \in A$, so A is complete.

$(\Rightarrow)$ Suppose that (A, d_A) is complete.
Let $\{x_n\} \subset A$ with $x_n \to x_0 \in X$.
Then $\{x_n\}$ is Cauchy in X and Cauchy in A.
By completeness, $x_n \to y_0 \in A$, implying $x_0 = y_0$.
Hence A is closed. ■

Definition 3.3.9. Given a metric space (X, d_X), a completion of $(X, d_X)$is a pair $((Y, d_y), \varphi)$, where (Y, d_Y) is complete and $\varphi : X \to Y$ is an isometry, i.e. $d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, with $\overline{\varphi(X)} = Y$.

Theorem 3.3.10. Every metric space (X,d) has a completion.

Proof: Observe that the function $\Gamma_{x_0}(x) = d(x, x_0)$ is continuous on X for all $x_0 \in X$.
Choose $a \in X$, and for every $v \in X$, define

$$\begin{array}{rccl} f_v : & X & \to & \mathbb{R} \\ & x & \mapsto & d(v,x) - d(x,a) \end{array}$$

Note that f_v is continuous, and

$$|f_v(x)| = |d(v,x) - d(v,a)| \leqslant d(v,a) \implies f_v \in C_b(X)$$

Define a function $\varphi : X \to C_b(X)$ by $\varphi(v) = f_v$.
Then for $v, w \in X$, we have that

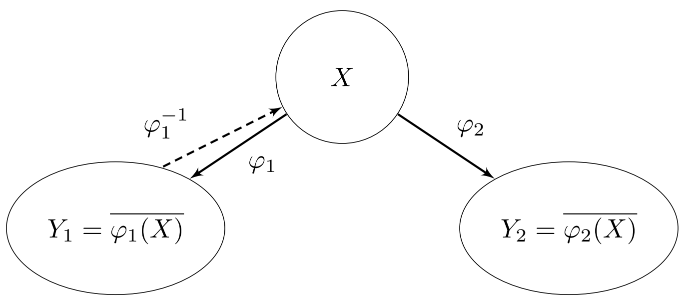
$$|f_v(x) - f_w(x)| = |(d(v,x) - d(v,a)) - (d(w,x) - d(w,a))| = |d(v,x) - d(v,w)| \leqslant d(v,w)$$

As the above holds for each $x \in X$, we have that $\|f_v - f_w\|_\infty \leqslant d(v,w)$, and letting $x = v$, we find that

$$|f_v(v) - f_w(v)| = |d(v,v) - d(v,w)| = d(v,w) \implies \|f_v - f_w\| = d(v,w)$$

Let $Y = \overline{\varphi(X)} \subset C_b(X)$, completing the completion. ■

Remark 3.3.11. Using the same notation as in the theorem above, note that once one isometric function for a completion is found, they are all found. Consider two isometries φ_1, φ_2:



The function φ_1^{-1} exists as φ_1 is an isometry, necessitating an inverse. Then $\varphi_2 \circ \varphi_1^{-1} : Y_1 \to Y_2$ is an isometry itself, and an isomorphism.

3.4 Characterizations of completeness

Recall the nested interval theorem for $\mathbb{R}$:

Theorem 3.4.1. If $\{[a_n, b_n]\}$ is a sequence with $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$, then

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$$

Is there a generalization of this for complete spaces?

Definition 3.4.2. Given a non-empty set $A \subset (X, d)$, denote the <u>diameter</u> of A to be

$$\mathsf{diam}(A) = \sup\{d(x,y) \mid x, y \in A\}$$

Proposition 3.4.3. Given a non-empty set $A \subset (X, d)$, $\mathsf{diam}(A) = \mathsf{diam}(\overline{A})$.

Proof: If $\mathsf{diam}(A) = \infty$, the proposition holds, so assume that $\mathsf{diam}(A) < \infty$.

Clearly $\mathsf{diam}(A) \leqslant \mathsf{diam}(\overline{A})$, as $A \subset \overline{A}$.

Let $\epsilon > 0$ and $x, y \in \overline{A}$.

Then there exist $w, v \in A$ with $d(x, w) < \frac{\epsilon}{2}$ and $d(v, y) < \frac{\epsilon}{2}$, so

$$\begin{aligned} d(x,y) &\leqslant d(x,w) + d(w,v) + d(v,y) \\ &< \frac{\epsilon}{2} + \mathsf{diam}(A) + \frac{\epsilon}{2} \\ &= \mathsf{diam}(A) + \epsilon \end{aligned}$$

As ϵ was arbitrary, $d(x, y) \leqslant \mathsf{diam}(A)$, so $\sup\{d(x, y) \mid x, y \in A\} \leqslant \mathsf{diam}(A)$. ■

Theorem 3.4.4. [Cantor's intersection theorem]
Let (X, d) be a metric space. Then the folowing are equivalent:

1. (X, d) is complete

2. If $\{F_n\}_{n=1}^{\infty}$ is a sequence of non-empty closed subsets of X with $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} [\mathsf{diam}(F_n)] = 0$, then

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

Proof: (**1.** ⇒ **2.**) Assume that $\{F_n\}$ is as in the assumption of **2**.

For each $n \in \mathbb{N}$, choose any $x_n \in F_n$.

Let $\epsilon > 0$.

Then there exists $N_0 \in \mathbb{N}$ such that $\mathsf{diam}(F_{N_0}) < \epsilon$.

Further, for all $m, n \geqslant N_0$, we have that $d(x_n, x_m) < \epsilon$, hence $\{x_n\}$ is Cauchy.

Since X is complete, $x_n \to x_0$ for some $x_0 \in X$.

However, note that $\{x_i\}_{i=n}^{\infty} \subset F_n$, and $\{x_i\}_{i=n}^{\infty} \to x_0$.

As F_n is closed, $x_0 \in F_n$ for all $n \in \mathbb{N}$, thus

$$x_0 \in \bigcap_{n=1}^{\infty} F_n \qquad \left(\{x_0\} = \bigcap_{n=1}^{\infty} F_n\right)$$

(**2.** ⇒ **1.**) Assume **2.** and let $\{x_n\} \subset X$ be Cauchy.

For each $n \in \mathbb{N}$, let $A_n = \{x_i\}_{i=n}^{\infty}$ and let $F_n = \overline{A}_n$.

As $\{x_n\}$ is Cauchy, $\mathsf{diam}(A_n) \to 0$, implying that $\mathsf{diam}(F_n) \to 0$.

Clearly $F_n \neq \emptyset$ and $F_{n+1} \subset F_n$, hence there exists $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

Let $\epsilon > 0$ and choose $N_0 \in \mathbb{N}$ so that $\mathsf{diam}(F_{N_0}) < \epsilon$.

Then $A_{N_0} = \{x_i\}_{i=N_0}^{\infty} \subset F_{N_0} \subset B(x_0, \epsilon)$.

Hence for all $n \geqslant N_0$, $d(x_n, x_0) < \epsilon$, implying $x_n \to x_0$. ■

Remark 3.4.5. There are some counterexamples to why the limit of $\mathsf{diam}(F_n)$ must go to 0 rather than something else. In the first we use the 1-norm on $\mathbb{R}$, and in the second example we apply the discrete metric.

1. $F_n = [n, \infty) \subset \mathbb{R}$, so $\mathsf{diam}(F_n) = \infty$ for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} F_n = \emptyset$

2. $F_n = \{i\}_{i=n}^{\infty} \subset \mathbb{N}$, so $\mathsf{diam}(F_n) = 1$, and $\bigcap_{n=1}^{\infty} F_n = \emptyset$

Definition 3.4.6. Let $(X, \|\cdot\|)$ be a normed linear space. If X is complete with respect to the metric induced by $\|\cdot\|$, then $(X, \|\cdot\|)$ is termed a Banach space.

Definition 3.4.7. Let $(X, \|\cdot\|)$ be a normed linear space. Given $\{x_n\} \subset X$, for each $k \in \mathbb{N}$, the kth partial sum of $\sum_{n=1}^{\infty} x_n$ is defined as $S_k = \sum_{n=1}^{k} x_n$.

The sum $\sum_{n=1}^{\infty} x_n$ is said to converge iff $\{S_k\}_{k=1}^{\infty}$ converges. Otherwise, the sum is said to diverge.

Theorem 3.4.8. [Generalized Weierstrass M-test]
Let $(X, \|\cdot\|)$ be a normed linear space with $\{x_n\} \subset X$. Then the following are equivalent:

1. $(X, \|\cdot\|)$ is a Banach space

2. If $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges in X

Proof: ($\mathbf{1.} \implies \mathbf{2.}$) Suppose that $\sum_{n=1}^{\infty} \|x_n\|$ converges in X.

Let $T_k = \sum_{n=1}^{k} \|x_n\|$ for all $k \in \mathbb{N}$, so $\{T_k\}_{k=1}^{\infty}$ is Cauchy.
So for $\epsilon > 0$ we can find $N_0 \in \mathbb{N}$ such that if $k > m \geqslant N_0$, then

$$\sum_{n=m+1}^{k} \|x_n\| = |T_k - T_m| < \epsilon$$

Let $S_k = \sum_{n=1}^{k} x_n$ for all $k \in \mathbb{N}$, so for $k > m \geqslant N_0$ as above,

$$\|S_k - S_m\| = \left\| \sum_{k=m+1}^{n} x_n \right\| \leqslant \sum_{k=m+1}^{n} \|x_n\| < \epsilon$$

Hence $\{S_k\}$ is Cauchy, and therefore convergent.

