Compact course notes

Pure Math 365, Winter 2012

Elementary differential geometry

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Contents

1 Regularity

1.1 Curves

Definition 1.1.1. Let $I \subset \mathbb{R}$ be an interval. A parametrized curve (or simply curve) is a continuous function $\alpha : I \to \mathbb{R}^n$.

Definition 1.1.2. A curve α is termed regular if $\alpha \in C^1$ and $\alpha'(t) \neq 0$ for all t.

Proposition 1.1.3. A curve is expressed $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$. The first derivative $\alpha'(t)$ exists if and only if $\alpha_i(t)$ exists for all $1 \leqslant i \leqslant n$,

Definition 1.1.4. The vector $\alpha'(t)$ is termed the velocity vector of α at t. Its length, denoted $s'(t) = \|\alpha'(t)\|$, is termed the speed.

Definition 1.1.5. Given a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, define:

$$\text{the unit tangent vector:} \quad T := \tfrac{\alpha'}{\|\alpha'\|}$$

$$\text{the principal normal vector:} \quad P := \tfrac{T'}{\|T'\|}$$

$$\text{the binormal vector:} \quad B := T \times P$$

The curvature κ of the curve is defined to be

$$\kappa := \frac{\|T'\|}{s'} = \frac{\|T'\|}{\|\alpha'\|}$$

The torsion τ of the curve is the unique function τ such that

$$B' = -s'P\tau$$

The set (T, P, B, κ, τ) is then called the Frenet frame.

Remark 1.1.6. The Frenet equations may be rewritten as:

$$\frac{d}{dt}\begin{bmatrix} T \\ P \\ B \end{bmatrix} = s' \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ P \\ B \end{bmatrix}$$

Definition 1.1.7. A parametrized curve $\alpha : I \to \mathbb{R}^3$ is said to have arc-length parametrization iff for any $t_0 \in I$ and s given by $s(t) := \int_{t_0}^{t} \|\alpha'(z)\| dz$, we have $s(t) = t - t_0$. That is, $\|\alpha'\| = 1$.

· Note that the length of an interval I is given by $\int_I ds$, where $ds = \|\alpha'(t)\|dt$. Moreover, to integrate a quantity f over a curve α, we write $\int_\alpha f ds = \int_a^b f(\alpha(t))\|\alpha'(t)\|dt$ for α running from a to b.

Theorem 1.1.8. The Frenet frame, the curvature and torsion depend only on the image and orientation of the curve. More precisely, for a curve $\alpha : I \to \mathbb{R}^3$ with associated Frenet frame (T, P, B, κ, τ), if for a monotonic function $\varphi : I \to I$ the curve $\beta = \alpha \circ \varphi$ has a Frenet frame $(\tilde{T}, \tilde{P}, \tilde{B}, \tilde{\kappa}, \tilde{\tau})$, then

$$\begin{aligned} \tilde{T}(t) &= \pm T(\varphi(t)) \\ \tilde{P}(t) &= P(\varphi(t)) \\ \tilde{B}(t) &= \pm B(\varphi(t)) \\ \tilde{\kappa}(t) &= \kappa(t) \\ \tilde{\tau}(t) &= \pm\tau(\alpha(t)) \end{aligned} \qquad \text{where} \quad \pm = \begin{cases} + & \text{if } \varphi' > 0 \\ - & \text{if } \varphi' < 0 \end{cases}$$

Theorem 1.1.9. [Fundamental theorem of Space curves in $\mathbb{R}^3$]
For κ, τ C^1 functions on an interval $J \ni 0$, there exists a unique curve, up to rigid motions, $\alpha : I \to \mathbb{R}^3$ with curvature κ and torsion τ, for $I \subset J$ an interval with $0 \in I$.

Definition 1.1.10. A curve is termed closed iff it may be parametrized by a piecewise regular function $\alpha : [a, b] \to \mathbb{R}^n$ with $\alpha(a) = \alpha(b)$.

Theorem 1.1.11. [Fenchel]
Let C be a regular closed space curve. Then

$$\int_C \kappa \, ds \geqslant 2\pi$$

Equality holds iff C is a convex plane curve.

Remark 1.1.12. If C is a regular plane curve, then $\int_C \kappa \, ds \in 2\pi\mathbb{N}$.

Theorem 1.1.13. [Frey, Milnor]
If C is a knot, then $\int_C \kappa \, ds \geqslant 4\pi$.

1.2 Surfaces

Definition 1.2.1. An n-dimensional smooth manifold is a Hausdorff topological space M equipped with a collection of charts $\varphi_i : U_i \to V_i$ covering M for $U_i \subset M$ open and $V_i \subset \mathbb{R}^n$ open, and φ_i continuous and bijective with smooth transfer functions $\varphi_j \circ \varphi_i^{-1}$.

