**Compact course notes**

# PURE MATH 352, WINTER 2011

*Complex Analysis*

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# 1 Preliminaries

**Definition 1.0.1.** For $p \in \mathbb{C}$ and $r \in \mathbb{R}$ with $r > 0$, the <u>(open) disk</u> of center $p$ and radius $r$ is given by $D_p(r) = \{z \in \mathbb{C} \,\big|\, |z-p| < r\}$.

**Definition 1.0.2.** A set $\Omega \in \mathbb{C}$ is termed <u>open</u> if for every $p \in \Omega$, there exists $r > 0$ such that $D_p(r) \subseteq \Omega$.

**Remark 1.0.3.** These are some common geometric shapes:
- An annulus: $A = \{z \,\big|\, r < |z-p| < R\}$ with $0 \leqslant r < R \leqslant \infty$
- A half-plane: $H = \{x + iy \,\big|\, x > 0\}$

**Definition 1.0.4.** For $f : \Omega \to \mathbb{C}$ and $p \in \mathbb{C}$, we have that $f(z) \to w$ as $z \to p$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - w| < \epsilon$ when $0 < |z-p| < \delta$ and $z \in \mathbb{C}$.

**Definition 1.0.5.** If a function $f : \mathbb{C} \to \mathbb{C}$ is such that for all real scalars $c$, $f(cz) = cf(z)$, then $f$ is <u>$\mathbb{R}$-linear</u>. If this holds for all complex scalars $c$, then $f$ is <u>$\mathbb{C}$-linear</u>.

**Remark 1.0.6.** A differentiable function $f : \Omega \to \mathbb{C}$ compared with $f : \Omega \to \mathbb{R}^2$ for $\Omega$ an open subset of $\mathbb{C}$ and $\mathbb{R}^2$ respectively, is stronger in the first case, due to complex linearity.

# 2 Complex differentiability

## 2.1 Derivatives

**Definition 2.1.1.** If $f : I \to \mathbb{R}$ is a function defined on an open interval $I$ with $p \in I$, then $f$ is (complex) <u>differentiable</u> at $p$ with derivative $m$ provided $\frac{f(x)-f(p)}{x-p} \to m$ as $x \to p$. This function is then <u>holomorphic</u>.

**Definition 2.1.2.** A function is <u>entire</u> if it is holomorphic on the whole complex plane.

**Remark 2.1.3.** Now let $f : \Omega \to \mathbb{R}^2$ and $p \in \Omega$. Then we say that $f$ is real differentiable at $p = \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)$ with derivative $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ provided for $\epsilon > 0$ there exists $\delta > 0$ such that if $0 \leqslant \| \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) - \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) \| < \delta$, then $\| f\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) - f\left(\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\right) - \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \left(\begin{smallmatrix} x-s \\ y-t \end{smallmatrix}\right) \| < \epsilon \| \left(\begin{smallmatrix} x-s \\ y-t \end{smallmatrix}\right) \|$.

**Proposition 2.1.4.** Let $f : \Omega \to \mathbb{C}$ be a function. Then $f$ has complex derivative $w = a + ib$ at some $p = s + it \in \Omega$ provided $f$ has a real derivative at $p$ of the type $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$

**Remark 2.1.5.** If $f : \Omega \to \mathbb{R}^2$ is real differentiable at $p \in \Omega$, then its derivative at $p$ is $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \begin{bmatrix} \frac{\partial u}{\partial x}\Big|_p & \frac{\partial u}{\partial y}\Big|_p \\ \frac{\partial v}{\partial x}\Big|_p & \frac{\partial v}{\partial y}\Big|_p \end{bmatrix}$
This is termed the Jacobian, with $f = u + iv$.

**Proposition 2.1.6.** $*$ [Cauchy-Riemann]
Let $f : \Omega \to \mathbb{C}$ be a function such that $f$ is complex differentiable at $p = s + it$ with complex derivative $w = a + ib$. Then the partial derivative of $u$ and $v$ exist with

$$\frac{\partial u}{\partial x}\Big|_p = \frac{\partial v}{\partial y}\Big|_p \quad \text{and} \quad \frac{\partial u}{\partial y}\Big|_p = -\frac{\partial v}{\partial x}\Big|_p$$

**Remark 2.1.7.** If $f : \Omega \to \mathbb{C}$ is differentiable at $p \in \Omega$, then $f$ is continuous at $p$.

**Proposition 2.1.8.** If for $f = u + iv : \Omega \to \mathbb{C}$ all partials of $u$ and $v$ exist and are continuous and the Cauchy-Riemann equations hold on $\Omega$, then $f$ is complex differentiable on $\Omega$.

**Proposition 2.1.9.** [Chain rule]
Let $\Omega, \Gamma$ be open sets in $\mathbb{C}$ with $f : \Omega \to \mathbb{C}$ and $g : \Gamma \to \mathbb{C}$. For $p \in \Omega$ if $f'(p)$ and $g'(f(p))$ exist, then $g \circ f : \Omega \to \mathbb{C}$ is differentiable at $p$ with $(g \circ f)'(p) = g'(f(p))f'(p)$.

## 2.2 Exponentials and logarithms

**Definition 2.2.1.** The exponential function on $\Omega$ is given below. Its range is $\Omega \setminus \{0\}$.

$$f(z) = f(x+iy) = e^x \cos(y) + ie^x \sin(y) = e^{x+iy} = e^z$$

**Remark 2.2.2.** Every complex number $z \neq 0$, $z = x+iy$ is of the form $z = e^{s+i\theta}$ for some $s, \theta \in \mathbb{R}$.
$s = \frac{1}{2}\ln(x^2+y^2)$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
In this case, there is exactly one $\theta \in (-\pi/2, \pi/2)$ such that $z = e^{s+i\theta}$

## 2.3 Curves and regions

**Definition 2.3.1.** A curve in $\Omega$ is a continuous function $\alpha : [a,b] \to \Omega$.

**Definition 2.3.2.** The trajectory of a curve $\alpha$ is the image set of the function, and is denoted by $\alpha^*$.

**Definition 2.3.3.** Given two curves $\alpha : [a,b] \to \Omega$ running from $p$ to $q$, and $\beta : [c,d] \to \Omega$ running from $q$ to $r$, replace $\beta$ with $\gamma : [b,e] \to \Omega$ that also runs from $q$ to $r$ and has the same trajectory as $\beta$.

Then a splice of the two curves is the curve $\delta : [a,e] \to \Omega$ where $\delta(t) = \begin{cases} \alpha(t) & t \in [a,b] \\ \gamma(t) & t \in [b,e] \end{cases}$

**Definition 2.3.4.** A curve $\alpha = x+iy : [a,b] \to \Omega$ is termed smooth whenever its complex derivative $\alpha'(t) = x'(t) + iy'(t)$ exists and is continuous on $[a,b]$.

**Proposition 2.3.5.** If $\alpha : [a,b] \to \Omega$ is smooth and $f : \Omega \to \mathbb{C}$ is holomorphic, then for all $t \in [a,b]$, $(f \circ \alpha)'(t) = f'(\alpha(t))\alpha'(t)$.

**Definition 2.3.6.** Then curve $\alpha : [a,b] \to \Omega$ is termed piecewise-smooth if there exists a partition of $[a,b] : a = a_0 < a_1 < \cdots < a_n = b$ such that $\alpha$ is smooth on each of $[a_{j-1}, a_j]$ for all $j = \{1, \ldots, n\}$.

**Definition 2.3.7.** An open set $\Omega$ is connected when $\Omega$ is not the disjoint union of two nonempty open subsets of $\mathbb{C}$.