($\mathbf{2.} \implies \mathbf{1.}$) Let $\{x_n\} \subset X$ be Cauchy.
For all $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that if $i, j \geqslant n_k$, then $\|x_i - x_j\| < \frac{1}{2^k}$.
Let $g_k = x_{n_k} - x_{n_{k+1}}$, and note that $\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}$, so that

$$\sum_{k=1}^{\infty} \|g_k\| = \sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

By the assumption, the sequence $\{S_k\} = \left\{ \sum_{j=1}^{k} \left(x_{n_j} - x_{n_{j+1}} \right) \right\}$ also converges.
The sequence $\{S_k\}$ may be simplified to

$$S_k = \sum_{j=1}^{k} \left(x_{n_j} - x_{n_{j+1}} \right) = (x_{n_1} - x_{n_2}) + (x_{n_2} - x_{n_3}) + \cdots + (x_{n_k} - x_{n_{k+1}}) = x_{n_1} - x_{n_{k+1}}$$

It follows directly that

$$x_{n_{k+1}} \xrightarrow{k \to \infty} x_{n_1} - \sum_{j=1}^{\infty} \left(x_{n_j} - x_{n_{j+1}} \right)$$

Since the right hand side is finite, we have that $\{x_{n_{k+1}}\}$ converges in $(X, \|\cdot\|)$.
Since $\{x_n\}$ is Cauchy, $\{x_n\}$ converges in $(X, \|\cdot\|)$. ■

3.5 The Banach contractive mapping theorem

Definition 3.5.1. Let (X, d) be a metric space with $\Gamma : X \to X$. Then for all $x, y \in X$,

· x is termed a fixed point of Γ iff $\Gamma(x) = x$

· Γ is termed Lipschitz iff there exists a constant $\alpha \geqslant 0$ such that $d(\Gamma(x), \Gamma(y)) \leqslant \alpha d(x, y)$

· Γ is termed a contraction iff there exists a constant $k \in [0, 1)$ such that $d(\Gamma(x), \Gamma(y)) \leqslant k d(x, y)$

Theorem 3.5.2. [Banach contractive mapping theorem]
Let (X, d) be a complete metric space and $\Gamma : X \to X$ a contraction. Then Γ has a unique fixed point $x_0 \in X$.

Proof: Let $x_1 \in X$, and $x_{i+1} = \Gamma(x_i)$ for $i \in \mathbb{N}$, and observe that

$$\begin{aligned} d(x_3, x_2) &= d(\Gamma(x_2), \Gamma(x_1)) \leqslant kd(x_2, x_1) \\ d(x_4, x_3) &= d(\Gamma(x_3), \Gamma(x_2)) \leqslant kd(x_3, x_2) \leqslant k^2 d(x_2, x_1) \\ d(x_5, x_4) &= d(\Gamma(x_4), \Gamma(x_3)) \leqslant kd(x_4, x_3) \leqslant k^3 d(x_2, x_1) \\ &\vdots \\ d(x_{n+1}, x_n) &= d(\Gamma(x_n), \Gamma(x_{n-1})) \leqslant kd(x_n, x_{n-1}) \leqslant k^{n-1}(x_2, x_2) \end{aligned}$$

Hence for all $m > n \in \mathbb{N}$,

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leqslant k^{m-2} d(x_2, x_1) + \cdots + k^{n-1} d(x_2, x_1) \\ &= k^{n-1} d(x_2, x_1) \left(k^{m-n-1} + \cdots + k + 1\right) \\ &< \frac{k^{n-1} d(x_2, x_1)}{1-k} \end{aligned}$$

Since $k^n \to 0$ as $n \to \infty$, it follows that $\{x_n\}$ is Cauchy.
As (X, d) is complete, $\{x_n\}$ converges to some $x_0 \in X$.
It is clear that Γ is continuous, and hence $\Gamma(x_n) \to \Gamma(x_0)$.
But $\Gamma(x_n) = x_{n+1} \to x_0$, and so $\Gamma(x_0) = x_0$.

Now suppose that also $\Gamma(y_0) = y_0$, so for all $n \in \mathbb{N}$,

$$d(x_0, y_0) = d(\Gamma(x_0), \Gamma(y_0)) \leqslant kd(x_0, y_0) \implies d(x_0, y_0) \leqslant k^n d(x_0, y_0)$$

And as $k \in [0, 1)$, $k^n d(x_0, y_0) \to 0$, and so $x_0 = y_0$. ■

Remark 3.5.3. If $k = 1$, then the above theorem will not hold, as $f : [1, \infty) \to [1, \infty)$ given by $f(x) = x + \frac{1}{x}$ shows.

Theorem 3.5.4. [Picard, Lindelof]
Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous and Lipschitz in y. Equivalently, suppose that there exists $\alpha \geqslant 0$ such that for all $y, z \in \mathbb{R}$ and $t \in [0, 1]$,

$$|f(t, y) - f(t, z)| \leqslant \alpha |y - z|$$

Then for a fixed $y_0 \in \mathbb{R}$, there exists a unique function $y(t) \in C[0, 1]$ with

$$\begin{aligned} y(0) &= y_0 \\ y'(t) &= f(t, y(t)) \quad \text{for all } x \in (0, 1) \end{aligned}$$

3.6 The Baire category theorem

Remark 3.6.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{n}{m} \in \mathbb{Q}, m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, n) = 1 \end{cases}$$

Then f is continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$ and discontinuous at every $x \in \mathbb{Q}$. However, the reverse type of function, one that is continuous at every $\mathbb{Q}$ and discontinuous at every $\mathbb{R} \setminus \mathbb{Q}$, is impossible to construct.

Definition 3.6.2. Let (X,d) be a metric space with $A \subset X$.

- A is termed F_σ iff there exist closed sets $\{F_n\}_{n=1}^\infty$ with $A = \bigcup_{n=1}^\infty F_n$
- A is termed G_δ iff there exist open sets $\{U_n\}_{n=1}^\infty$ with $A = \bigcap_{n=1}^\infty U_n$
- A is termed nowhere dense iff $\mathsf{int}(\overline{A}) = \emptyset$
- A is of first category in X iff there exist nowhere dense sets $\{A_n\}_{n=1}^\infty$ with $A = \bigcup_{n=1}^\infty A_n$
- A is of second category in X iff A is not of first category
- A is termed residual iff A^c is of first category

Remark 3.6.3.

- A is F_σ iff A^c is G_δ
- $[0,1) = \bigcup_{n=1}^\infty [0, 1-\frac{1}{n}] = \bigcap_{n=1}^\infty (-\frac{1}{n}, 1)$ is both F_σ and G_δ
- If (X,d) is a metric space and $F \subset X$ is closed, then F is G_δ implies F^c is F_σ
- $\mathbb{Q}$ is of first category in $\mathbb{R}$
- The Cantor set is nowhere dense in $\mathbb{R}$
- A is nowhere-dense in X iff $\overline{A}$ is nowhere dense in X

Definition 3.6.4. For metric spaces (X, d_X) and (Y, d_Y), let $f : (X, d_X) \to (Y, d_Y)$ be a function. Define

$$D(f) = \{x_0 \in X \mid f(x) \text{ is discontinuous at } x_0\}$$
$$D_n(f) = \{x_0 \in X \mid \text{ for every } \delta > 0 \text{ there exists } y, z \in B_x(x_0, \delta) \text{ such that } d_Y(f(y), f(z)) \geqslant \tfrac{1}{n}\}$$

Proposition 3.6.5. For metric spaces (X, d_X) and (Y, d_Y), let $f : (X, d_X) \to (Y, d_Y)$ be a function. Then for each $n \in \mathbb{N}$, $D_n(f)$ is closed. Moreover,

$$D(f) = \bigcup_{n=1}^\infty D_n(f)$$

hence $D(f)$ is F_σ.

Theorem 3.6.6. [Baire category theorem I]
Let (X,d) be a complete metric space. If $\{U_n\}_{n=1}^\infty$ is a sequence of open dense subsets of X, then $\bigcap_{n=1}^\infty U_n$ is dense in X.

Proof: Let $W \subset X$ be non-empty and open.
Then $W \cap U_1$ is non-empty and open.
Then there exists $x_1 \in X$ and $r_1 \in (0,1]$ with $B(x_1, r_1) \subset B[x_1, r_1] \subset W \cap U_1$.
We can further find $x_2 \in X$ and $r_2 \in (0, \frac{1}{2}]$ with $B(x_2, r_2) \subset B[x_2, r_2] \subset (B(x_1, r_1) \cap U_2)$.
Proceeding inductively, we get sequences $\{x_n\} \subset X$ and $\{r_n\} \subset (0,1]$ with $r_i \in (1, \frac{1}{i}]$, and

$$B(x_{n+1}, r_{n+1}) \subset B[x_{n+1}, r_{n+1}] \subset (B(x_n, r_n) \cap U_{n+1})$$

Let $F_n = B[x_n, r_n]$.
Then $F_{n+1} \subset F_n$ and $\mathsf{diam}(F_n) = 2r_n \leqslant \frac{2}{n} \to 0$ as $n \to \infty$.
By Cantor's intersection theorem, $\{x_0\} = \bigcap_{n=1}^\infty F_n = \bigcap_{n=1}^\infty B[x_n, r_n]$.
Hence $x_0 \in B[x_1, r_1] \subset W$, meaning that $x_0 \in W$ and $x_0 \in B[x_n, r_n] \subset U_n$ for all $n \in \mathbb{N}$.
Hence $x_0 \in W \cap (\bigcap_{n=1}^\infty U_n)$. ■

Remark 3.6.7. Note that U is open and dense iff $F = U^c$ is closed and nowhere dense.

Theorem 3.6.8. [Baire category theorem II]
If (X,d) is a complete metric space, then X is of 2nd category in itself.