Definition 1.2.2. A parametrized surface of class C^r is a subset $S \subset \mathbb{R}^3$ with some C^r maps $X_i : U_i \to \mathbb{R}^3$ for $U_i \subset \mathbb{R}^2$ such that for all $p \in S$, there is some U_i with $p \in V \subset \mathbb{R}^3$ open and $X_i(U_i) = V \cap S$.

Definition 1.2.3. A vector $v \in \mathbb{R}^3$ is tangent to a surface S at p if there exists $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ for $\varepsilon > 0$ such that

$$\begin{aligned} \alpha(t) &\in S \text{ for all } t \\ \alpha(0) &= p \\ \alpha'(0) &= v \end{aligned}$$

Lemma 1.2.4. If v is tangent to a surface S at p, then so is λv for all $\lambda \in \mathbb{R}$.

Definition 1.2.5. If the set of tangent vectors of a surface S at a point p form a 2-dimensional vector space, it is then termed a tangent space or tangent plane and denoted $T_p(S)$.

Proposition 1.2.6. Suppose that a surface S is given in the form $S = \{(x, y, z) \mid f(x, y, z) = 0\}$ for some f. Then the tangent plane to S at $(x_0, y_0, z_0) \in S$ is given by

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

Definition 1.2.7. A function $f : S \to \mathbb{R}^3$ is termed a tangent vector field if $f(p) \in T_p(S)$.

· Suppose $X : U \to S$ is a parametrization, and $X^{-1}(p) = \{(u_0, v_0)\}$. Then every curve in U through (u_0, v_0) yields a curve in S through p. Let $(u(0), v(0)) = (u_0, v_0)$. Then if $\gamma(t) = (u(t), v(t))$, we have

$$\gamma'(t) = X_u(u(t), v(t))u'(t) + X_v(u(t), v(t))v'(t)$$

Therefore $\text{span}\{(X_u(u_0, v_0), X_v(u_0, v_0)\} \subset T_p(S)$. Moreover, if $X_u \times X_v \neq 0$, then $\text{span}\{X_u, X_v\} = T_p(s)$.

Definition 1.2.8. A subset $S \subset \mathbb{R}^3$ is termed a regular surface if for all $p \in S$ there exists $U \subset \mathbb{R}^2$, $V \subset \mathbb{R}^3$ both open with $p \in V$ and a surjective continuous function $X : U \to V \cap S$ such that

1. X is C^1

2. X is a homeomorphism (bijective, and inverse is continuous)

3. for all $(u, v) \in U$, $dX_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective

Definition 1.2.9. With respect to the above defintion, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then

$$\begin{array}{rcl} F : \mathbb{R}^n & \to & \mathbb{R}^m \\ x & \mapsto & (f_1(x), \ldots, f_m(x)) \end{array} \quad \text{with} \quad dF_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

Moreover, F is differentiable at x_0 if

$$\lim_{x \to x_0} \left[\frac{F(x) - F(x_0) - dF_{x_0}(x - x_0)}{|x - x_0|} \right] = 0$$

Proposition 1.2.10. Let $U \subset \mathbb{R}^2$ open with a C^1 function $f : U \to \mathbb{R}$ and $S = \{(x, y, z) \mid z = f(x, y)\}$. Then S is a regular surface.

Proposition 1.2.11. Let $U \subset \mathbb{R}^3$ open with a C^1 function $f : U \to \mathbb{R}$. If r is a regular value of f, then $S = \{(x, y, z) \mid f(x, y, z) = r\}$ is a regular surface.

Definition 1.2.12. Let $f : X \to Y$ be a function with $r \in \operatorname{im}(f)$. Then r is termed a regular value of f iff for all $p \in f^{-1}(r)$, $df_p \neq 0$.

Definition 1.2.13. For vectors $v, w \in \mathbb{R}^n$, define $v^\perp := \{w \mid v \cdot w = 0\}$.

Lemma 1.2.14. If $S = \{(x, y, z) \mid f(x, y, z) = r\}$ is a regular surface for r a regular value of f, then $T_p(S) = \nabla f(p)^\perp$.

Proof: Suppose $\gamma : (-\varepsilon, \varepsilon) \to S$ with $\gamma(0) = p$.

Then $f \circ \gamma$ is constant, and

$$(f \circ \gamma)'(0) = f_x(\gamma(0))\gamma_1'(0) + f_y(\gamma(0))\gamma_2'(0) + f_z(\gamma(0))\gamma 1_3(0) = \nabla f(p) \cdot \gamma'(0)$$

Since $T_p(S) = \{\gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to S \text{ with } \gamma(0) = p\}$, we have $T_p(S) \subset \nabla f(p)^\perp$.

Since S is regular, $\dim(T_p(S)) = 2$, and since $df_p \neq 0$ for all $p \in f^{-1}(r)$, as r is regular, $\nabla f(p) \neq 0$.