**Proposition 2.3.8.** $*$ A set $\Omega$ is connected if and only if for all $p, q \in \Omega$, there exists a piecewise-smooth curve $\alpha$ that runs from $p$ to $q$.

**Definition 2.3.9.** A region is a connected open set. Hereinafter $\Omega$ always refers to a region.

**Proposition 2.3.10.** For $\Omega$ a region and $f : \Omega \to \mathbb{C}$ holomorphic, if $f' = 0$ on $\Omega$, then $f$ is constant on $\Omega$.

**Definition 2.3.11.** A function $f : \Omega \to \mathbb{C}$ is termed a primitive for a function $g$ if $f' = g$ on $\Omega$.

## 2.4 Power series

**Definition 2.4.1.** A sequence $z_n \in \mathbb{C}$ converges if for any $\epsilon > 0$ , there exists $n_0 \in \mathbb{N}$ and $p \in \mathbb{C}$ such that $|z_n - p| < \epsilon$ for all $n \geqslant n_0$. In this case, $z_n$ converges to $p$.

**Definition 2.4.2.** A sequence $z_n \in \mathbb{C}$ is Cauchy if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|z_m - z_n| < \epsilon$ for all $n \geqslant n_0$.

**Proposition 2.4.3.** A sequence converges if and only if it is Cauchy.

**Proposition 2.4.4.** If $\sum_{k=1}^{\infty} |z_n|$ converges in $\mathbb{R}$, then $\sum_{k=1}^{\infty} z_n$ converges in $\mathbb{C}$.

**Proposition 2.4.5.** $*$ For any power series $\sum_{k=1}^{\infty} a_n z^n$, there exists $R \in \mathbb{R}^*$ (where $\mathbb{R}^* = \mathbb{R} \cup \infty$) with $R \geqslant 0$ such that the power series converges absolutely if $|z| < R$ and diverges if $|z| > R$.

In this case, $R = \text{lub}\{r \geqslant 0 \,\big|\, |z_n| r^n \text{ is bounded}\}$.

**Definition 2.4.6.** The $R$ described above is termed the <u>radius</u> of the sequence.

**Theorem 2.4.7.**$*$ [Hadamard]

For a series $\displaystyle\sum_{k=1}^{\infty} a_n z^n$, if $\limsup |a_n|^{1/n}$ is nonzero and finite, then $R = \dfrac{1}{\limsup |a_n|^{1/n}}$

**Proposition 2.4.8.** Let $\sum_{k=1}^{\infty} a_n z^n$ be a power series with radius $R$.

**1.** If $\limsup |a_n|^{1/n} < \infty$, then $R = \dfrac{1}{\limsup |a_n|^{1/n}}$.

**2.** If $\limsup |a_n|^{1/n} = \infty$, then $R = 0$.

**3.** If $\limsup |a_n|^{1/n} = 0$, then $R = \infty$.

**Theorem 2.4.9.** $*$ [Differentiation theorem, pt.1]

The series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and and its differentiated series $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ have equal radii.

**Theorem 2.4.10.** [Differentiation theorem, pt.2]

Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ have radius $R > 0$. Then for every $p \in D_0(R)$, $f'(p) = \displaystyle\sum_{n=1}^{\infty} n a_n p^{n-1}$.

**Definition 2.4.11.** A function is termed <u>entire</u> if the radius of its power series is $R = \infty$.

# 3 Integrability

## 3.1 Fundamentals

**Definition 3.1.1.** The integral of a curve $\alpha = x + iy : [a, b] \to \mathbb{C}$ given by $t \mapsto x(t) + iy(t)$ is defined as:

$$\int_a^b \alpha = \int_a^b x(t)dt + i \int_a^b y(t)dt$$

**Remark 3.1.2.** The integration of curves has the properties of complex linearity and triangle inequality:

$$\int_a^b (\alpha \pm \beta) = \int_a^b \alpha \pm \int_a^b \beta \quad \text{and} \quad \left| \int_a^b \alpha \right| \leqslant \int_a^b |\alpha|$$

**Theorem 3.1.3.** The integral of a continuous function $f : \Omega \to \mathbb{C}$ along a smooth curve $\gamma : [a, b] \to \Omega$ is:

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt$$

**Proposition 3.1.4.** If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma : [a, b] \to \mathbb{C}$ is smooth, then

$$\begin{aligned} \left| \int_\gamma f \right| &\leqslant \|f\|_{\gamma^*} \int_a^b |\gamma'(t)|dt \\ &= \|f\|_{\gamma^*} \, length(\gamma) \end{aligned}$$

where $\|f\|_{\gamma^*} = \max\limits_{t \in [a,b]} \{|f(\gamma(t))|\}$.

**Theorem 3.1.5.** The series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and and its integrated series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ have equal radii.

## 3.2 Paths

**Definition 3.2.1.** If $\gamma : [a, b] \to \mathbb{C}$ is smooth and $h : [c, d] \to [a, b]$ has a continuous derivative and $h(c) = a$, $h(d) = b$, then $\gamma \circ h : [c, d] \to \mathbb{C}$ is termed a <u>reparametrization</u> of $\gamma$.

**Proposition 3.2.2.** If $f : \Omega \to \mathbb{C}$ is continuous and $\gamma \circ h$ is a reparametrization of $\gamma$, then $\int_{\gamma \circ h} f = \int_\gamma f$.

**Definition 3.2.3.** If $\gamma : [a, b] \to \mathbb{C}$ is smooth, then its <u>opposite</u> is $\tilde{\gamma} : [a, b] \to \mathbb{C}$ given by $t \mapsto \gamma(a + b - t)$. Then $\int_{\tilde{\gamma}} f = -\int_\gamma f$ for $f$ continuous.

**Definition 3.2.4.** A <u>path</u> in $\Omega$ that runs from $p$ to $q$ is a set of smooth curves

$$\{\gamma_1 : [a_1, b_1] \to \Omega, \gamma_2 : [a_2, b_2] \to \Omega, \ldots, \gamma_n : [a_n, b_n] \to \Omega\}$$

with $\gamma(b_{i-1}) = \gamma(a_i)$ for all $i$, and $\gamma(a_1) = p$, $\gamma(a_n) = q$.

**Proposition 3.2.5.** If $f : \Omega \to \mathbb{C}$ is continuous and $f$ has a primitive on $\Omega$ and $\gamma$ is a path from $p$ to $q$ in $\Omega$:

$$\int_\gamma f = g(q) - g(p)$$

for $g$ the primitive of $f$ on $\Omega$.

**Definition 3.2.6.** The integrals of $f : \Omega \to \mathbb{C}$ continuous are termed <u>path independent</u> if for any two paths $\gamma, \beta \in \Omega$, both running from $p$ to $q$ in $\Omega$, $\int_\gamma f = \int_\beta f$.

**Proposition 3.2.7.** Suppose $f : \Omega \to \mathbb{C}$ is continuous and $\int_\gamma f = 0$ for all closed paths $\gamma$ in $\Omega$. Then $f$ has path independent integrals.

## 3.3 Cauchy

**Proposition 3.3.1.** $*$ If $f : \Omega \to \mathbb{C}$ is continuous with $\int_\gamma f = 0$ for all closed paths $\gamma \in \Omega$, then $f$ has a primitive in $\Omega$.

**Remark 3.3.2.** The line from $p$ to $q$ is denoted by $\overline{pq}$. Explicitly, $\overline{pq} : [0, 1] \to \mathbb{C}$ is defined by $t \mapsto (1-t)p+tq$.