Proof: Suppose that X is of 1st category in X.
Then for nowhere dense sets A_n, we have that

$$X = \bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty \overline{A_n}$$

Then for $U_n = \left(\overline{A_n}\right)^c$, we have that U_n is dense and open in X, implying that

$$\bigcap_{n=1}^{\infty} U_n = X^c = \emptyset$$

As this contradicts BCTI, this is false. ■

Corollary 3.6.9. $\mathbb{Q} \subset \mathbb{R}$ is not G_δ.

Proof: Suppose that $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ for each U_n open.
Since $\mathbb{Q} \subset U_n$ for each $n \in \mathbb{N}$, U_n must be dense.
Let $F_n = U_n^c$.
Then $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed and nowhere dense.
For $\mathbb{Q} = \{r_1, r_2, \dots\}$, let $F_n' = F_n \cup \{r_n\}$.
Then as F_n' is closed and nowhere dense, $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n'$ is of 1st category, a contradiction. ■

Corollary 3.6.10. There is no function $f : \mathbb{R} \to \mathbb{R}$ with $D(f) = \mathbb{R} \setminus \mathbb{Q}$.

One wonders if the converse is true, i.e. given an F_σ set $A \subset \mathbb{R}$, is it possible to find a function $f : \mathbb{R} \to \mathbb{R}$ with $D(f) = A$. It turns out that such a function does always exist, given that A is of first category in $\mathbb{R}$.

Definition 3.6.11. Let (X, d_x) and (Y, d_Y) be metric spaces and $\{f_n : X \to Y\}$ a sequence of functions with $f_n \to f_0 : X \to Y$ pointwise on X. Then $\{f_n\}$ converges to f_0 uniformly at x_0 iff for ever $\epsilon > 0$ there exists a $\delta > 0$ and $N_0 \in \mathbb{N}$, such that for $x \in B(x_0, \delta)$ we have $d_Y(f_n(x), f_0(x_0)) < \epsilon$.

Theorem 3.6.12. Let (X, d_x) and (Y, d_Y) be metric spaces and $\{f_n : X \to Y\}$ a sequence of functions with $f_n \to f_0 : X \to Y$ pointwise on X and uniformly at x_0. If each f_n is continuous at x_0, then f_0 is also continuous at x_0.

Theorem 3.6.13. Let $f_n(a, b) \to \mathbb{R}$ with $f_n \to f_0$ pointwise on (a, b). If each f_n is continuous on (a, b), then $f_n \to f_0$ uniformly at some $x_0 \in (a, b)$.

Corollary 3.6.14. If $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is a sequence of continuous functions with $f_n \to f_0$ pointwise on $\mathbb{R}$, then there exists a dense G_δ set $A \subset \mathbb{R}$ with $f_0(x)$ continuous at each $x_0 \in A$.

Remark 3.6.15. It immediately follows that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, then $f'(x)$ is continuous at each point in a dense G_δ subset of $\mathbb{R}$.

Theorem 3.6.16. If $\{f_n : (a, b) \to \mathbb{R}\}$ is a sequence of continuous functions that converge pointwise on (a, b), then there exists $x_0 \in (a, b)$ such that $f_n \to f_0$ uniformly at x_0.

Proof: Claim: There exists $\alpha_1 < \beta_1 \in (a, b)$ and $N_1 \in \mathbb{N}$ such that if $x \in [\alpha_1, \beta_1]$ and $n, m \geqslant \mathbb{N}$, then $|f_n(x) - f_m(x)| \leqslant 1$.

Suppose that the claim fails, so there exists $a < t_1 < b$ and $n_1, m_1 \in \mathbb{N}$ such that $|f_{n_1}(t_1) - f_{m_1}(t_1)| > 1$.
Since $f_{n_1} - f_{m_1}$ in continuous, we can find an open interval I_1 with $\overline{I_1} \subset (a, b)$ and $|f_{n_1}(x) - f_{m_1}(x)| > 1$ for all $x \in I_1$.
As the claim does not hold, we can find $t_1 \in I_1$ and $n_2, m_2 > \max\{n_1, m_1\}$ such that $|f_{n_2}(x) - f_{m_2}(x)| > 1$.
Again by the continuity of $f_{n_2} - f_{m_2}$, we can find an open interval I_2 with $I_2 \subset \overline{I_2} \subset I_1 \subset \overline{I_1} \subset (a, b)$ for which $|f_{n_2}(x) - f_{m_2}(x)| > 1$ for all $x \in I_2$.

Proceed now inductively to choose a sequence $\{I_n\}$ of open intervals and $\{n_k\}, \{m_k\} \subset \mathbb{N}$ such that $(a, b) \supset \overline{I_1} \supset I_1 \supset \overline{I_2} \supset I_2 \supset \overline{I_3} \supset \cdots$ and $n_{k+1}, m_{k+1} > \max\{n_k, m_k\}$ with $|f_{n_k}(x) - f_{m_k}(x)| > 1$ for all $x \in I_k$.
By the Weierstrass M-test, there exists $t_0 \in \bigcap_{k=1}^{\infty} \overline{I_k} = \bigcap_{k=1}^{\infty} I_k$.

Then $|f_{n_k}(t_0) - f_{m_k}(t_0)| > 1$ for all k, so $\{f_n(t_0)\}$ is not Cauchy, a contradiction.
Hence the claim holds.

By a similar inductive procedure, we can construct $\{[\alpha_k, \beta_k]\}$ with $(a, b) \subset (\alpha_1, \beta_1) \subset [\alpha_1, \beta_1] \supset (\alpha_2, \beta_2) \supset [\alpha_2, \beta_2] \supset (\alpha_3, \beta_3) \supset \cdots$ and $\{N_k\} \subset \mathbb{N}$ with $N_1 < N_2 < N_3 < \cdots$ such that if $n, m \geqslant k$, then $|f_n(x) - f_m(x)| < \frac{1}{k}$ for all $x \in [\alpha_k, \beta_k]$.
Let $x_0 \in \bigcap_{k=1}^{\infty}[\alpha_k, \beta_k] = \bigcap_{k=1}^{\infty}(\alpha_k, \beta_k)$.
Let $\epsilon > 0$ and choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$
If $x \in (\alpha_k, \beta_k)$ and $n, m \geqslant k$, then $|f_n(x) - f_m(x)| \leqslant \frac{1}{k} < \epsilon$.
And as $x_0 \in (\alpha_k, \beta_k)$, we can find $\delta > 0$ so that $B(x_0, \delta) \subset (\alpha_k, \beta_k)$. ■

4 Compactness

4.1 Compact metric spaces

Definition 4.1.1. Let (X, d) be a metric space. A collection $\{U_\alpha\}_{\alpha \in I}$ of open sets in X is termed an open cover (or cover) of X iff $X = \bigcup_{\alpha \in I} U_\alpha$.

Similarly, for $A \subset X$, a collection of sets $\{U_\alpha\}_{\alpha \in I}$ is said to cover A iff $A \subset \bigcup_{\alpha \in I} U_\alpha$.

Given a cover $\{U_\alpha\}_{\alpha \in I}$ of X, a subcover of X is a collection $\{U_\alpha\}_{\alpha \in J}$ for $J \subset I$ and $X = \bigcup_{\alpha \in J} U_\alpha$.

A subcover $\{U_\alpha\}_{\alpha \in J}$ is termed a finite subcover iff J is finite.

Definition 4.1.2. A metric space (X, d) is termed compact iff every cover $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover. For $A \subset X_1$, A is compact iff every cover of A in X has a finite subcover. That is, A is compact in X iff (A, d_A) is compact.

Definition 4.1.3. A metric space (X, d) is termed sequentially compact iff every sequence $\{x_n\} \subset X$ has a convergent subsequence. A subset $A \subset X$ is termed sequentially compact iff every sequence $\{x_n\} \subset A$ has a subsequence that converges to an element of A.

Definition 4.1.4. A metric space (X, d) has the Bolzano-Weierstrass property (or BWP) iff every infinite subset of X has a limit point.

Theorem 4.1.5. Let (X, d) be a metric space. Then the folowing are equivalent:
1. (X, d) is sequentially compact
2. (X, d) has the BWP

Proof: (**1.** $\implies$ **2.**) Let $A \subset X$ be infinite, so we can find $\{x_n\} \subset A$ with $x_n \neq x_m \iff n \neq m$.
Then there exists $\{x_{n_k}\} \subset \{x_n\}$ with $x_{n_k} \to x_0$.
Let $\epsilon > 0$ so that $B(x_0, \epsilon)$ contains infinitely many terms of $\{x_{n_k}\}$, hence $x_0 \in \mathsf{Lim}(A)$.

(**2.** $\implies$ **1.**) Let $\{x_n\} \subset X$.
If there is an element in $\{x_n\}$ that appears infinitely many times, then clearly $\{x_n\}$ has a convergent subsequence.
If this is not true, then $\{x_n\}$ as a subset of X is infinite.
We may also assume WLOG by (potentially) replacing $\{x_n\}$ with a subsequence $\{x_{n_k}\}$ that $x_n \neq x_m \iff n \neq m$.
Then $A = \{x_n\}$ has a limit point $x_0 \in X$.
Let $\epsilon = 1$, so there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} \in B(x_0, 1)$.
Similarly we can find $n_2 > n_1$ such that $x_{n_2} \in B(x_0, \frac{1}{2})$.
Proceeding inductively, we find $\{n_k\} \subset \mathbb{N}$ increasing and $\{x_{n_k}\}$ with $d(x_{n_k}, x_0) < \frac{1}{k}$.
Hence $x_{n_k} \to x_0$. ■

Proposition 4.1.6. Let (X, d) be a metric space and $A \subset X$. Then

1. If A is compact, then A is closed and bounded.
2. If A is closed and (X, d) is compact, then A is compact.
3. If A is sequentially compact, then A is closed and bounded.
4. If A is closed and X is sequentially compact, then A is sequentially compact.
5. If X is sequentially compact, then X is closed.