Therefore $\dim(\nabla f(p)^\perp) = 2$ as well, and so $\nabla f(p)^\perp \subset T_p(S)$, and so $\nabla f(p)^\perp = T_p(S)$. ■

Proposition 1.2.15. If $S \ni p$ is a regular surface and $X : U \to S$ is a regular parametrization (i.e. $\operatorname{rank}(dX_q) = 2$ for all $q \in U$) and $X^{-1}(p) = \{q\}$, then $T_p(S) = \operatorname{Im}(dX_q)$.

Suppose that we have two parametrizations X, Y with a diffeomorphism $F : U \to V$. Then we have that

$$\begin{array}{rcl} X_u & = & Y_u \frac{\partial F_1}{\partial u} + Y_v \frac{\partial F_2}{\partial u} \\ X_v & = & Y_u \frac{\partial F_1}{\partial v} + Y_v \frac{\partial F_2}{\partial v} \end{array} \qquad dF = {}_{\{Y_u, Y_v\}} I_{\{X_u, X_v\}} = \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_2}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix}$$

So then for $w = aX_u + bX_v \in \mathbb{R}^3$ such that $[w] = [a\ b]$ and $q \in U = \operatorname{dom}(X) = \operatorname{dom}(Y)$,

$$dF_q \left({}_{\{X_u(q), X_v(q)\}}[w] \right) = {}_{\{Y_u(q), Y_v(q)\}}[w]$$

2 Orientability

Definition 2.0.1. Heuristically, orientability is the ability to decide on a well-defined definition of clockwise direction on a surface in Euclidean space.

2.1 The normal vector

Definition 2.1.1. Given a regular surface S and a parametrization $X : U \to S$ in (u, v), the normal vector to S at $p = X(q)$ is defined to be

$$N(q) := \frac{X_u \times X_v}{\|X_u \times X_v\|}(q)$$

Moreover, if we have two parametrizations X_1, X_2 of S with a transition map F such that $X_1 = X_2 \circ F$, then $N_1(q) = \mathrm{sgn}(\det(dF_q))N_2(q)$.

Theorem 2.1.2. The following are equivalent definitions of an orientable surface S:

· It is possible to cover S with open sets R_i given by the images of regular parametrizations $X_i : U_i \to R_i \subset \mathbb{R}^3$ such that if $R_i \cap R_j \neq \emptyset$, then there exists a diffeomorphism $F : X_i^{-1}(R_i \cap R_j) \to X_j^{-1}(R_i \cap R_j)$ with $X_i = X_j \circ F$.

· It is possible to cover S with open sets R_i given by the images of regular parametrizations $X_i : U_i \to R_i \subset \mathbb{R}^3$ such that for $N_i(q) = \frac{X_{i_u} \times X_{i_v}}{\|X_{i_u} \times X_{i_v}\|}(q)$, we have $N_i(q_i) = N_j(q_j)$ if $p \in R_i \cap R_j$ for $X_i(q_i) = X_j(p_j) = p$.

· There exists a continuous function $n : S \to \mathbb{S}^2$ such that $n(p)^\perp = T_p(S)$.

Corollary 2.1.3. If S is a regular surface and $X : U \to R \subset \mathbb{R}^3$ parametrizes $S \ni p$ smoothly around p, so that $X(u_0, v_0) = p$, then around p

$$n \circ X = \pm N$$

This n is termed the Gauss map of S.

2.2 The first fundamental form

Definition 2.2.1. An inner product on a vector space V over $\mathbb{F}$ is a map $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ that is

1. bilinear
2. symmetric
3. positive definite

Theorem 2.2.2. If $W \subset V$ for vector spaces and $\langle\ ,\ \rangle$ is an inner product on V, then $\langle\ ,\ \rangle|_{W\times W}$ is an inner product on W.

Proposition 2.2.3. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for a vector space V over $\mathbb{F}$, and $\langle\ ,\ \rangle$ an inner product on V. Then there exists $g \in M_{n\times n}$ such that for all $u, w \in V$, we have

$$\langle u, w\rangle \;=\; {}_{\mathcal{B}}[u]^T\ g\ {}_{\mathcal{B}}[w] \;=\; {}_{\mathcal{B}}[u]^T \left[\langle v_i, v_j\rangle\right] {}_{\mathcal{B}}[w]$$

Definition 2.2.4. The metric tensor, or first fundamental form, in the coordinates (u, v) is given by

$$g = \begin{bmatrix} X_u \cdot X_u & X_u \cdot X_v \\ X_v \cdot X_u & X_v \cdot X_v \end{bmatrix}$$

· The metric on the xy-plane is unique, but its matrix representation depends on choice of parametrization.