**Definition 3.3.3.** For points $p, q, r$, the set of lines $\overline{pq}, \overline{qr}, \overline{rp}$ is termed a <u>triangle</u>, denoted by $\partial \triangle (p, q, r)$.

**Theorem 3.3.4.** [Cauchy-Goursat]
If $f$ is holomorphic on a region $\Omega$ and $\triangle$ is any triangle completely inside $\Omega$, then $\int_{\partial\triangle} f = 0$.

**Theorem 3.3.5.** If $f$ is holomorphic on a region $\Omega$, except (possibly) on a finite set of points, on which $f$ remains continuous, and $\triangle$ is any triangle inside $\Omega$, then $\int_{\partial\triangle} f = 0$.

**Definition 3.3.6.** A region is <u>convex</u> from a point $p$ in $\Omega$ if for all $z \in \Omega$, the segment $\overline{pz}$ is in $\Omega$.

**Theorem 3.3.7.** $*$ If $\Omega$ is convex from a point $p$, and $f : \Omega \to \mathbb{C}$ is holomorphic except (possibly) at a single point, then $f$ has a primitive on $\Omega$, or equivalently, $\int_\gamma f = 0$ for all closed paths $\gamma$ in $\Omega$.

**Theorem 3.3.8.** [Cauchy integral formula]
If the following hold:

$$\left.\begin{array}{l} \Omega \text{ is convex from a point } p \\ \gamma \text{ is a closed path in } \Omega \\ f \text{ is holomorphic on } \Omega \\ p \in \Omega \setminus \gamma^* \end{array}\right\} \quad \text{then} \quad \int_\gamma \frac{f(z)}{z-p} dz = \int_\gamma \frac{f(p)}{z-p} dz$$

**Definition 3.3.9.** If $\gamma$ is a closed path in $\mathbb{C}$ and $w \notin \gamma^*$, define the <u>index</u> of $\gamma$ around $w$ to be

$$\mathrm{ind}_\gamma(w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-w} dz$$

**Theorem 3.3.10.** If $\gamma$ is a closed path in $\mathbb{C}$, then $\operatorname{ind}_\gamma(w) \in \mathbb{Z}$ for all $w \notin \gamma^*$.

**Theorem 3.3.11.** The function $\operatorname{ind}_\gamma : \mathbb{C} \setminus \gamma^* \to \mathbb{Z}$ is continuous.

**Proposition 3.3.12.** On the unbounded component of $\mathbb{C} \setminus \gamma^*$, $\operatorname{ind}_\gamma = 0$.

**Theorem 3.3.13.** If the following hold:

$$\left.\begin{array}{l} \gamma \text{ is a closed path in } \Omega \\ \operatorname{ind}_\gamma(w) = 0 \ \forall \ w \notin \Omega \\ f \text{ is holomorphic on } \Omega \\ z \in \Omega \setminus \gamma^* \end{array}\right\} \quad \text{then} \quad \operatorname{ind}_\gamma(z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

## 3.4 Implications of the Cauchy integral formula

**Definition 3.4.1.** A sequence of functions $f_n : A \to \mathbb{C}$ tends to a function $f : A \to \mathbb{C}$ uniformly on $A$ if $\|f_n - f\|_A \to 0$ as $n \to \infty$.

**Remark 3.4.2.** If $f_n : A \to \mathbb{C}$ are continuous for all $n$ and $f_n \to f$ uniformly, then $f$ is continuous.

**Remark 3.4.3.** If $\gamma$ is a path in $\Omega$ and $f_n, f$ are defined and continuous on $\gamma^*$ for all $n \in \mathbb{N}$, and if $f_n \to f$ uniformly as $n \to \infty$ on $\gamma^*$, then $\int_\gamma f_n \to \int_\gamma f$ as $n \to \infty$.

**Theorem 3.4.4.** [Weierstrass M-test]
Let $A \subseteq \mathbb{C}$ and $f_n : A \to \mathbb{C}$ be a sequence of functions. Let $M_n \geqslant 0$ with $\displaystyle\sum_{n=1}^{\infty} M_n$ a convergent series and $\|f_n\|_A \leqslant M_n$ for all $n$. Then $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on $A$.

**Theorem 3.4.5.** $*$ If $f$ is holomorphic on $\Omega$ and $R > 0$ with $D_p(R) \subseteq \Omega$ for some $p \in \Omega$, then for all $z \in D_p(R)$ there exists a power series $\displaystyle\sum_{n=0}^{\infty} a_n (z-p)^n = f(z)$.

**Corollary 3.4.6.** A holomorphic function is equivalent to an analytic function.

**Corollary 3.4.7.** If $f$ is holomorphic on $\Omega$ and $D_p(r) \subseteq \Omega$ with $\gamma : [0, 2\pi] \to D_p(r)$ the circle of radius $r$, then $f^{(n)}(p)$ exists for all $n$ with $\displaystyle\frac{f^{(n)}(p)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-p)^{n+1}} dz$.

**Corollary 3.4.8.** Every holomorphic function has a primitive on some disk.

**Corollary 3.4.9.** For $f$ analytic in $\Omega$ and $D_p(R) \subseteq \Omega$ with $M$ an upper bound for $|f|$ on $D_p(R)$,

$$\left| \frac{f^{(n)}(p)}{n!} \right| \leqslant \frac{M}{R^n}$$

**Theorem 3.4.10.** $*$ [Liouville]
If a function $f$ is entire and bounded by some $M$ on $\mathbb{C}$, then $f$ is constant on $\mathbb{C}$.

**Theorem 3.4.11.** $*$ [Fundamental theorem of algebra]
If $f$ is a polynomial over $\mathbb{C}$ and $\deg(f) \geqslant 1$, then $f$ has at least one root in $\mathbb{C}$.

**Definition 3.4.12.** A point $p \in A \subseteq \mathbb{C}$ is termed a cluster / limit / accumulation point of $A$ if for any $\epsilon > 0$ there exists $q \in D_p(\epsilon) \subseteq A$ with $q \neq p$.

Otherwise, there exists $\epsilon > 0$ such that $D_p(\epsilon) \cap A = \{p\}$, and $p$ is termed isolated.

**Proposition 3.4.13.** For $f : \Omega \to \mathbb{C}$ non-constant and analytic, every $p \in \Omega$ such that $f(p) = 0$ is isolated.

**Theorem 3.4.14.** [Identity theorem]
If $f, g$ are analytic on $\Omega$ and $f(z) = g(z)$ for all $z \in A \subseteq \Omega$ with at least 1 cluster point in $A$, $f = g$ on $\Omega$.

**Theorem 3.4.15.** $*$ [MORERA]
If $f$ is continuous on $\Omega$ and $\int_{\partial\triangle} f = 0$ for every triangle $\triangle \subset \Omega$, then $f$ is analytic on $\Omega$.

**Definition 3.4.16.** A sequence of functions $f_n : \Omega \to \mathbb{C}$ is said to converge uniformly on compact sets to a function $f : \Omega \to \mathbb{C}$ if for every compact compact set $A \subseteq \Omega$, $f_n \to f$ uniformly on $A$.

**Theorem 3.4.17.** $*$ If $f_n : \Omega \to \mathbb{C}$ are analytic and $f_n \to f$ uniformly on compact subsets of $\Omega$, then $f$ is analytic on $\Omega$.

**Theorem 3.4.18.** [MAXIMUM MODULUS PRINCIPLE]
If $f$ is non-constant and analytic on $\Omega$, then $|f|$ has no local maximum on $\Omega$.