Proof: **1.** Let $X_0 \in X$ and let $U_n = B(x_0, n)$ for all $n \in \mathbb{N}$.
Then $\{U_n\}_{n=1}^{\infty}$ is a cover of A.
Hence there is a finite subcover $\{U_{n_i}\}_{i=1}^{k}$ of A with $\{n_k\}$ increasing.
Thus $A \subset B(x_0, n_k)$, and if A is not closed, we can find $x_0 \in \mathsf{bdy}(A) \supset A$.
Let $V_n = B[x_0, \frac{1}{n}]^c$.
Then $A \subset \bigcup_{n=1}^{\infty} V_n$ and $\{V_n\}_{n=1}^{\infty}$ is a cover with no finite subcover.

2. Suppose that X is compact and $A \subset X$ is closed.
Let $\{U_\alpha\}_{\alpha \in I}$ be a cover of A, so $\{U_\alpha\}_{\alpha \in I} \cup \{A^c\}$ is a cover of X.
Hence there is a finite subcover $\{U_\alpha\}_{\alpha \in J} \cup \{A^c\}$ of X and $A \subset \{U_\alpha\}_{\alpha \in J}$.

3. Suppose that A is sequentially compact.
Let $\{x_n\} \subset A$ with $x_n \to x_0$.
By sequential compactness, we have a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to y_0 \in A$.
Hence $x_0 = y_0$ and $x_0 \in A$, so A is closed.

Suppose that A is not bounded.
Then we can find $\{x_n\} \subset A$ with $d(x_n, x_m) \geqslant 1$ for all $n \neq m$.
Therefore $\{x_n\}$ has no Cauchy subsequence, so A cannot be sequentially compact.

4. Suppose that A is closed and X is sequentially compact with $\{x_n\} \subset A$.
Then there exists $\{x_{n_k}\} \subset \{x_n\}$ with $x_{n_k} \to x_0 \in X$.
Since A is closed, $x_0 \in A$.

5. Let $\{x_n\} \subset X$ be Cauchy.
Then $\{x_n\}$ has a convergent subsequence, so $\{x_n\}$ converges. ■

Remark 4.1.7.

· If $A \subset \mathbb{R}$ is closed and bounded, then A is sequentially compact.
· A sequence $\{x_k\} \subset \mathbb{R}^n$ converges iff $\{x_{n,i}\} \subset \mathbb{R}$ converges for all $1 \leqslant i \leqslant n$.

Definition 4.1.8. A cell in $\mathbb{R}^n$ is a set $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Theorem 4.1.9. [Heine, Borel]
A set $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.

Proof: $(\Rightarrow)$ Trivial.

$(\Leftarrow)$ Assume that A is closed and bounded, but that $\{U_\alpha\}_{\alpha \in I}$ is a cover of A with no finite subcover.
Since A is bounded, there exists a closed cell $J_1 = [a_1, b_1] \times \cdots \times [a_n, b_n]$ with $A \subset J_1$.
Bisecting each of the component 1-cells $[a_i, b_i]$ to subdivide A into 2^n closed subcells.
Then at least one of those is such that its intersection with A cannot be covered by finitely many U_α.
Call this closed subcell J_2, and note $\mathsf{diam}(J_2) = \frac{1}{2}\mathsf{diam}(J_1)$.
Proceed inductively to construct a sequence $\{J_k\}$ of closed cells such that $J_{k+1} \subset J_k$.
Then $\mathsf{diam}(J_{k+1}) = \frac{1}{2}\mathsf{diam}(J_k)$.
Let $F_k = A \cap J_k$, so F_k cannot be covered by finitely many sets U_α.
Note that $\mathsf{diam}(J_k) = \frac{1}{2^{k-1}}\mathsf{diam}(J_1) \to 0$.

Hence F_k is a sequence of non-empty nested closed sets with disappearing diameter.
Hence by Cantor's intersection theorem, $\bigcap_{k=1}^{\infty} F_k = \{x_0\} \subset A$.
Since $x_0 \in A$, $x_0 \in U_{\alpha_0}$ for some $\alpha_0 \in I$.
Therefore there exists $\epsilon > 0$ such that $B(x_0, \epsilon) \subset U_{\alpha_0}$.
If k is large enough so that $\mathsf{diam}(F_k) < \frac{\epsilon}{2}$, then $F_k \subset B(x_0, \epsilon) \subset U_{\alpha_0}$.
Now we have a finite subcover of F_k, a contradiction, so $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover. ■

Now we know what compactness is in $\mathbb{R}^n$. Hence we can make the following observations.

Remark 4.1.10. Let $A \subset \mathbb{R}^n$. Then equivalently
- A is compact
- A is sequentially compact
- A has the BWP
- A is closed and bounded

Definition 4.1.11. Let $\{A_\alpha\}_{\alpha \in I} \subset \mathbf{P}(X) \setminus \{\emptyset\}$. Then $\{A_\alpha\}_{\alpha \in I}$ has the finite intersection property (FIP) iff given $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$, we have that $\bigcap_{i=1}^{n} A_{\alpha_i} \neq \emptyset$.

Theorem 4.1.12. Let (X, d) be a metric space. Then equivalently
1. X is compact
2. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of non-empty closed sets with FIP, then $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$

Proof: (**1.** $\Rightarrow$ **2.**) Suppose X is compact and $\{F_\alpha\}_{\alpha \in I}$ is as in **2.**
If $\bigcap_{\alpha \in I} F_\alpha = \emptyset$ and $U_\alpha = F_\alpha^c$, then $\bigcup_{\alpha \in I} U_\alpha = X$, so $\{U_\alpha\}_{\alpha \in I}$ is a cover.
By compactness, there exists $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ a finite subcover.
Hence $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$, contradicting the FIP.

(**2.** $\Rightarrow$ **1.**) Suppose that **2.** holds but X is not compact.
Then there exists a cover $\{U_\alpha\}_{\alpha \in I}$ with no finite subcover.
Let $F_\alpha = U_\alpha^c$, so then $\{F_\alpha\}_{\alpha \in I}$ has the FIP, so $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.
This contradicts the fact that $\{U_\alpha\}_{\alpha \in I}$ is a cover. ■

Corollary 4.1.13. If (X, d) is compact and $\{F_n\}_{n=1}^{\infty}$ is a sequence of non-empty and closed sets with $F_{n+1} \subset F_n$ for all n, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Corollary 4.1.14. If (X, d) is compact, then it has the BWP. In particular, (X, d) is sequentially compact.

Proof: Let $A \subset X$ be infinite.
Let $\{x_1, x_2, \dots\} \subset A$ be a sequence of distinct elements, and $F_n = \{x_n, x_{n+1}, \dots\}$.
By the previous corollary, there exists $x_0 \in \bigcap_{n=1}^{\infty} F_n$.
Hence for every $\epsilon > 0$ we have $B(x_0, \epsilon) \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset$.
Therefore $B(x_0, \epsilon) \cap A$ is infinite, so $x_0 \in \mathsf{Lim}(A)$. ■

Definition 4.1.15. Let (X, d) be a metric space. Then (X, d) is termed totally bounded iff for any $\epsilon > 0$ there exist finitely many points $\{x_1, \dots, x_n\} \subset X$ with $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

Given a collection of points $\{x_\alpha\}_{\alpha \in I} \subset X$ with $X = \bigcup_{\alpha \in I} B(x_\alpha, \epsilon)$, the set is termed a ϵ-net for X.

A set $A \subset X$ is termed totally bounded iff (A, d_a) is totally bounded.

Remark 4.1.16.
- If X is totally bounded, then X is bounded.
- The metric space $(\mathbb{N}, d)$ for d the discrete metric, is bounded, but has no finite $\frac{1}{2}$-net.

Theorem 4.1.17. If (X, d) is sequentially compact, then (X, d) is totally bounded.

Proof: Suppose that (X, d) is not totally bounded.
Then there exists $\epsilon > 0$ such that X has no finite ϵ-net.
From this we may construct a sequence $\{x_n\} \subset X$ with $d(x_n, x_m) \geqslant \epsilon > 0$ for $n \neq m$.
Then $\{x_n\}$ cannot have a convergent subsequence, so (X, d) can not be sequentially compact. ■

Remark 4.1.18. For $A \subset (X, d)$, A is totally bounded iff $\overline{A}$ is totally bounded. Given $\epsilon > 0$, a $\frac{\epsilon}{2}$-net for A is an ϵ-net for $\overline{A}$.

Theorem 4.1.19. Let (X, d_X) be sequentially compact, and $f : (X, d_X) \to (Y, d_Y)$ continuous. Then $f(X)$ is sequentially compact in (Y, d_Y).

Proof: Let $\{y_n\} \subset f(X)$.
Then there exists $\{x_n\} \subset X$ with $y_n = f(x_n)$ for all $n \in \mathbb{N}$.
By sequential compactness, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \to x_0 \in X$.
Let $y_0 = f(x_0) \in f(X)$.
Then we have that $y_{n_k} = f(x_{n_k}) \to f(x_0)$, and so $f(X)$ is sequentially compact. ■

Corollary 4.1.20. [Extreme value theorem]
If (X, d) is sequentially compact and $f : X \to \mathbb{R}$ is continuous, then there exist $c, d \in X$ wih $f(c) \leqslant f(x) \leqslant f(d)$ for all $x \in X$.