Theorem 2.2.5. Let S be a regular surface in $\mathbb{R}^3$ with $X : U \to R \subset \mathbb{R}^3$ injective parametrizing a part of S. Let $Q \subset U$ be compact. Suppose $X|_{Q^\circ}$ is a homeomorphism on its image and is such that its differential has everywhere maximal rank. Then

$$\mathrm{area}(X(Q)) = \iint_Q \sqrt{\det(g)}\ du\ dv$$

Remark 2.2.6. With respect to the above, if $\gamma : [a,b] \to S$ is a curve on S such that $X \circ \beta = \gamma$, then

$$\text{length}(\gamma) = \int_a^b \sqrt{(\beta')^T g \beta'}\ dt$$

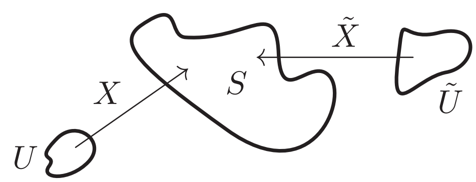
Remark 2.2.7. With respect to the above, if $f : S \to \mathbb{R}$, then

$$\iint_{X(Q)} f\ dS = \iint_Q (f \circ X)\sqrt{\det(g)}\ du\ dv$$

· We will see that g is a tensor, as $g_{ij} = \dfrac{\partial x_i}{\partial x_k}\dfrac{\partial x_j}{\partial x_\ell} g_{k\ell}$.

Remark 2.2.8. Given $q \in U$ and $p = X(q) \in S$, we have a basis $\{X_u(q), X_v(q)\}$ of $T_p(S)$.

Proposition 2.2.9. Suppose for a regular surface S we have parametrizations $X : U \to R \subset S$ and $\tilde{X} : \tilde{U} \to \tilde{R} \subset S$ with $R \cap \tilde{R} \neq 0$ as illustrated below.



Then the 2-tensors g, $\tilde{g}$ are related by

$$g_{ij} = \sum_{k,\ell=1}^{2} \frac{\partial \tilde{x}_k}{\partial x_i}\frac{\partial \tilde{x}_\ell}{\partial x_j}\tilde{g}_{k\ell}$$

Definition 2.2.10. For a regular surface and $p \in S$, the <u>differential</u> of the Gauss map n is the map

$$dn_p : T_p(S) \to T_{n(p)}(\mathbb{S}^2)$$

This map is defined as follows: if there exists $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ for $\varepsilon > 0$ such that

$$\alpha(t) \in S \text{ for all } t$$
$$\alpha(0) = p$$
$$\alpha'(0) = v$$

and $n \circ \alpha$ is a path in $\mathbb{S}^2$, then $(n \circ \alpha)(0) = n(p)$ implies $dn_p(v) = (n \circ \alpha)'(0)$.

Definition 2.2.11. Given a regular surgface S and a regular parametrization X with $X(q) = p$ satisfying the conditions of **1.2.8**, for another regular surface $M \subset \mathbb{R}^3$, a function $f : S \to M$ is termed <u>smooth</u> at p if $f \circ X$ is smooth at q.

Definition 2.2.12. Given an oriented surface in $\mathbb{R}^3$ with $n : S \to \mathbb{S}^2$, as $T_{n(p)}(\mathbb{S}^2) \cong T_p(S)$, define the map

$$-dn_p : T_p(S) \to T_p(S)$$

to be the the <u>shape operator</u>, and denote it by S_p. Later we will find out that this map is self-adjoint (symmetric), and thus diagonalizable. Moreover, for a parametrization $X : U \to S$ such that $X(q) = p$,

$$dn_q(X_u(q)) = N_u(q) \qquad\qquad dn_q(X_v(q)) = N_v(q)$$

2.3 The second fundamental form

Definition 2.3.1. Given a regular surface $S \subset \mathbb{R}^3$ oriented by $n : S \to \mathbb{S}^2$, define the first and second fundamental forms by

$$\begin{array}{rrcl} I_p: & T_p(S) \times T_p(S) & \to & \mathbb{R} \\ & (a,b) & \mapsto & a \cdot b \end{array} \qquad \begin{array}{rrcl} II_p: & T_p(S) \times T_p(S) & \to & \mathbb{R} \\ & (a,b) & \mapsto & -dn_p(a) \cdot b \end{array}$$

Lemma 2.3.2. II_p is a symmetric bilinear form.

Proof: Let $a = a_1 X_1 + a_2 X_2$ and $b = b_1 X_1 + b_2 X_2$, elements in $T_p(S)$.
Then

$$\begin{aligned} II_p(a,b) &= -dn_p\left(\sum a_i X_i\right) \sum b_j X_j \\ &= \sum_{i,j} a_i(-dn_p(X_i)X_j)b_j \\ &= \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} N \cdot X_{11} & N \cdot X_{12} \\ N \cdot X_{21} & N \cdot X_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= (a_1\ a_2)L(b_1\ b_2)^T \end{aligned}$$

Since the matrix representation L of II_p in the basis X_1, X_2 is symmetric, the result follows. ■

Lemma 2.3.3. If $B = \{X_1(q), X_2(q)\}$ is a basis for $T_p(S)$, then $_B[-dn_p]_B = g^{-1}L$.