**Corollary 3.4.19.** For $f : \Omega \to \mathbb{C}$ analytic and non-constant and $\Omega \supseteq A$ compact, $|f|$ attains its maximum over $A$ on the boundary of $A$.

# 4 Meromorphic functions

**Definition 4.0.1.** A meromorphic function $f : \Omega \to \mathbb{C}$ is a holomorphic function that (possibly) has non-essential singularities on a set of measure zero $S \subset \Omega$.

## 4.1 Singularities

**Definition 4.1.1.** Define the punctured disk of radius $r > 0$ centered at $p \in \mathbb{C}$ be described by $D_p^*(r) = \{z \in \mathbb{C} \,\big|\, r < |z - p| < r\}$.

**Definition 4.1.2.** If $f$ is analytic on $\Omega$, a singularity of $f$ at $p$ is termed removable if $\lim\limits_{z\to p} [f(z)]$ exists, so that $f^*(z) = \begin{cases} f(z) & z \neq p \\ \lim\limits_{z\to p} [f(z)] & z = p \end{cases}$ is analytic on $\Omega$.

**Proposition 4.1.3.** If $f$ is analytic on $\Omega$ with singularity at $p$, and $f$ is bounded on some $D_p^*(r) \subseteq \Omega$, then $f$ has a removable singularity at $p$.

**Definition 4.1.4.** For $f : \Omega \to \mathbb{C}$ holomorphic and $\epsilon > 0$ , $f(D_p^*(\epsilon))$ is not dense in $\mathbb{C}$ if there exists $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| \geqslant \delta$ for all $z \in D_p^*(\epsilon) \subseteq \Omega$.

**Proposition 4.1.5.** For $f$ a non-constant and entire function, $f(\mathbb{C})$ is dense in $\mathbb{C}$.

**Theorem 4.1.6.** $*$ [CASORATI-WEIERSTRASS]
Let $f$ be analytic on $\Omega$ with non-removable singularity at $p$. Then only one of the following conditions hold:
- **i.** For every $D_p^*(\epsilon) \subseteq \Omega$ with $\epsilon > 0$ , $f(D_p^*(\epsilon))$ is dense in $\mathbb{C}$.
- **ii.** There exists a positive integer $m$ such that $(z - p)^m f(z)$ has a removable singularity at $p$.

**Definition 4.1.7.** With respect to the above definition, in case **i**, $p$ is termed an essential singularity of $f$. In case **ii**, $p$ is termed a pole of $f$.

**Theorem 4.1.8.** [PICARD]
For $f : \Omega \to \mathbb{C}$ analytic with $p \in \Omega$ an essential singularity, for any $\epsilon > 0$ either
- **i.** $f(D_p^*(\epsilon)) = \mathbb{C}$
- **ii.** $f(D_p^*(\epsilon)) = \mathbb{C} \setminus \{w\}$ for some $w \in \mathbb{C}$

Moreover, for every $y \in f(D_p^*(\epsilon))$, there are infinitely many $z \in D_p^*(\epsilon)$ such that $f(z) = y$.

**Proposition 4.1.9.** If $f$ is analytic on $\Omega$ with a pole at $p$, then there exist:

**i.** an analytic function $h(z)$ on $\Omega \cup \{p\}$

**ii.** an integer $m \geqslant 1$

**iii.** scalars $b_1, b_2, \ldots, b_m$ with $b_m \neq 0$ such that $f(z) = h(z) + \underbrace{\frac{b_1}{z-p} + \frac{b_2}{(z-p)^2} + \cdots + \frac{b_m}{(z-p)^m}}_{\text{the principal part of } f \text{ at } p}$

**Definition 4.1.10.** With respect to the above, $b_1$ is termed the residue of $f$ at $p$ and is denoted $\operatorname{res}(f,p)$. The integer $m$ is termed the order of the pole $p$.

**Theorem 4.1.11.** For $\gamma$ a closed curve in $\Omega$ and $p \in \Omega$ with $f$ holomorphic on $\Omega$, $\displaystyle\int_\gamma f = b_1 2\pi i \cdot \operatorname{ind}_\gamma(p)$

**Proposition 4.1.12.** If $f$ has a singularity at $p$ and $f$ is analytic on $\Omega$, then $p$ is a pole of order 1 $\iff$ $(z-p)f(z) \to b \neq 0$ and finite. Moreover, $b$ is the residue of $f$ at $p$.

**Remark 4.1.13.** Let $f, g : \Omega \to \mathbb{C}$ be analytic on $\Omega \setminus \{p\}$ with $g(p) = 0$ but $f(p) \neq 0$ and $g'(p) \neq 0$. Then $f/g$ has a pole of order 1 at $p$ with $\operatorname{res}(f/g, p) = f(p)/g'(p)$.

## 4.2 Cauchy's theorem

**Definition 4.2.1.** A chain in $\Omega$ is a finite list of closed paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ in $\Omega$ denoted $\gamma = \gamma_1 + \gamma_2, + \cdots + \gamma_n$. The image of this chain is defined as $\gamma^* = \displaystyle\bigcup_{i=1}^n \gamma_i^*$.

**Definition 4.2.2.** Two chains $\alpha, \beta$ are homologous in $\Omega$ if $\operatorname{ind}_\alpha(w) = \operatorname{ind}_\beta(w)$ for all $w \notin \Omega$.

**Remark 4.2.3.** If $\Omega$ is convex from a point, then every chain in $\Omega$ is homologous to 0 in $\Omega$.

**Theorem 4.2.4.** [Cauchy]
For $f : \Omega \to \mathbb{C}$ holomorphic, a chain $\gamma$ is homologous to 0 on a region $\Omega \subseteq \mathbb{C}$ if and only if

$$\int_\gamma f = 0$$

in which case the Cauchy integral formula holds:

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)\operatorname{ind}_\gamma(z) \qquad \text{for every } z \in \Omega \setminus \gamma^*$$

**Corollary 4.2.5.** For $f : \Omega \to \mathbb{C}$ analytic and chains $\alpha$ homologous to $\beta$ in $\Omega$,

$$\int_\alpha f = \int_\beta f$$

**Theorem 4.2.6.** $*$ [Residue theorem]
Let $f$ be analytic on $\Omega$ and $p_1, p_2, \ldots, p_n$ be poles of $f$ and $\gamma$ a chain in $\Omega$ that is homologous to 0 in $\Omega \cup \{p_1, p_2, \ldots, p_n\}$. Then

$$\int_\gamma f = 2\pi i \left( \sum_{j=1}^n \operatorname{ind}_\gamma(p_j) \operatorname{res}(f, p_j) \right)$$

**Proposition 4.2.7.** A function $f$ at $p$ has a pole of order $n$ if and only if $\lim\limits_{z \to p} [(z-p)^n f(z)] = b \neq 0$ and finite. Then

$$\operatorname{res}(f,p) = \lim_{z \to p} \left[ \frac{d^{n-1}}{dz^{n-1}} (z-p)^n f(z) \right] \frac{1}{(n-1)!}$$

**Definition 4.2.8.** A region $\Omega$ is simply connected if $\operatorname{ind}_\gamma(w) = 0$ for every closed path $\gamma \in \Omega$ and all $w \notin \Omega$.

**Proposition 4.2.9.** If $f : \Omega \to \mathbb{C}$ is holomorphic for $\Omega$ simply connected, then $f$ has a primitive on $\Omega$.