Proof: As $f(X)$ is sequentially compact in $\mathbb{R}$, $f(X)$ is closed and bounded.
Let $\alpha = \text{glb}(f(X))$ and $\beta = \text{lub}(f(X))$.
Then $\alpha, \beta \in f(X)$ so there exist $c, d \in X$ such that $\alpha = f(c)$ and $\beta = f(d)$. ■

Theorem 4.1.21. [Lebesgue]
Let (X, d) be sequentially compact and $\{U_\alpha\}_{\alpha \in I}$ an open cover of X. Then there exists $\epsilon_0 > 0$ such that if $0 < \delta < \epsilon_0$ and $x_0 \in X$, then there exists $\alpha_0 \in I$ with $B(x_0, \delta) \subset U_{\alpha_0}$.

Proof: Given $x \in X$, define $\varphi(x) = \sup\{r > 0 \mid \text{there exists } \alpha_0 \in I \text{ with } B(x, r) \subset U_{\alpha_0}\}$.
If $U_{\alpha_0} = X$ for some α_0, the theorem is trivial, so assume $U_{\alpha_0} \neq X$ for all $\alpha_0 \in I$.
With this assumption, given that X is bounded, we have that $\varphi(x) < \infty$ for all $x \in X$.
By the triangle inequality for $x, y \in X$, we find that $\varphi(x) \leqslant d(x, y) + \varphi(y)$.
This implies that $|\varphi(x) - \varphi(y)| \leqslant d(x, y)$.
Hence φ is uniformly continuous.
By the EVT, φ attains its minimum value $\epsilon_0 > 0$ on X. ■

Note that the ϵ_0 found above is termed the Lebesgue number for the cover $\{U_\alpha\}_{\alpha \in I}$.

Theorem 4.1.22. [Lebesgue, Borel]
Let (X, d) be a metric space. Then equivalently
1. X is compact
2. X has the BWP
3. X is sequentially compact

Proof: We already know **1.** $\Rightarrow$ **2.** $\iff$ **3.**, hence it remains to prove **3.** $\Rightarrow$ **1.**.
Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X.
Let ϵ_0 be the Lebesgue number for this cover, and find δ with $0 < \delta < \epsilon_0$.
Since X is totally bounded, there exist finitely many points $\{x_1, \ldots, x_n\} \subset X$ with $X = \bigcup_{i=1}^n B(x_i, \delta)$.
Since $\delta < \epsilon_0$, there exists $\alpha_i \in I$ with $B(x_i, \delta) \subset U_{\alpha_i}$ for all i.
Therefore $X = \bigcup_{i=1}^n U_{\alpha_i}$, and so X is compact. ■

Theorem 4.1.23. Let (X, d) be a metric space. Then X is compact iff X is complete and totally bounded.

Proof: (⇒) Already known.

(⇐) Let $\{x_n\} \subset X$ and X be totally bounded, so X has a finite $\frac{1}{k}$-net for all $k \in \mathbb{N}$.
Then there exists an open ball $S_1 = B(y_1, 1)$ of radius 1 that contains infinitely many terms of $\{x_n\}$.
And there exists an open ball $S_1 = B(y_2, \frac{1}{2})$ of radius $\frac{1}{2}$ that contains infinitely many terms in $\{x_n\} \cap S_1$.
Proceed to construct a sequence $\{S_k\} = \{B(y_k, \frac{1}{k}\}$ with infinitely many terms of $\{x_n\}$ in $S_1 \cap S_2 \cap \cdots \cap S_k$.
Then there exist $n_1 < n_2 < \cdots$ such that $x_{n_k} \in S_1 \cap \cdots \cap S_k$.
Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that if $k \geqslant N$, then $\mathsf{diam}(S_k) = \frac{2}{k} < \epsilon$.
If $k > m \geqslant N$, then $x_{n_m}, x_{n_k} \in S_N$, which implies that $d(x_{n_k}, x_{n_m}) < \epsilon$, so $\{x_n\}$ is Cauchy.
Hence X is seuentially compact, so X is compact. ■

Definition 4.1.24. Let $(X, d_X), (Y, d_Y)$ be metric spaces. Then $f : X \to Y$ is termed uniformly continuous iff for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.

Remark 4.1.25. Uniform continuity implies continuity.

Theorem 4.1.26. [Sequential characterization of uniform continuity]
Let $(X, d_X), (Y, d_Y)$ be metric spaces. Then equivalently

1. f is uniformly continuous

2. If $\{x_n\}, \{z_n\} \subset X$ are such that $d_X(x_n, z_n) \to 0$, then $d_Y(f(x_n), f(z_n)) \to 0$.

Proof: (**1.** ⇒ **2.**) Let $\epsilon > 0$.
By uniform continuity, there exists $\delta > 0$ such that if $x, z \in X$ with $d_X(x, z) < \delta$, then $d_Y(f(x), f(z)) < \epsilon$.
We can find $N \in \mathbb{N}$ sugh that if $n \geqslant N$, then $d_X(x_n, z_n) < \delta$.
Hence if $n \geqslant N$, then $d_Y(f(x_n), f(z_n)) < \epsilon$.

(**2.** ⇒ **1.**) Suppose f is not uniformly continuous.
Then for some $\epsilon_0 > 0$ and each $\delta > 0$, we can find $x_\delta, z_\delta \in X$ with $d_X(x_\delta, z_\delta) < \delta$.
This gives us that $d_Y(f(x_\delta), f(z_\delta)) \geqslant \epsilon_0$.
Let $\delta = \frac{1}{n}$ to get two sequences $\{x_n\}, \{z_n\} \subset X$ with $d_X(x_n, z_n) < \frac{1}{n}$ and $d_Y(f(x_n), f(z_n)) \geqslant \epsilon_0$.
Hence **2.** fails. ■

Theorem 4.1.27. Let (X, d_X) be a compact metric space. If $f : (X, d_X) \to (Y, d_Y)$ is continuous, then f is uniformly continuous.

Proof: Suppose that f is not uniformly continuous.
Then there exist $\{x_n\}, \{z_n\} \subset X$ with $d_X(x_n, z_n) \to 0$, but $d_Y(f(x_n), f(z_n)) \geqslant \epsilon_0 > 0$ for all $n \in \mathbb{N}$.
Since (X, d_X) is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x_0 \in X$.
Therefore also $z_{n_k} \to x_0$ for some subsequence $\{z_{n_k}\}$ of $\{z_n\}$.
By continuity, $f(x_{n_k}) \to f(x_0)$ and $f(z_{n_k}) \to f(x_0)$, but from above $d_Y(f(x_{n_k}), f(z_{n_k})) \not\to 0$.
This is a contradiction. ■

Definition 4.1.28. If $(X, d_X), (Y, d_Y)$ are metric spaces, then a homeomorphism between X and Y is a bijection $\varphi : X \to Y$ with φ and φ^{-1} continuous.

Remark 4.1.29. If φ is a homeomorphism, then $U \subset X$ is open iff $\varphi(U)$ is open. Hence (X, d_X) and (Y, d_Y) are essentially the same as topological spaces.

Theorem 4.1.30. Let $(X, d_X), (Y, d_Y)$ be metric spaces and X be compact. If $\varphi : X \to Y$ is bijective and continuous, then φ^{-1} is continuous.

Proof: We need to show that if $U \subset X$ is open, then $\varphi(U)$ is open.
Let $F = U^c$, so F is closed, and further compact.
Hence $\varphi(F)$ is compact, and further closed.
As $\varphi(U)^c = \varphi(F)$, we have that $\varphi(U)$ is open. ■

4.2 Finite dimensional normed linear spaces

Definition 4.2.1. Let W be an n-dimensional vector space with basis $\{v_1, \dots, v_n\}$ and $\Gamma_n : \mathbb{R}^n \to W$ defined by

$$\Gamma_n((a_1, \dots, a_n)) = a_1 v_1 + \cdots + a_n v_n$$

Then Γ_n is termed a vector space isomorphism, and $\Gamma^{-1} : W \to \mathbb{R}^n$ is also an isomorphism.Let $(W, \| \cdot \|_W), (V, \| \cdot \|_V)$ be normed linear spaces. Let $T_V \to W$ be linear. Let $\|T\| = \sup\{\|T(v)\|_W \mid v \in V, \|v\|_V = 1\}$. Then T is termed <u>bounded</u> iff $\|T\| < \infty$.

Definition 4.2.2. If $T : V \to W$ is linear, then T is bounded iff T is continuous.

Remark 4.2.3.
· T is bounded iff T is uniformly continuous
· T is bounded iff T is continuous at $0 \in V$

Theorem 4.2.4. Let $(W, \| \cdot \|_W)$ be an n-dimensional normed linear space. Let $\Gamma_n : \mathbb{R}^n \to W$ be as before. Then Γ_n, Γ_n^{-1} are bounded.

<u>*Proof:*</u> Let $\{v_1, \dots, v_n\}$ be a basis of W.
Let $a = (a_1, \dots, a_n) \in (\mathbb{R}^n, \| \cdot \|_2)$ be such that $\|a\|_2 \leqslant 1$.
Then $\Gamma_n(a) = a_1 v_1 + \cdots + a_n v_n$, and

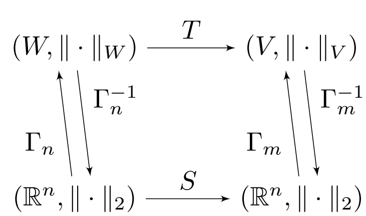
$$\|\Gamma_n(a)\|_W \leqslant \|a_1 v_1\|_2 + \cdots + \|a_n v_n\|_2 \leqslant \|v_1\|_2 a_1 + \cdots + \|v_n\|_2 a_n \implies \|\Gamma_n\| \leqslant \sum_{i=1}^{n} \|v_i\|_2$$

This shows that Γ_n is bounded.