Definition 2.3.4. We use the above results to compute the

$$\begin{array}{rl} \underline{\text{mean curvature}}: & H = \frac{1}{2}\mathrm{tr}(-dn_p) \\ \underline{\text{Gauss curvature}}: & \kappa = \det(-dn_p) \end{array}$$

Lemma 2.3.5. Suppose $T : V \to V$ is a linear map, and $B = \{v_1, \dots, v_n\}$ is a basis of V and $\langle\ ,\ \rangle$ is an inner product on V. Then for

$$\begin{aligned} g &= [\langle v_i, v_j \rangle]_{1 \leqslant i,j, \leqslant n} \\ L &= [\langle Tv_j, v_i \rangle]_{1 \leqslant i,j, \leqslant n} \\ A &=_B [T]_B \end{aligned}$$

we may relate them all together by $A = g^{-1}L$.

Proof: First note $T(v_i) = \sum_k A_{ki} v_k$, so

$$\begin{aligned} T(v_i) \cdot v_j &= \sum_k A_{ki} v_k \cdot v_j \\ &= \sum_k A_{ki} g_{kj} \\ &= \sum_k g_{jk} A_{ki} \\ &= (gA)_{ji} \end{aligned}$$

Therefore $L = gA$ and so $A = g^{-1}L$. ■

3 The Gauss curvature

3.1 Principal curvatures

Definition 3.1.1. Let S be an oriented regular surface in $\mathbb{R}^3$ and $\gamma : I \to S$ a parametrized curve. Then the normal curvature of S along γ is defined as

$$\kappa_n(t) = \frac{1}{s'(t)} T'(t) \cdot n(\gamma(t)) = \kappa P(t) \cdot (n \circ \gamma)(t) = \kappa P(t) \cdot N(t)$$

where T is the unit tangent vector, as defined earlier above.

Proposition 3.1.2. Let $p \in S$ and $\gamma : I \to S$ with $\gamma(0) = p$ and $\gamma'(0) = w$ where $\|w\| = 1$. Then $\kappa_n(0) = II_p(w, w)$.

Proof: Since the image of γ is in S and $\gamma'(t) \in T_{\gamma(t)}(S)$ for all t, $\gamma' \cdot (n \circ \gamma) = 0$ and $T \cdot (n \circ \gamma) = 0$,

$$T' \cdot (n \circ \gamma) + T \cdot (n \circ \gamma)' = 0$$

Then at $t = 0$, we have that

$$\begin{aligned}
\kappa_n(0) &= \frac{1}{s'(0)} T'(0) \cdot (n \circ \gamma)(0) \\
&= \frac{-1}{s'(0)} T(0) \cdot dn_{\gamma(0)}(\gamma'(0)) \\
&= \frac{-1}{\|w\|} \frac{w}{\|w\|} \cdot dn_p(w) \\
&= \frac{-dn_p(w) \cdot w}{\|w\|^2} \\
&= II_p\left(\frac{w}{\|w\|}, \frac{w}{\|w\|}\right)
\end{aligned}$$

■

Corollary 3.1.3. [Meusnier]
The normal curvature of S at p depends only on the direction, not on the curves through p.

Definition 3.1.4. The quantities

$$k_1 = \max_{\|w\|=1} \{II_p(w, w)\} \qquad k_2 = \min_{\|w\|=1} \{II_p(w, w)\}$$

are termed the principal curvatures of S at p. Their corresponding directions are termed the principal directions.

Proposition 3.1.5. Let S be a regular surface with $p \in S$. Then there exists $\{c_1, c_2\}$ a basis of $T_p(S)$ such that $dn_p(c_i) = -k_i c_i$ for $i = 1, 2$. Thus the principal curvatures are eigenvalues of the shape operator.

The above demontrates a common phenomenon - if $T : V \to V$ is a self-adjoint linear operator, then

$$\max_{\|v\|=1} \{\langle T(v), v\rangle\} \qquad \min_{\|v\|=1} \{\langle T(v), v\rangle\}$$

are the largest and smallest eigenvalues, respectively, when they exist. Recall that an operator is self-adjoint iff it is equal to its conjugate transpose.

Definition 3.1.6. For S a regular surface, if all directions at $p \in S$ are principal, then p is termed umbilical.

Proposition 3.1.7. Let S be a regular connected surface with all points $p \in S$ umbilical. Then either S is contained in a plane or on the surface of $\mathbb{S}^2$.