## 4.3 Fourier series

**Definition 4.3.1.** For $f : \mathbb{R} \to \mathbb{R}$ continuous and $\omega \in \mathbb{R}$, the Fourier transform of $f$ is

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx \\ &= \lim_{r\to\infty}\left[\int_{-r}^{r} f(x)\cos(\omega x)dx + i\int_{-r}^{r} f(x)\sin(\omega x)dx\right]\end{aligned}$$

**Proposition 4.3.2.** For $f$ analytic on $\Omega = \mathbb{C} \setminus \{p_1, p_2, \ldots, p_m\}$ and poles $p_i$ of $f$ with $\mathrm{Im}(p_i) \neq 0$ for all $i$, if $|zf(z)| \leqslant M$ when $|z| \geqslant R$ for some values $M, R$, then $\hat{f}(\omega)$ exists for all $\omega > 0$ and

$$\hat{f}(\omega) = 2\pi i \sum_{\substack{i=1 \\ \mathrm{Im}(p_i)>0}}^{m} \mathrm{res}(f(z)e^{i\omega z}, p_i)$$

**Proposition 4.3.3.** Let $f$ be analytic on $\Omega = \mathbb{C} \setminus \{p_1, p_2, \ldots, p_m\}$ where the $p_i$'s are the poles of $f$. Then if $|z|^\lambda |f(z)| \leqslant M$ for some $M$ and $\lambda > 1$ and all $z \in \Omega$ with $|z| \geqslant |z_0|$ for some $|z_0|$ large enough, then

$$\int_{\gamma_N} f(z)\pi\frac{\cos(\pi z)}{\sin(\pi z)}dz \to 0 \qquad \text{as} \qquad N \to \infty$$

where $\gamma_N$ is the rectangular path of width $2N + 1$ and height $2N$ centered at the origin for $N \in \mathbb{N}$.

Moreover, in this case

$$\sum_{\substack{n=-\infty \\ n\neq p_i}}^{\infty} f(n) = -\sum_{i=1}^{m} \mathrm{res}\left(f(z)\pi\frac{\cos(\pi z)}{\sin(\pi z)}, p_i\right)$$

## 4.4 Rouché

**Proposition 4.4.1.** Let $f : \Omega \to \mathbb{C}$ be analytic and non-constant. Then $f'/f$ has poles at the zeros of $f$. If $p \in \Omega$ is a zero of $f$ of multiplicity $m \geqslant 1$, then $\mathrm{res}(f'/f, p) = m$.

**Definition 4.4.2.** A path $\gamma$ has interior of for all $w \notin \gamma^*$, $\mathrm{ind}_\gamma(w) \in \{0, 1\}$.

Then the interior is defined to be the set $\{w \,|\, \mathrm{ind}_\gamma(w) = 1\}$.

**Remark 4.4.3.** A path with interior is equivalent to a simple closed path.

**Proposition 4.4.4.** If the following hold:

$$\left.\begin{array}{l} f \text{ is analytic and non-constant on } \Omega \\ \gamma \text{ is a simple closed path in } \Omega \\ \gamma \text{ is homologous to 0 in } \Omega \\ p \in \Omega \text{ is a zero of } f \text{ with multiplicity } m_p \end{array}\right\} \quad \text{Then} \quad \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{\substack{\text{all } p \\ \mathrm{ind}_\gamma(p)=1}} \mathrm{res}(f'/f, p) = \sum_{\substack{\text{all } p \\ \mathrm{ind}_\gamma(p)=1}} m_p$$

**Proposition 4.4.5.** $*$ Let $\alpha, \beta : [0, 1] \to \mathbb{C}$ be closed paths such that $|\alpha(t) - \beta(t)| < |\beta(t)|$ for all $t \in [0, 1]$. Then $\mathrm{ind}_\alpha(0) = \mathrm{ind}_\beta(0)$.

**Theorem 4.4.6.** [Rouché]
If the following hold:

$$\left.\begin{array}{l} f, g \text{ are analytic on } \Omega \\ \gamma : [0,1] \to \Omega \text{ is a simple closed path} \\ \gamma \text{ is homologous to 0 in } \Omega \\ |g(z) - f(z)| < |f(z)| \text{ for all } z \in \gamma^* \end{array}\right\}$$

Then $f$ and $g$ have the same number of zeros in the interior of $\gamma$, counting multiplicities.

**Theorem 4.4.7.** For $f : \Omega \to \mathbb{C}$ analytic and non-constant, $f(\Gamma)$ is open for every open set $\Gamma \subseteq \Omega$.

## 4.5 Laurent

**Definition 4.5.1.** The <u>annulus</u> around $p \in \mathbb{C}$ is the set of points in the open set between an inner radius $r$ and outer radius $R$, denoted $A_p(r,R) := \{z \,\big|\, r < |z-p| < R\}$, with $0 \leqslant r < R \leqslant \infty$.

**Definition 4.5.2.** The <u>Laurent series</u> of a function $f$ defined on an annulus $A_p(r,R)$ is the series of coefficients for integer powers of $z-p$, when $f$ is expressed as

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-p)^n} + \sum_{m=0}^{\infty} a_m (z-p)^m$$

**Proposition 4.5.3.** Suppose $f$ has a Laurent expansion in a region $\Omega$, and $p \in \Omega$ is a singularity of $f$. Then

**1.** $p$ is removable $\iff$ $b_i = 0$ for all $i$
**2.** $p$ is a pole $\iff$ $b_i = 0$ for infinitely many $i$
**3.** $p$ is essential $\iff$ $b_i = 0$ for finitely many $i$

**Theorem 4.5.4.** [Laurent]
If $f$ is analytic on $A_p(r,R)$, then $f$ has a Laurent expansion on $A_p(r,R)$.

**Remark 4.5.5.** Note that Laurent expansions are unique.

## 4.6 Univalency

**Definition 4.6.1.** Let $f : \Omega \to \mathbb{C}$ be analytic and one-to-one. Then $f$ is termed <u>univalent</u>.
This is equivalent to stating that $f(z_1) = f(z_2) \iff z_1 = z_2$ for all $a_1, z_2 \in \Omega$.

**Theorem 4.6.2.** Let $f : \Omega \to \mathbb{C}$ be univalent, and let $f(\Omega) = \Gamma$. Then $\Gamma$ is an open region also, and the inverse function $g : \Gamma \to \Omega$ is also analytic. Moreover, if $p \in \Omega$ with $f(p) = q$, then

$$g'(q) = \frac{1}{f'(p)}$$

# 5 Selected proofs

**Proposition 2.1.6.** [Cauchy-Riemann] Let $f : \Omega \to \mathbb{C}$ be a function such that $f$ is complex differentiable at $p = s + it$ with complex derivative $w = a + ib$. Then the partial derivatives of $u$ and $v$ exist with

$$\frac{\partial u}{\partial x}\bigg|_p = \frac{\partial v}{\partial y}\bigg|_p \quad \text{and} \quad \frac{\partial u}{\partial y}\bigg|_p = -\frac{\partial v}{\partial x}\bigg|_p$$

**Proof:** Suppose that $f = u + iv$ is complex differentiable at $p$ with derivative $w$.
Let $p = s + it$, $z = x + iy$, $w = a + ib$.
Then $\frac{f(z)-f(p)}{z-p} \to w$ as $z \to p$.
Equivalently: for every $\epsilon > 0$ , there exists $\delta > 0$ such that if $z \in \Omega$ and $0 < |z - p| < \delta$, then

$$|f(z) - f(p) - w(z - p)| < \epsilon |z - p|$$

Equivalently: for $\epsilon > 0$ , there exists $\delta > 0$ such that if $x + iy \in \Omega$ and $0 < |(x + iy) - (s + it)| < \delta$, then

$$|f(x + iy) - f(s + it) - (a + ib)((x - s) + i(y - t))| < \epsilon |(x - s) + i(y - t)|$$