Now let $S = \{a \in \mathbb{R}^n \mid \|a\| = 1\}$.
As S is compact, $\Gamma_n(S)$ is compact on W.
The map $w \to \|w\|_2$ is continuous on W, so $\Gamma_n(S)$ has an element w_0 of least norm.
However, $\|w_0\|_2 > 0$.
Let $\alpha = \min\{\|\Gamma_n(a) \mid a \in S\} > 0$.
If $w \in W$ and $\|w\|_2 \leqslant \alpha$, then $\|\Gamma_n^{-1}(w)\|_2 \leqslant 1$ and further $\|\Gamma_n^{-1}\| \leqslant \frac{1}{\alpha}$. ■

Theorem 4.2.5. If $(W, \| \cdot \|_W)$ is n-dimensional and $(V, \| \cdot \|_V)$ is m-dimensional and $T : V \to W$ is linear, then T is bounded.

<u>*Proof:*</u> Consider the following diagram.



The map $S = \Gamma_m^{-1} \circ T \circ \Gamma_n : \mathbb{R}^n \to \mathbb{R}^m$ is necessarily bounded and continuous.
Therefore the map $T = \Gamma_m \circ S \circ \Gamma_n^{-1}$ is similarly bounded and continuous. ■

Corollary 4.2.6. For the spaces as above, if the map $T : W \to (V, \| \cdot \|_V)$ is linear, then it is bounded.

Remark 4.2.7. As Γ_n is a homeomorphism, $(W, \| \cdot \|_W) \simeq (\mathbb{R}^n, \| \cdot \|_2)$, for W n-dimensional. Moreover, if $w \in W$, then

$$\|\Gamma_n^{-1}(w)\|_2 \leqslant \|\Gamma_n^{-1}\| \|w\|_W \implies \|w\|_W = \|\Gamma_n(\Gamma_n^{-1}(w))\|_W \leqslant \|\Gamma_n\| \|\Gamma_n^{-1}(w)\|_2 \leqslant \|\Gamma_n\| \|\Gamma_n^{-1}\| \|w\|_W$$

This means that there exist $\alpha, \beta \in W$ such that for all $w \in W$,

$$\alpha \|\Gamma_n^{-1}(w)\|_2 \leqslant \|w\|_W \leqslant \beta \|\Gamma_n^{-1}(w)\|_2$$

Hence we come to the following conclusions.

· $U \subset W$ is open iff $\Gamma_n^{-1}(U)$ is open in $\mathbb{R}^n$
· $A \subset W$ is bounded iff $\Gamma_n^{-1}(A)$ is bounded
· $F \subset W$ is closed iff Γ_n^{-1} is closed

This implies that $\{w_n\} \subset A$ is Cauchy iff $\{\Gamma_n^{-1}(w_n)\}$ is Cauchy, which in turn implies $(W, \|\cdot\|_W)$ is complete.

Remark 4.2.8. If $(V, \|\cdot\|_V)$ is a normed linear space and $W \subset V$ is a finite-dimensional subspace, then W is closed. Further, if $(X, \|\cdot\|_X)$ is an infinite-dimensional Banach space and $\{U_\alpha\}_{\alpha \in I}$ is a basis of X, then I is uncountable.

Remark 4.2.9. If $(V, \|\cdot\|_V)$ is a normed linear space and $W \subset V$ is a proper subspace of V, then $\mathsf{int}(V) = \emptyset$.

4.3 The Weierstrass approximation theorem

Proposition 4.3.1. The set of polynomials is dense in $C[a, b]$.

To prove this, we first show how to normalize functions, so that we are only considering the interval $[0, 1]$, and $f(0) = f(1) = 0$. Let $\varphi : [a, b] \to [0, 1]$ be defined by

$$\varphi(x) = \frac{x - a}{b - a}$$

Then φ, φ^{-1} are continuous bijections, and the linear isometric operator $\Gamma : C[a, b] \to C[0, 1]$ with $\Gamma(f)(t) = f \circ \varphi(t)$ normalizes all functions to $[0, 1]$. Let $\Upsilon : C[0, 1] \to C[0, 1]$ be defined by

$$\Upsilon(f)(x) = f(x) - ((f(1) - f(0))x + f(0))$$

Then Υ is a linear isometric operator and enforces that $f(0) = f(1) = 0$. Hence we may assume that all $f \in C[0, 1]$ with $f(0) = f(1) = 0$.

Lemma 4.3.2. For any $x \in [0, 1]$ and $n \in \mathbb{N}$, $(1 - x^2)^n \geqslant 1 - nx^2$.

Proof: Let $h(x) = (1 - x^2)^n - 1 + nx^2$, so $h(0) = 0$, and

$$h'(x) = 2nx(1 - (1 - x^2)^{n-1}) \geqslant 0$$

Hence $h(x)$ is always increasing, and the result follows. ■

Theorem 4.3.3. [Approximation theorem - Weierstrass]
For $f \in C[a, b]$ there exists a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on $[a, b]$

Proof: First we assume that $[a, b] = [0, 1]$ and $f(0) = f(1) = 0$.
From here we may extend f to a uniformly continuous function on $\mathbb{R}$, by $f(x) = 0$ for all $x \notin [0, 1]$.
For each $n \in \mathbb{N}$, define

$$Q_n(t) = c_n(1 - t^2)^n, \quad \int_{-1}^{1} Q_n(t)dt = 1$$

Then we have that

$$\begin{aligned}
\int_{-1}^{1}(1-x^2)^n dx &= 2\int_0^1 (1-x^2)^n dx \\
&\geqslant 2\int_0^{1/\sqrt{n}} (1-nx^2)dx \\
&= 2\left(\frac{2}{3\sqrt{n}}\right) \\
&= \frac{4}{3}\cdot\sqrt{n} \\
&> \frac{1}{\sqrt{n}}
\end{aligned}$$

Hence $c_n \leqslant \sqrt{n}$.
For $n \in \mathbb{N}$, let

$$\begin{aligned}
P_n(x) &= \int_{-1}^{1} f(x+t)Q_n(t)dt \\
&= \int_{-x}^{1-x} f(x+t)Q_n(t)dt \\
&= \int_0^1 f(u)Q_n(u-x)du
\end{aligned}$$

Above the change $u = x+t$ was made.
Now apply The Liebniz rule to get

$$\frac{d^{2n+1}}{dx^{2n+1}}P_n(x) = \int_0^1 f(u)\frac{\partial^{2n+1}}{\partial x^{2n+1}}Q_n(u-x)du = 0$$

Hence P_n is a polynomial of degree at most $2n$.
So if $\delta \in (0,1)$, then $c_n(1-x^2)^n \leqslant \sqrt{n}(1-\delta^2)^n$ on $[-1,-\delta]\cup[\delta,1]$.
Let $\epsilon > 0$ and $\delta \in (0,1)$ such that if $|t| < \delta$, then $|f(x+t)-f(x)| < \frac{\epsilon}{2}$ for all $x \in \mathbb{R}$.
Let $x \in [0,1]$, so then

$$\begin{aligned}
|P_n(x)-f(x)| &= \left|\int_{-1}^{1}(f(x+t)-f(x))Q_n(t)dt\right| \\
&\leqslant \int_{-1}^{1}|f(x+t)-f(x)|Q_n(t)dt \\
&\leqslant \int_{-1}^{-\delta}|f(x+t)-f(x)|Q_n(t)dt + \int_{-\delta}^{\delta}|f(x+t)-f(x)|Q_n(t)dt + \int_{\delta}^{1}|f(x+t)-f(x)|Q_n(t)dt \\
&\leqslant 2\|f\|_\infty\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} + 2\|f\|_\infty\sqrt{n}(1-\delta^2)^n \\
&= 4\|f\|_\infty\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2}
\end{aligned}$$

Let n be large enough so that $4\|f\|_\infty\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2}$, and the result will follow. ■

Corollary 4.3.4. Let $f \in C[0,1]$, and assume that $\int_0^1 f(t)t^n dt = 0$ for all $n \in \mathbb{N}\cup\{0\}$. Then $f = 0$.

Corollary 4.3.5. $C[a,b]$ is separable.

Proof: Define the following sets:

$$P_n = \{p(x) \in C[a,b] \mid p(x) \text{ is a polynomial of degree } n \text{ over } \mathbb{R}\}$$
$$Q_n = \{p(x) \in C[a,b] \mid p(x) \text{ is a polynomial of degree } n \text{ over } \mathbb{Q}\}$$

Note that $\overline{Q_n} = P_n$.
Since $\bigcup_{n=1}^{\infty} P_n$ is dense in $C[a,b]$, we have that $\bigcup_{n=1}^{\infty} Q_n$ is dense in $C[a,b]$. ■

Proposition 4.3.6. The collection of nowhere-differentiable functions in $C[0,1]$ is residual.

Lemma 4.3.7. For each $n \in \mathbb{N}$ define

$$\mathcal{F}_n = \{f(x) \in C[0,1] \mid \exists\, x_0 \in [0, 1-\tfrac{1}{n}] \text{ such that } |f(x_0+h) - f(x_0)| \leqslant nh \;\forall\; 0 < h < 1 - x_0\}$$

Then $\mathcal{F}_n$ is closed and nowhere dense in $C[0,1]$.