Proof: Let $\lambda : S \to \mathbb{R}$ be the principal curvature function.
Let $X : U \to S \ni p$ be a parametrization with $X(q) = p$ for U connected.
Recall that $dn \cdot dX = dN$, in particular $dn(X_u) = N_u$ and $dn(X_v) = N_v$.
For scalars a, b, consider $aX_u + bX_v \in T_p(S)$, so then

$$dn(aX_u + bX_v) = aN_u + bN_v \qquad \text{or} \qquad \lambda a X_u - \lambda b X_v = aN_u + bN_v$$

Which means that

$$\begin{aligned} (a,b) = (1,0) &\implies N_u = -\lambda X_u \\ (a,b) = (0,1) &\implies N_v = -\lambda X_v \end{aligned}$$

This allows us to state that

$$N_u + \lambda X_u = N_v + \lambda X_v = 0$$

Differentiation gives us that

$$\begin{aligned} 0 &= \frac{\partial}{\partial v}(N_u + \lambda X_u) = N_{uv} + \lambda_v X_u + \lambda X_{uv} \\ 0 &= \frac{\partial}{\partial u}(N_v + \lambda X_v) = N_{vu} + \lambda_u X_v + \lambda X_{vu} \end{aligned}$$

Directly implying

$$\lambda_v X_u - \lambda_u X_v = 0$$

Since X_u, X_v are linearly independent, $\lambda_u = \lambda_v = 0$.
Therefore λ is locally constant.
Since U is connected, S is connected, and so λ is constant.

If $\lambda = 0$, then $N_u = N_v = 0$, and N is locally constant, so $S \subset$ plane.
If $\lambda \neq 0$, then for $Y = X + \frac{1}{\lambda}N$, we have $Y_u = Y_v = 0$, so $\|X - Y\| = \frac{1}{|\lambda|}$, and thus $S \subset \mathbb{S}^2$ with radius $\frac{1}{|\lambda|}$. ■

Lemma 3.1.8. The Gauss curvature is the unique function κ satisfying, for any parametrization X,

$$N_u \times N_v = (\kappa \circ X)(X_u \cdot X_v)$$

for every parametrization X of a regular surface S.

Theorem 3.1.9. Let $S \ni p$ be a regular surface oriented by n and $V_\varepsilon = S \cap \{x \mid |x - p| < \varepsilon\}$. Then

$$|\kappa(p)| = \lim_{\varepsilon \to 0} \left[\frac{\text{area}(n(V_\varepsilon))}{\text{area}(V_\varepsilon)} \right]$$

3.2 Surfaces of revolution

Definition 3.2.1. Let $f = (g(u), h(u))$ be a curve parametrized in $\mathbb{R} \times \mathbb{R}$. Then the surface of revolution S of f in $\mathbb{R}^3$ around the x_1-axis is parametrized by

$$X(u,v) = (g(u), h(u)\cos(v), h(u)\sin(v))$$

for $(u,v) \in U = \mathbb{R} \times [0, 2\pi)$. The first and second fundamental forms of S are

$$g = \begin{bmatrix} (g')^2 + (h')^2 & 0 \\ 0 & h^2 \end{bmatrix} \qquad\qquad L = \frac{1}{\sqrt{(g')^2 + (h')^2}} \begin{bmatrix} h'g'' - g'h'' & 0 \\ 0 & g'h \end{bmatrix}$$

The shape operator is given by

$$g^{-1}L = \frac{1}{\sqrt{(g')^2+(h')^2}}\begin{bmatrix} \frac{h'g''-g'h''}{(g')^2+(h')^2} & 0 \\ 0 & \frac{g'}{h} \end{bmatrix} = \begin{bmatrix} k_\mu & 0 \\ 0 & k_\pi \end{bmatrix}$$

Since $X_u(X_v)$ is an eigenvector, the lines on S where $u(v)$ is constant are termed parallels (meridians) and curvature along that line is $k_\pi(k_\mu)$.

· From above, we have that $\kappa = k_\mu k_\pi$.

Definition 3.2.2. An asymptotic direction at a point $p \in S$ for S a regular surface is a direction with normal curvature 0. An asymptotic curve is a curve tangent everywhere to an asymptotic dicection.

For example, the curves

$$\begin{aligned} v &\mapsto (a, b\cos(v), b\sin(v)) \\ v &\mapsto (-a, b\cos(v), b\sin(v)) \end{aligned}$$

are asymptotic curves on a torus parametrized by $X(u,v) = (a\cos(u), (b+a\sin(u))\cos(v), (b+a\sin(u))\sin(v))$.

3.3 Intrinsicity

Definition 3.3.1. An oriented surface S with Gauss map n is termed minimal if $H = \frac{1}{2}\mathrm{tr}(-dn) = 0$.

Proposition 3.3.2. If the area is minimized on S, that is, for all $h : U \to \mathbb{R}$ the variation S_h of S is such that $\mathrm{Area}(S_h) \geqslant \mathrm{Area}(S)$, then $H = 0$.

Proof: Fix a parametrization X and a map h.