Equivalently: for $\epsilon > 0$ , there is $\delta > 0$ such that if $\binom{x}{y} \in \Omega$ and $0 \leqslant \left\| \binom{x}{y} - \binom{s}{t} \right\| < \delta$, then

$$\left\| f\begin{pmatrix} x \\ y \end{pmatrix} - f\begin{pmatrix} s \\ t \end{pmatrix} - \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{pmatrix} x - s \\ y - t \end{pmatrix} \right\| < \epsilon \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} s \\ t \end{pmatrix} \right\|$$

Then the derivative of $f$ at $p$ is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
For $f = u + iv$ with $u, v$ real functions, the derivative of $f$ at $p$ is given by the 2-dimensional Jacobian,

$$\begin{bmatrix} \frac{\partial u}{\partial x}\Big|_p & \frac{\partial u}{\partial y}\Big|_p \\ \frac{\partial v}{\partial x}\Big|_p & \frac{\partial v}{\partial y}\Big|_p \end{bmatrix}$$

Since the two matrices must be equal, we have

$$\frac{\partial u}{\partial x}\bigg|_p = a = \frac{\partial v}{\partial y}\bigg|_p \quad \text{and} \quad \frac{\partial u}{\partial y}\bigg|_p = -b = -\frac{\partial v}{\partial x}\bigg|_p$$

■

**Proposition 2.3.8.** A set $\Omega$ is connected if and only if for all $p, q \in \Omega$, there exists a piecewise-smooth curve $\alpha$ that runs from $p$ to $q$.

**Proof:** Suppose that there does not exist a piecewise-smooth curve from $p$ to $q$ in $\Omega$.
Let $A = \{z \,|\, \exists \text{ a pw-sc in } \Omega \text{ from } p \text{ to } z\}$ and $B = \{z \,|\, \nexists \text{ a pw-sc in } \Omega \text{ from } p \text{ to } z\}$.
Clearly $p \in A$, $q \in B$ and $A \cap B = \emptyset$ with $A \cup B = \Omega$, a disjoint union.

---

Pick $w \in A$ and take $r > 0$ such that $D_w(r) \subseteq \Omega$.
Then there exists a pw-sc $\alpha$ in $\Omega$ from $p$ to $w$.
For each $z \in D_w(r)$ there exists a pw-sc $\beta$ to $w$ in $D_w(r)$ and hence in $\Omega$.
Splice $\alpha$ with $\beta$ to get a pw-sc from $p$ to $z$ inside $\Omega$.
Therefore $A$ is open.

---

Take $w \in B$ and $r > 0$ such that $D_w(r) \subseteq \Omega$.
Suppose that $D_w(r) \nsubseteq B$ and there exists $z \in D_w(r)$ with $z \in A$.
Then there exists a pw-sc $\alpha$ from $p$ to $z$ in $\Omega$.
Note there also exists a pw-sc $\beta$ from $z$ to $w$ in $D_w(r)$ and hence in $\Omega$.
Splice $\alpha$ with $\beta$ to get a pw-sc from $p$ to $w$ in $\Omega$.
Then $w \notin B$, a contradiction.
Therefore $D_w(r) \subseteq B$ and $B$ is open.
Thus $\Omega$ is the disjoint union of non-empty open sets. ■

**Proposition 2.4.5.** For any power series $\sum_{k=1}^{\infty} a_n z^n$, there exists $R \in \mathbb{R}^*$ (where $\mathbb{R}^* = \mathbb{R} \cup \infty$) with $R \geqslant 0$ such that the power series converges absolutely if $|z| < R$ and diverges if $|z| > R$.

**Proof:** Consider $B = \{r \geqslant 0 \,\big|\, |a_n| r^n \text{ is a bounded sequence } \}$.
Let $R = \text{lub}\{S\}$.
If $|z| > R$, then $|a_n z^n| = |a_n||z^n| = |a_n||z|^n$ is not bounded.
If $|z| < R$, then there exists $r \in B$ such that $|z| < r < R$.
Hence all $|a_n| r^n \leqslant$ some bound $M$.
Then $|a_n z^n| = |a_n| r^n \left|\frac{z^n}{r^n}\right| \leqslant M \left|\frac{z}{r}\right|^n$ and $\left|\frac{z}{r}\right| < 1$.
Then by the geometric series test, $\sum M \left|\frac{z}{r}\right|^n$ converges.
By the comparison test, $\sum a_n z^n$ also converges. ∎

**Theorem 2.4.7.** [Hadamard]
For a series $\sum_{k=1}^{\infty} a_n z^n$, if $\limsup |a_n|^{1/n}$ is nonzero and finite, then $R = \dfrac{1}{\limsup |a_n|^{1/n}}$

**Proof:** Suppose that $0 < L = \limsup |a_n|^{1/n} < \infty$.
It will be shown that $|a_n| r^n$ is bounded when $r < \frac{1}{L}$ and unbounded when $r > \frac{1}{L}$.

$$\begin{aligned} \text{Then } 0 < r < \tfrac{1}{L} &\Longrightarrow 0 < L < \tfrac{1}{r} \\ &\Longrightarrow |a_n|^{1/n} < \tfrac{1}{r} \text{ eventually} \\ &\Longrightarrow |a_n|^{1/n} < 1 \text{ eventually} \\ &\Longrightarrow |a_n| r^n \text{ is bounded} \end{aligned}$$

$$\begin{aligned} \text{Next, let } 0 < \tfrac{1}{L} < r &\Longrightarrow 0 < \tfrac{1}{r} < L \\ &\Longrightarrow \exists\ s \text{ such that } \tfrac{1}{r} < s < L \\ &\Longrightarrow \tfrac{1}{r} < s < |a_n|^{1/n} \text{ infinitely often} \\ &\Longrightarrow 1 < (sr)^n < |a_n| r^n \text{ infinitely often} \\ &\Longrightarrow |a_n| r^n \text{ is unbounded.} \end{aligned}$$

Therefore the radius of the series is $\frac{1}{L}$. ∎

**Theorem 2.4.9.** [Differentiation theorem, pt.1]
The series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and and its differentiated series $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ have equal radii.

**Proof:** The former series has radius $\dfrac{1}{\limsup |a_n|^{1/n}}$.

The latter series has radius
$$\begin{aligned} \frac{1}{\limsup |n a_n|^{1/n}} &= \frac{1}{\limsup |n|^{1/n} |a_n|^{1/n}} \\ &= \frac{1}{\limsup |n|^{1/n} \limsup |a_n|^{1/n}} \\ &= \frac{1}{\limsup |a_n|^{1/n}} \end{aligned}$$

Therefore the two series have equal radii. ∎

**Proposition 3.3.1.** If $f : \Omega \to \mathbb{C}$ is continuous with $\int_\gamma f = 0$ for all closed paths $\gamma \in \Omega$, then $f$ has a primitive in $\Omega$.