Proof: Let $n \in \mathbb{N}$ and $\{f_k\} \subset \mathcal{F}_n$ with $f_k \to f$ in $\|\cdot\|_\infty$.
For each k, we can find $x_k \in [0, 1-\frac{1}{n}]$ with $|f_k(x_k+h) - f_k(x_k)| \leqslant nh$ for all $0 < h < 1 - x_k$.
WLOG, assume that, by choosing a subsequence if necessary, $x_k \to x_0 \in [0, 1-\frac{1}{n}]$.
Let $0 < h < 1 - x_0$ and $\epsilon > 0$.
We can choose $N_0 \in \mathbb{N}$ such that if $k \geqslant N_0$, then $0 < h < 1 - x_k$, and $N_1 > N_0$, such that if $k \geqslant N_1$, then

1. $|f(x_0) - f(x_k)| < \frac{\epsilon}{4}$
2. $|f(x_0+h) - f(x_k+h)| < \frac{\epsilon}{4}$
3. $\|f_k - f\|_\infty < \frac{\epsilon}{4}$

Now note that

$$\begin{aligned}
|f(x_0) - f(x_0+h)| &\leqslant |f(x_0) - f(x_k)| + |f(x_k) - f_k(x_k)| + |f_k(x_k) - f_k(x_k+h)| + |f_k(x_k+h) - f(x_k+h)| + |f(x_k+h) - f(x_0+h)| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + nh + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&= nh + \epsilon
\end{aligned}$$

Since ϵ was arbitrary, $|f(x_0) - f(x_0+h)| \leqslant nh$, hence $f \in \mathcal{F}_n$, and $\mathcal{F}_n$ is closed.

Now let $f \in C[0,1]$ and $\epsilon > 0$.
Then we can find a polynomial $p(x)$ with $\|f - p\|_\infty < \frac{\epsilon}{2}$.
Define functions

$$\varphi(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2 - x & \text{if } x \in [1,2] \end{cases} \qquad g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x) \qquad F(x) = g|_{[0,1]}$$

Choose $\alpha > 0$ such that $\|\alpha F\|_\infty < \frac{\epsilon}{2}$.
Then $p(x) + \alpha F(x) \in \mathcal{F}_n^c$ for each n, so $\|f - (p + \alpha F)\|_\infty < \epsilon$. ■

Theorem 4.3.8. [Banach, Mazurkiewicz]
Let $ND[0,1]$ be the set of continuous nowhere-differentiable functions on $[0,1]$. Then $ND[0,1]$ is residual in $(C[0,1], \|\cdot\|_\infty)$.

Proof: Let $f \in C[0,1]$ be differentiable at $x_0 \in C[0,1]$.
Then $f \in \mathcal{F}_n$ for some $n \in \mathbb{N}$, and hence

$$ND[0,1] \supset \left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)^c \implies ND[0,1]^c \subset \underbrace{\bigcup_{n=1}^{\infty} \mathcal{F}_n}_{\text{1st category}}$$

■

4.4 The Stone-Weierstrass theorem

Definition 4.4.1. Let (X, d) be a compact metric space. Then $\Phi \subset C(X)$ is termed point separating iff whenever $x, y \in X$ with $x \neq y$, there exists $\varphi \in \Phi$ with $\varphi(x) \neq \varphi(y)$.

Proposition 4.4.2. If (X, d) is a compact metric space, then $C(X)$ is point separating.

Proof: Let $a, b \in X$ with $a \neq b$.
Let $f(x) = d(a, x)$, so $f(a) = 0$ and $f(b) \neq 0$. ■

Remark 4.4.3. Suppose that $\Phi \subset C(X)$ is such that $f(x) = f(y)$ for all $f \in \Phi$. If $g \in \overline{\Phi}$, then $g(x) = g(y)$ as well. Hence if Φ is dense in $C(X)$, it must be point-separating.

Definition 4.4.4. A linear subspace $\Phi \subset C(X)$ is termed a lattice iff for each $f, g \in \Phi$,

i. $f \vee g \in \Phi$, for $(f \vee g)(x) = \max\{f(x), g(x)\}$
ii. $f \wedge g \in \Phi$, for $(f \wedge g)(x) = \min\{f(x), g(x)\}$

Remark 4.4.5. First note that the subspace of all piecewise linear functions is a lattice. Further, note that condition **ii.** above is superfluous, as

$$f \wedge g = -(-f \vee -g)$$

Next, observe that condition **i.** above may be replaced with simply having the absolute value of any function in the space, as

$$f \vee g = \frac{1}{2}(f + g - |f - g|)$$

Theorem 4.4.6. [Stone, Weierstrass - Lattice version]
Let (X, d) be a compact metric space, and Φ a linear subspace of $C(X)$ such that

i. Φ is point separating
ii. $1 \in \Phi$
iii. If $f, g \in \Phi$, then $f \vee g \in \Phi$ (i.e. Φ is a lattice)

Then $\overline{\Phi} = C(X)$.

Proof: Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$ with $x \neq y$.
Then there exists $g \in \Phi$ with $g(x) = \alpha$ and $g(y) = \beta$.
Since Φ is point separating, there is $\varphi \in \Phi$ with $\varphi(x) \neq \varphi(y)$, so define

$$g(t) = \alpha + (\beta - \alpha)\frac{\varphi(t) - \varphi(x)}{\varphi(y) - \varphi(x)}$$

This function satisfies the conditions.
So now let $f \in C(X)$ and let $\epsilon > 0$.

Step 1: Fix $x \in X$.
We know that for all $y \in X$, we can find $\varphi_{x,y} \in \Phi$ with $\varphi_{x,y}(x) = f(x)$, and $\varphi_{x,y}(y) = f(y)$.
For each $y \in X$, $\varphi_{x,y}(t) - f(t)$ is continous, with $\varphi_{x,y}(y) - f(y) = 0$.
We can find $\delta_y > 0$ such that if $z \in B(y, \delta_y)$, then $|\varphi_{x,y}(z) - f(z)| < \epsilon$.
Then $\{B(y, \delta_y)\}_{y \in X}$ is a cover of X.
Then there exists $\{y_1, \ldots, y_n\}$ with $\{B(y_i, \delta_{y_i})\}_{i=1}^n$ covering X.
Let $\varphi_x(t) = \varphi_{x,y_1} \vee \varphi_{x,y_2} \vee \cdots \vee \varphi_{x,y_n}$.
Then $\varphi_x \in \Phi$ with $\varphi_x(x) = x$, and $f(z) - \epsilon < \varphi_x(z)$ for all $z \in X$.

Step 2: Note that $\varphi_(t) - f(t)$ is continuous and $\varphi_x(x) - f(x) = 0$.
So for each $x \in X$, we can find $\delta_x > 0$ such that $z \in B(x, \delta_x)$, and hence $|\varphi_x(z) - f(z)| < \epsilon$.
As before, we can find $\{x_1, \ldots, x_m\}$ with $\{B(x_j, \delta_{x_j})\}_{j=1}^m$ a cover of X.
Let $\varphi = \varphi_{x_1} \wedge \varphi_{x_2} \wedge \cdots \wedge vp_{x_m} \in \Phi$.
Then for any $z \in X$, we have that $f(z) - \epsilon < \varphi(z) < f(z) + \epsilon$. ■

Definition 4.4.7. A linear space $\Phi \subset C(X)$ is termed an <u>algebra</u> iff $f, g \in \Phi$ implies $fg \in \Phi$, for $(fg)(x) = f(x)g(x)$.

Remark 4.4.8. Let $\Phi \subset C(X)$ be an algebra. Then $\overline{\Phi}$ is also an algebra. To see this, let $f, g \in \Phi$, and $\{f_n\}, \{g_n\} \subset \overline{\Phi}$ with $f_n \to f$ and $g_n \to g$. Then

$$\begin{aligned}\|fg - f_n g_n\|_\infty &\leqslant \|fg - f_n g\|_\infty + \|f_n g - f_n g_n\|_\infty \\ &\leqslant \|g\|_\infty \|f - f_n\|_\infty = \|f_n\|_\infty \|g - g_n\|_\infty \\ &\leqslant \|g\|_\infty \|f - f_n\|_\infty + M\|g - g_n\|_\infty \\ &\xrightarrow{n\to\infty} 0\end{aligned}$$

The M above is such that $\|f_n\|_\infty \leqslant M$ for all $n \in \mathbb{N}$.

Theorem 4.4.9. [Stone, Weierstrass - Subalgebra version]
Let (X, d) be a compact metric space, and Φ a linear subspace of $C(X)$ such that

- **i.** Φ is point separating
- **ii.** $1 \in \Phi$
- **iii.** If $f, g \in \Phi$, then $fg \in \Phi$

Then $\overline{\Phi} = C(X)$.

<u>*Proof:*</u> Since $\overline{\Phi}$ also satisfies the above conditions, assume that Φ is closed.
Let $f \in \Phi$ and $\epsilon > 0$.
Then f is bounded, so there exists $M > 0$ such that $f(x) \in [-M, M]$ for all $x \in X$.
By the Weierstrass approximation theorem, we can find $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ with

$$||t| - p(t)| < \epsilon \ \forall\ t \in [-M, M]$$

Let $p \circ f = a_0 + a_1 f + \cdots + a_n f^n \in \Phi$, so then

$$||f(x)| - (p \circ f)(x)| < \epsilon \ \forall\ x \in X$$

Hence $|f| \in \Phi$.
As $f \vee g = \frac{1}{2}(f + g + |f - g|)$, Φ is a lattice and is dense in $C(X)$.
As Φ is closed, $\Phi = C(X)$. ■

Example 4.4.10.
$\cdot$ $X = [a, b]$
A function $f \in C(X)$ is piecewise linear (or polynomial) iff there is a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ such that on $[t_{i-1}, t_i]$, $f(x) = m_i x + b_i$ (or $f(x) = p_i(x)$ a polynomial). Moreover, if we let

$$\Phi = \{f \in C[a, b] \mid f \text{ is piecewise linear (or polynomial)}\}$$

then Φ is a lattice, and hence $\overline{\Phi} = C[a, b]$.