Let $Z^t = X + thN$ for $t \in \mathbb{R}$, so Z^t parametrizes, for each t, a surface S^t near S, and Z^0 parametrizes S. For $A_h(t)$ the area of S^t, we want to prove that $A_h'(0) = 0$ if $H = 0$, so we will have a critical point. Now consider

$$Z_u^t = X_u + th_uN + thN_u \qquad\qquad Z_v^t = X_v + th_vN + thN_v$$

$$Z_i^t \cdot Z_j^t = (X_i + th_iN + thN_i) + (X_j + th_jN + thN_j) = X_i \cdot X_j + X_i \cdot thN_j + X_j \cdot thN_i + \underbrace{t^2h^2N_i \cdot N_j + t^2h_i \cdot h_j}_{F}$$

If g^t is a metric for Z^t, then

$$\begin{aligned} g^t &= g + th\left(X_i \cdot N_j + X_j \cdot N_i\right) + F \\ &= g - th\left(X_{ij} \cdot N + X_{ji} \cdot N\right) + F \\ &= g - 2thX_{ij} \cdot N + F \\ &= g + 2thL + F \end{aligned}$$

Recall that

$$\begin{aligned} A_h(t) &= \iint_U \sqrt{\det(g^t)}\; du\; dv \\ &= \iint_U \sqrt{\det(g)\det(g^{-1}g^t)}\; du\; dv \\ &= \iint_U \sqrt{\det\left(1 + 2thg^{-1}L + g^{-1}F\right)}\sqrt{\det(g)}\; du\; dv \\ A_h'(t) &= \iint_U \frac{\mathrm{tr}(2hg^{-1}L) + \mathcal{O}(t)}{2(\cdots\cdots)}\sqrt{\det(g)}\; du\; dv \\ A_h'(0) &= \iint_U \mathrm{tr}(g^{-1}L)h\sqrt{\det(g)}\; du\; dv \end{aligned}$$

Above we showed that if $H = 0$, then $A_h'(0) = 0$.
It is also clear from the calculations that if $A_h'(0) = 0$ for all h, then $H = 0$. ■

Theorem 3.3.3. The equation for a critical point of an area functional is $H = 0$.

Theorem 3.3.4. [Cristoffel]
Let $X : U \to S$ parametrize part of a smooth surface $S \subset \mathbb{R}^3$. Then for $i, j = u, v$

$$X_{ij} = L_{ij} \cdot N + \Gamma^1_{ij} X_1 + \Gamma^2_{ij} X_2$$

where Γ^k_{ij} is termed the Cristoffel symbol and is given explicitly by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell=1}^{2} (g^{-1})_{k\ell} \left(\frac{\partial g_{i\ell}}{\partial x_j} + \frac{\partial g_{\ell j}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell} \right) = \Gamma^k_{ji}$$

Corollary 3.3.5. With respect to the definitions above,

$$\Gamma^k_{ij} = \sum_\ell (g^{-1})_{k\ell} (X_{ij} \cdot X_\ell)$$

Theorem 3.3.6. [Codazzi]
With respect to the definitions above, the Codazzi equations,

$$\frac{\partial L_{ij}}{\partial x_\ell} - \sum_k \Gamma^k_{i\ell} L_{kj} = \frac{\partial L_{i\ell}}{\partial x_j} - \sum_k \Gamma^k_{ij} L_{k\ell}$$

for any choice of $i, j, \ell \in \{1, 2\}$, are satisfied.

Theorem 3.3.7. If $g, L : U \to \mathrm{Sym}(2)$ satisfy the Codazzi equations, then there exists a map $X : U \to \mathbb{R}^3$ that parametrizes a surface for which g is the 1st fundamental form and L is the second fundamental form.

Theorem 3.3.8. [Theorema Egregium - Gauss]
The Gaussian curvature, under isometry, is an intrinsic property of a surface. That is, it only depends on the metric g and its derivatives.

Corollary 3.3.9. If surfaces S_1, S_2 are isometric via $f : S_1 \to S_2$, then $\kappa(f(p)) = f(p)$ for all $p \in S_1$.

Definition 3.3.10. A surface S_1 is termed isometric to another surface S_2 if there exists an isometry $f : S_1 \to S_2$. Such a map f is termed an isometry iff:

1. f is a diffeomorphism
2. the map $df_p : T_p(S_1) \to T_p(S_2)$ is a vector-space isometry

Given two vector spaces V_1, V_2 with associated inner products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, a map $L : (V_1, \langle\ ,\ \rangle_1) \to (V_2, \langle\ ,\ \rangle_2)$ is termed a vector space isometry iff for all $u, v \in V_1$,

$$\langle u, v \rangle_1 = \langle Lu, Lv \rangle_2$$

4 The Gauss-Bonnet theorem

4.1 Geodesics

Definition 4.1.1. A closed path γ on $\mathbb{S}^2$ is termed a great circle iff for every point p on γ there exists p' on γ that is antipodal to p.