**Proof:** Pick $p \in \Omega$.
For $z \in \Omega$, let $g(z) = \int_\gamma f$ where $\gamma$ is any path that runs from $p$ to $z$.
Note that all such $\gamma$ from $p$ to $z$ give the same value for $g(z)$.
Check that $g'(w) = f(w)$ for all $w \in \Omega$.
It will be shown that for $z$ arbitrarily close to $w$, there exists a function $\varphi(z)$ with

$$|g(z) - g(w) - f(w)(z-w)| \leqslant \varphi(z)|z-w| \text{ with } \varphi(z) \to 0 \text{ as } z \to p$$

Pick $r > 0$ so that $D_w(r) \subseteq \Omega$.
For $z \in D_w(r)$, let $\ell$ be the straight line from $w$ to $z$.
Take any path $\gamma \in \Omega$ from $p$ to $w$.
Then $\gamma + \ell$ is a path in $\Omega$ from $p$ to $z$.
Then

$$\begin{aligned}
g(z) - g(w) - f(w)(z-w) &= \int_{\gamma+\ell} f - \int_\gamma f - f(w)(z-w) \\
&= \int_\gamma f + \int_\ell f - \int_\gamma f - f(w)(z-w) \\
&= \int_\ell f - f(w)(z-w) \\
&= \int_\ell (f - f(w)) \\
&= \int_\ell (f(\mu) - f(w)) d\mu
\end{aligned}$$

So then $\left| \int_\ell (f - f(w)) \right| \leqslant \|f - f(w)\|_\ell |z-w|$.
Now check that $\|f - f(w)\|_\ell \to 0$ as $z \to w$.
Let $\epsilon > 0$ .
Need $\delta > 0$ such that $\|f - f(w)\|_\ell < \epsilon$ when $|z-w| < \delta$.
So we need $\delta > 0$ such that $|f(w + t(z-w)) - f(w)| < \epsilon$ for all $t \in [0,1]$ when $|z-w| < \delta$.
Since $f$ is continuous at $w$, we get $\delta > 0$ such that $|z-w| < \delta \Longrightarrow |f(z) - f(w)| < \epsilon$.
In particular, for every $t \in [0,1]$ and $|z-w| < \delta$, $|w + t(z-w) - w| = t|z-w| < \delta$.
For $\mu = w + t(z-w)$, we get $|f(\mu) - f(w)| < \epsilon$.
That is, $\|f - f(w)\|_\gamma \to 0$ as $z \to w$. ∎

**Theorem 3.3.7.** If $\Omega$ is convex from a point $p$, and $f : \Omega \to \mathbb{C}$ is holomorphic except (possibly) at a single point, then $f$ has a primitive on $\Omega$, or equivalently, $\int_\gamma f = 0$ for all closed paths $\gamma$ in $\Omega$.

**Proof:** Let $\Omega$ be convex from a point $p$.
Define $g(w) = \int_{\overline{pw}} f$ for all $w \in \Omega$.
Take $r > 0$ such that $D_w(r) \subseteq \Omega$.
For every $z \in D_w(r)$, the triangle $\triangle(p, w, z) \subset \Omega$, since $\Omega$ is convex from $p$.
From Cauchy-Goursat, we have that

$$\int_{\partial\triangle(p,w,z)} f = \int_{\overline{pw}} f + \int_{\overline{wz}} f + \int_{\overline{zp}} f = 0$$

Reversing path endpoints and rearranging,

$$\int_{\overline{pz}} f - \int_{\overline{pw}} f = \int_{\overline{wz}} f$$

Then for every $z \in D_w(r)$,

$$\begin{aligned}
|g(z) - g(w) - f(w)(z - w)| &= \left| \int_{\overline{pz}} f - \int_{\overline{pw}} f - f(w)(z - w) \right| \\
&= \left| \int_{\overline{wz}} f - f(w)(z - w) \right| \\
&= \left| \int_{\overline{wz}} f(\zeta) - f(w) d\zeta \right| \\
&\leqslant \|f - f(w)\|_{\overline{wz}} |z - w| \\
\left| \frac{g(z) - g(w)}{z - w} - f(w) \right| &\leqslant \|f - f(z)\|_{\overline{wz}}
\end{aligned}$$

Since the right hand size goes to zero as $z \to w$, so does the left hand side.
Thus $g$ is the primitive of $f$ on $\Omega$. ∎

**Theorem 3.4.5.** If $f$ is holomorphic on $\Omega$ and $R > 0$ with $D_p(R) \subseteq \Omega$ for some $p \in \Omega$, then for all $z \in D_p(R)$ there exists a power series $\displaystyle\sum_{n=0}^{\infty} a_n(z - p)^n = f(z)$.

**Proof:** Pick $0 < r < R$.
Let $\gamma : [0, 2\pi] \to D_p(R)$ be given by $t \mapsto p + re^{it}$.
By the Cauchy integral theorem, for every $z \in D_p(r)$ we have $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$
For $\zeta \in \gamma^*$ and $z \in D_p(r)$,

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - p - (z - p)} = \frac{1}{\zeta - p} \left( \frac{f(\zeta)}{1 - \left(\frac{z-p}{\zeta-p}\right)} \right) = \frac{f(\zeta)}{\zeta - p} \sum_{n=0}^{\infty} \left( \frac{z - p}{\zeta - p} \right)^n = \sum_{n=0}^{\infty} \frac{(z - p)^n}{(\zeta - p)^{n+1}} f(\zeta)$$

The above summation is correct, as $|z - p| < |\zeta - p|$ for all $z \in D_p(r)$.
Observe that

$$\left\| \frac{(z - p)^n}{(\zeta - p)^{n+1}} f(\zeta) \right\|_{\zeta\in\gamma^*} = \frac{|z - p|^n}{r^n} \cdot \frac{\|f\|_{\gamma^*}}{r}$$

Then $\displaystyle\sum_{n=0}^{\infty} \left( \frac{|z - p|}{r} \right)^n \frac{\|f\|_{\gamma^*}}{r}$ converges, as $\frac{|z-p|}{r} < 1$.

By the Weierstrass M-test, $\displaystyle\sum_{n=0}^{\infty} \frac{(z - p)^n}{(\zeta - p)^{n+1}} f(\zeta)$ converges uniformly on $\gamma^*$.

Due to this, the integral can be passed on to the series terms to get

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-p)^{n+1}}d\zeta(z-p)^n$$

Notice that $a_n = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta-p)^{n+1}}d\zeta$ does not depend on $z \in D_p(r)$.

Moreover, $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta-p)^{n+1}}d\zeta = \frac{f^{(n)}(p)}{n!}$.

Thus the integrals do not depend on $r$.

So for any $z \in D_p(R)$, pick $r$ such that $0 < |z-p| < r < R$ to get $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z-p)^n$ for all $z \in D_p(R)$. ∎

**Theorem 3.4.10.** [Liouville]
If a function $f$ is entire and bounded by some $M$ on $\mathbb{C}$, then $f$ is constant on $\mathbb{C}$.

**Proof:** Let $M$ be a bound for $|f|$ over $\mathbb{C}$.

Then $f$ has a power series representation $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ with $|a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leqslant \left|\frac{M}{R^n}\right|$

This is from the Cauchy derivative estimates for any $R > 0$.

As $R \to \infty$, $a_n = 0$ for all $n \in \mathbb{N}$.

Thus $f(z) = 0$ for all $z \in \mathbb{C}$. ∎

**Theorem 3.4.11.** [Fundamental theorem of algebra]
If $f$ is a polynomial over $\mathbb{C}$ and $\deg(f) \geqslant 1$, then $f$ has at least one root in $\mathbb{C}$.

**Proof:** Suppose for a contradiction that $f(z)$ has no root in $\mathbb{C}$, or equivalently that $\frac{1}{f(z)}$ is entire.

As $f$ is a polynomial,

$$\begin{aligned}
|f(z)| &= |z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\geqslant |z^n|\left(1 - \left(\frac{|a_{n-1}|}{|z|} + \frac{|a_{n-2}|}{|z^2|} + \cdots + \frac{|a_0|}{|z^n|}\right)\right) \\
&\to \infty \quad \text{as } |z| \to \infty
\end{aligned}$$

In particular, $|f(z)| > 1$ for $|z| > R$ for some radius $R$.