$\cdot$ $X = [0, 1] \times [0, 1]$
Then if we let

$$\Phi = \left\{ h = \sum_{i=1}^{n} f_i(x) g_i(x) \mid f_i, g_i \in C[0, 1], n \in \mathbb{N} \right\}$$

Then Φ is a subalgebra, and hence $\overline{\Phi} = C([0, 1] \times [0, 1])$.

Definition 4.4.11. Let (X, d) be a compact metric space. Then define

$$\begin{aligned}C(X, \mathbb{C}) &= \{f : X \to \mathbb{C} \mid f \text{ is continuous}\} \\ \|f\|_\infty &= \max_{x \in X}\{|f(x)|\}\end{aligned}$$

Remark 4.4.12. For $f \in C(X, \mathbb{C})$ with $f = \text{Re}(f) + i\text{Im}(f)$, we have that

$$\text{Re}(f) = \frac{f + \overline{f}}{2} \qquad \text{Im}(f) = \frac{f - \overline{f}}{2}$$

Where $\overline{f} = \text{Re}(f) - i\text{Im}(f)$.

Theorem 4.4.13. [Stone, Weierstrass - Complex version]
Let (X, d) be a compact metric space, and Φ a self-adjoint linear subspace of $C(X, \mathbb{C})$ such that

- **i.** Φ is point separating
- **ii.** $1 \in \Phi$
- **iii.** $f, g \in \Phi$ implies $fg \in \Phi$

Then $\overline{\Phi} = C(X, \mathbb{C})$.

Example 4.4.14. Let $X = \Pi = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, and $\phi : [0, 2\pi] \to \Pi$ given by $\varphi(\theta) = e^{i\pi\theta} = \cos(\theta) + i\sin(\theta)$. Define a metric on $[0, 2\pi)$ by the arc-length on Π. Then

$$C(\Pi) \simeq C([0, 2\pi)^*) = \{f \in C([0, 2\pi]) \mid f(0) = f(\pi)\}$$

which is the set of 2π-periodic functions. Then define the point separating algebra algebra of $C([0, 2\pi)^*)$ to be

$$\text{Trig}(\Pi) = \text{span}\{1, \cos(nx), \sin(mx) \mid m, n \in \mathbb{N}\} = \left\{h = \sum_{k=0}^{n} a_x \cos(kx) + b_k \sin(kx)\right\}$$
$$\text{Trig}_{\mathbb{C}}(\Pi) = \text{span}\{e^{in\theta} \mid n \in \mathbb{Z}\}$$

4.5 The Arzela-Ascoli theorem

Remark 4.5.1. Given $\mathcal{F} \subset C(X)$, for (X, d) a compact metric space, when is $\mathcal{F}$ compact?

Definition 4.5.2. Given a metric space (X, d), a set $A \subset X$ is termed relatively compact iff $\overline{A}$ is compact.

Note that an A is totally bounded iff $\overline{A}$ is totally bounded, it follows that $\mathcal{F} \subset C(X)$ is relatively compact iff $\mathcal{F}$ is totally bounded.

Definition 4.5.3. Let (X, d) be a compact metric space with $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is termed equicontinuous at x_0 iff for each $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, x_0) < \delta$, then $|f(x) - f(x_0)| < \epsilon$ for all $f \in \mathcal{F}$.

Similarly, $\mathcal{F}$ is termed equicontinuous iff $\mathcal{F}$ is equicontinuous at all $x_0 \in X$.

Further, $\mathcal{F}$ is termed uniformly equicontinuous iff for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, if $d(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$.

Example 4.5.4. Let $\mathcal{F} = \{x^n\}_{n=1}^{\infty}$. Then $\mathcal{F}$ is equicontinuous on $[0, \frac{1}{2}]$, but not on $[0, 1]$.

Remark 4.5.5. It follows from the definition that if $\mathcal{F}$ is finite, then it is uniformly equicontinuous.

Proposition 4.5.6. Let (X, d) be a compact metric space, and $\mathcal{F} \subset C(X)$ equicontinuous. Then $\mathcal{F}$ is uniformly equicontinuous.

Proof: Let $\epsilon > 0$.
For each $x_0 \in X$, there exists $\delta_{x_0} > 0$ such that if $d(x, x_0) < \delta_{x_0}$, then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$.
This holds for all $f \in \mathcal{F}$.
Note that $\{B(x_0, \delta_{x_0})\}_{x_0 \in X}$ is a cover of X.
Hence this cover has a Lebesgue number $\delta_1 > 0$, so choose $0 < \delta_0 < \delta_1$.

Hence for any $y \in X$ there is some $x_0 \in X$ so that $B(y, \delta_0) \subset B(x_0, \delta_{x_0})$.
So for $z \in B(y, \delta_0)$, we have that

$$\begin{aligned} |f(y) - f(z)| &\leqslant |f(y) - f(x_0)| + |f(x_0) - f(z)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

■

Definition 4.5.7. Let (X, d) be a compact metric space with $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is termed pointwise bounded iff for each $x_0 \in X$, $\{f(x_0) \mid f \in \mathcal{F}\}$ is bounded.

Proposition 4.5.8. Let (X, d) be a compact metric space and $\mathcal{F} \subset C(X)$ equicontinuous and pointwise bounded. Then $\mathcal{F}$ is uniformly bounded.

Proof: As $\mathcal{F}$ is uniformly equicontinuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < 1$.
The above holds for all $f \in \mathcal{F}$.
Let $\{x_1, \ldots, x_n\}$ be a δ-net for X, and suppose that $|f(x_i)| < M_i$ for each $f \in \mathcal{F}$.
Let $M_0 = \max\{M_1, \ldots, M_n\}$, so if $x \in X$, then there exists x_i with $d(x, x_i) < \delta$ implying

$$|f(x)| \leqslant |f(x) - f(x_i)| + |f(x_i)| \leqslant 1 + M_0$$

■

Theorem 4.5.9. [Arzela, Ascoli]
Let (X, d) be a compact metric space with $\mathcal{F} \subset (C(X), \|\cdot\|_\infty)$. Then equivalently:
1. $\mathcal{F}$ is relatively compact
2. $\mathcal{F}$ is equicontinuous and pointwise bounded

Proof: **1.** $\Rightarrow$ **2.** As $\mathcal{F}$ is relatively compact, it is bounded.
Hence it is both pointwise and totally bounded.
Let $\epsilon > 0$.
So there exists a finite $\frac{\epsilon}{3}$-net $\{f_1, \ldots, f_n\} \subset \mathcal{F}$ of $\mathcal{F}$.
Since $\{f_1, \ldots, f_n\}$ is uniformly equicontinuous, there exists $\delta > 0$ with $d(x, y) < \delta$ implying

$$|f_i(x) - f_i(y)| < \frac{\epsilon}{3} \ \forall\, x, y \in X \text{ and } i = 1, \ldots, n$$

Let $f \in \mathcal{F}$.
For $d(x, y) < \delta$, there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\|f - f_{i_0}\|_\infty < \frac{\epsilon}{3}$, so

$$\begin{aligned} |f(x) - f(y)| &\leqslant |f(x) - f_{i_0}(x)| + |f_{i_0}(x) - f_{i_0}(y)| + |f_{i_0}(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Hence $\mathcal{F}$ is equicontinuous.

2. $\Rightarrow$ **1.** Since (X, d) is compact, $\mathcal{F}$ is uniformly equicontinuous and uniformly bounded.
Hence there is $M > 0$ such that $f(x) \in [-M, M]$ for each $f \in \mathcal{F}$ and $x \in X$.
Let $\epsilon > 0$.
Let $P = \{-M = y_0 < y_1 < \cdots < y_m = M\}$ be a partition of $[-M, M]$, with

$$\|P\| = \max_j \{y_i - y_{i-1}\} < \frac{\epsilon}{3}$$

As $\mathcal{F}$ is uniformly equicontinuous, there exists $\delta > 0$ with $d(x,z) < \delta$ implies $|f(x) - f(z)| < \frac{\epsilon}{3}\ \forall\ f \in \mathcal{F}$.
Let $\{x_1, \ldots, x_n\}$ be a δ-net for X, and

$$\Phi = \{\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,m\}\}$$

Then $|\Phi| = m^n = \ell < \infty$, so for each $k = 1, \ldots, \ell$, let

$$\begin{aligned}\mathcal{F}_k &= \{f \in \mathcal{F} \mid f(x_i) \in [y_{\sigma_k(i)-i}, y_{\sigma_k(i)}]\ \forall\ i = 1, \ldots, n\} \\ \mathcal{F} &= \bigcup_{k=1}^{\ell} \mathcal{F}_k\end{aligned}$$

If possible, choose $f_k \in \mathcal{F}_k$ for every k.
Then for $f \in \mathcal{F}$, $f \in \mathcal{F}_k$ for some k, and for $w \in X$, $w \in B(x_i, \delta)$ for some i, so

$$\begin{aligned}|f(w) - f_k(w)| &\leqslant |f(w) - f(x_i)| + |f(x_i) - f_k(x_i)| + |f_k(x_i) - f_k(w)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

Hence $\|f - f_k\|_\infty < \epsilon$, and $\{f_k\}$ is an ϵ-net for $\mathcal{F}$. ■

Definition 4.5.10. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be metric spaces. Then a linear map $\Gamma : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is termed <u>compact</u> if $\Gamma(B_X[0,1]) \subset Y$ is relatively compact.

Theorem 4.5.11. [PEANO]
Let $D \subset \mathbb{R}^2$ be open and f continuous on D. Then for $(x_0, y_0) \in D$, the differential equation $y' = f(x,y)$ has a local solution passing through the point (x_0, y_0).

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