So $|g(z)| < 1$ for $|z| > R$.

Since $g$ is entire, it is also bounded on $D_0(R)$.

Thus $g$ is bounded on $\mathbb{C}$.

By Liouville, $g$ is constant.

This is a contradiction, as $\deg(f) \geqslant 1$.

Therefore $f$ has at least 1 root in $\mathbb{C}$. ∎

**Theorem 3.4.15.** [Morera]
If $f$ is continuous on $\Omega$ and $\int_{\partial\triangle} f = 0$ for every triangle $\triangle \subset \Omega$, then $f$ is analytic on $\Omega$.

**Proof:** Take any $D_p(R) \subseteq \Omega$.

From a previous proof, $g(z) = \displaystyle\int_{\overline{pz}} f$ is a primitive for $f$ for all $z \in D_p(R)$.

Since $g' = f$ on $D_p(R)$, $g$ is analytic on $D_p(R)$.

By differentiation of power series, $f$ is analytic on $D_p(R)$. ∎

**Theorem 3.4.16.** If $f_n : \Omega \to \mathbb{C}$ are analytic and $f_n \to f$ uniformly on compact subsets of $\Omega$, then $f$ is analytic on $\Omega$.

**Proof:** For every closed disk $\overline{D_p(r)} \subset \Omega$, $f_n \to f$ uniformly on $\overline{D_p(r)}$.
Since the $f_n$ are continuous on $\overline{D_p(r)}$, $f$ is continuous on $\overline{D_p(r)}$.
Hence $f$ is continuous on $\Omega$.
Since the $f_n$ are holomorphic, we know for every $\triangle \subset \Omega$, $\int_{\partial\triangle} f_n = 0$.
Also, $\int_{\partial\triangle} f_n \to \int_{\partial\triangle} f$, as $\partial\triangle$ is compact.
Hence $\int_{\partial\triangle} f = 0$.
By Morera, $f$ is analytic on $\Omega$. ∎

**Theorem 4.1.6.** [Casorati-Weierstrass]
Let $f$ be analytic on $\Omega$ with non-removable singularity at $p$. Then only one of the following conditions hold:
**i.** For every $D_p^*(\epsilon) \subseteq \Omega$ with $\epsilon > 0$ , $f(D_p^*(\epsilon))$ is dense in $\mathbb{C}$.
**ii.** There exists a positive integer $m$ such that $(z-p)^m f(z)$ has a removable singularity at $p$.

**Proof:** **ii.** $\implies \neg$ **i.** Suppose that **ii.** holds.
Let $m \geqslant 1$ be the smallest integer such that $(z-p)^m f(z)$ has a removable singularity at $p$.
Let $g(z) = \begin{cases} (z-p)^m f(z) & z \neq p \\ \lim\limits_{z\to p} [(z-p)^m f(z)] & z = p \end{cases}$ so that $g$ is analytic on $\Omega \cup \{p\}$.
Thus $g(z) = a_0 + a_1(z-p) + a_2(z-p)^2 + \cdots$ for all $z$ in some $D_p(r) \subseteq \Omega \cup \{p\}$.
If $g(p) = a_0 = 0$, then $(z-p)^m f(z) = (z-p)(a_1 + a_2(z-p) + \cdots)$ for all $z \in D_p^*(r)$.
However, then $(z-p)^{m-1} f(z)$ has a removable singularity at $p$, contradicting the minimality of $m$.
Thus $g(p) = a_0 \neq 0$.
As $g$ is continuous at $p$, for some $\epsilon > 0$ if $z \in D_p(\epsilon) \subseteq \Omega \cup \{p\}$, then $|g(z)| \geqslant B > 0$ for some constant $B$.
Thus for $z \in D_p^*(\epsilon)$,

$$|f(z)| = \left| \frac{g(z)}{(z-p)^m} \right| \geqslant \frac{B}{\epsilon^m} > 0$$

So $f(D_p^*(\epsilon))$ is bounded away from zero.
Thus $f$ is not dense in $\mathbb{C}$. ∎

**Theorem 4.2.6.** [Residue theorem]
Let $f$ be analytic on $\Omega$ and $p_1, p_2, \ldots, p_n$ be poles of $f$ and $\gamma$ a chain in $\Omega$ that is homologous to 0 in $\Omega \cup \{p_1, p_2, \ldots, p_n\}$. Then

$$\int_\gamma f = 2\pi i \left( \sum_{j=1}^n \mathrm{ind}_\gamma(p_j) \mathrm{res}(f, p_j) \right)$$

**Proof:** Set $c_j = \mathrm{ind}_\gamma(p_j)$.
Fix $r_j > 0$ such that $D_{p_j}(r_j) \subseteq \Omega \cup \{p_1, p_2, \ldots, p_n\}$ are non-overlapping.
Take $\beta_j : [0, 2\pi] \to \Omega$ given by $t \mapsto p_j + r_j e^{ic_j t}$.
Then the chain $\beta_1 + \beta_2 + \cdots + \beta_n$ is homologous to $\gamma$ on $\Omega$.
By Cauchy,

$$\int_\gamma f = \sum_{j=1}^n \int_{\beta_j} f = \sum_{j=1}^n 2\pi i \cdot c_j \cdot \mathrm{res}(f, p_j) = 2\pi i \sum_{j=1}^n \mathrm{ind}_\gamma(p_j) \mathrm{res}(f, p_j)$$

∎

**Proposition 4.4.5.** Let $\alpha, \beta : [0,1] \to \mathbb{C}$ be closed paths such that $|\alpha(t) - \beta(t)| < |\beta(t)|$ for all $t \in [0,1]$. Then $\text{ind}_\alpha(0) = \text{ind}_\beta(0)$.

**Proof:** Note that $0 \notin \beta^*$, and also $0 \notin \alpha^*$, because if $\alpha(s) = 0$ for some $0 \leqslant s \leqslant 1$, then

$$|\beta(s)| = |0 - \beta(a)| = |\alpha(s) - \beta(s)| < |\beta(s)|$$

For $t \in [0,1]$, let $\gamma(t) = \frac{\alpha(t)}{\beta(t)}$.
Note that

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\beta'(t)}{\alpha(t)} \cdot \frac{\alpha'(t)\beta(t) = \beta'(t)\alpha(t)}{(\beta(t))^2} = \frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)}$$

Then combining the above,

$$|\alpha(t) - \beta(t)| < |\beta(t)| \Longrightarrow \left| \frac{\alpha(t)}{\beta(t)} - 1 \right| < 1 \Longrightarrow |\gamma(t) - 1| < 1 \Longrightarrow \gamma(t) \in D_1(1) \;\; \forall \;\; t$$

Thus $\text{ind}_\gamma(0) = 0$.
From the parametrization of $\gamma$, we get that

$$0 = \int_\gamma \frac{1}{z-0} dz = \int_0^1 \frac{\gamma'(t)}{\gamma(t)} dt = \int_0^1 \frac{\alpha'(t)}{\alpha(t)} dt - \int_0^1 \frac{\beta'(t)}{\beta(t)} dt = \int_\alpha \frac{1}{z-0} dz - \int_\beta \frac{1}{z-0} dz$$

Therefore $\text{ind}_\alpha(0) = \text{ind}_\beta(0)$. ∎

# References


Rudin, Walter. *Real and Complex analysis.* Mc-Graw Hill: 2006

Ullrich, David C. *Complex made simple.* American Mathematical Society: 2008