Definition 4.1.2. Given a parametrized curve $\gamma : [a, b] \to S$, we define the length of γ on S by the induced norm on $\mathbb{R}^3$ to be

$$\ell(\gamma) := \int_a^b \|\gamma'(t)\| \, dt$$

Definition 4.1.3. Given a surface S, define a distance function d on S by

$$\begin{array}{rrcl} d: & S \times S & \to & \mathbb{R} \\ & (p, q) & \mapsto & \inf_\gamma \{\ell(\gamma) \mid \gamma : [0, 1] \to S \text{ is piecewise smooth, } \gamma(0) = p, \ \gamma(1) = q\} \end{array}$$

Definition 4.1.4. Given a surface S, heuristically a geodesic curve on S is a curve γ that minimizes length locally. With T, P, B from the Frenet frame and $N = n \circ \gamma$ the normal vector, define the geodesic curvature of γ on S to be

$$\kappa_g := \frac{1}{\|\gamma'\|} T' \cdot (N \times T)$$

Formally, a curve γ is a geodesic iff $\kappa_g = 0$ for the curve.

Proposition 4.1.5. A curve α may be expressed as

$$\begin{aligned} \alpha' &= vT \\ \alpha'' &= v'T + v^2 \kappa_n N + v^2 \kappa_g \,(N \times T) \end{aligned}$$

Lemma 4.1.6.

$$\kappa_g = \kappa P(N \times T)$$

Suppose that we have a curve $\alpha(t) = X(u(t), v(t))$ with arc-length parametrization, i.e. $\|\alpha'\| = 1$. Then

$$\begin{aligned} \alpha' &= X_u u' + X_v v' \\ \alpha'' &= X_{uu} \left(u'\right)^2 + X_{uv} u'v' + X_u u'' + X_{vu} v'u' + X_{vv} \left(v'\right)^2 + X_v v'' \end{aligned}$$

Moreover, if α is a geodesic, then $\alpha''(t)$ has no components tangent to $T_{\alpha(t)}(S)$. This means that X_u and X_v have no effect on α'', meaning that

$$\begin{aligned} \Gamma^1_{11} \left(u'\right)^2 + 2\Gamma^1_{12} u'v' + u'' + \Gamma^1_{22} \left(u'\right)^2 = 0 \\ \Gamma^2_{11} \left(u'\right)^2 + 2\Gamma^2_{12} u'v' + v'' + \Gamma^2_{22} \left(v'\right)^2 = 0 \end{aligned}$$

Theorem 4.1.7. Let $S \ni p$ be a regular surface, and fix $u \in T_p(S)$ such that $\|u\| = 1$. Then there exists a unique geodesic in S through p in the direction of u.

Proof: Let $X : U \to S$ be a regular parametrization of a neighborhood of S.
Consider the following ODE system:

$$\begin{array}{ll} x_1' = v_1 & v_1' = -\Gamma^1_{11} v_1^2 - 2\Gamma^1_{12} v_1 v_2 - \Gamma^1_{22} v_2^2 \\ x_2' = v_2 & v_2' = -\Gamma^2_{11} v_1^2 - 2\Gamma^2_{12} v_1 v_2 - \Gamma^2_{22} v_2^2 \end{array}$$

with initial conditions:

$$\begin{array}{rcl} (x_1(0), x_2(0)) & = & X^{-1}(p) \\ (v_1(0), v_2(0)) & = & dX^{-1}_{(x_1(0), x_2(0))} u \end{array}$$

Above we have $X(u, v) = X(x_1(u), x_2(v))$.
First-order ODEs with initial conditions have unique solutions for $t \in (-\varepsilon, \varepsilon)$ for ε sufficiently small.

It is left to verify that we have arc length parametrization (else we do not have a geodesic).
Define a function

$$f = \left(g_{11}(x_1')^2 + 2g_{12}x_1'x_2' + g_{22}(x_2')^2\right) \cdot (x_1, x_2)$$

By design, $f(0) = 1$ and we have left to prove that $f' = 0$.
Consider the derivative:

$$f' = g_{11,1}(x_1')^3 + g_{11,2}(x_1')^2 + 2g_{11}x_1'x_1'' + \cdots$$

Using the geodesic equations, we replace x_1'' by $-\sum_{j,k} \Gamma^i_{jk} x_j x_k$.
Employing the equality

$$\Gamma^k_{ij} = \frac{1}{2}\sum_\ell (g^{-1})_{k\ell}(g_{i\ell,j} + g_{\ell j,i} - g_{ij,\ell})$$

we simplify to get that $f' = 0$. ∎

4.2 A topological approach

Lemma 4.2.1. For a curve $\alpha(t) = X(u(t), v(t))$, recalling that $\|\alpha'\| = s'$, we have that

$$\frac{(s')^3 \kappa_g}{\sqrt{\det(g)}} = (u')^3 \Gamma^2_{11} - (u')^2 v' \Gamma^1_{11} + 2(u')^2 v' \Gamma^2_{12} - 2u'(v')^2 \Gamma^1_{12} - v'u'' + u'v'' + u'(v')^2 \Gamma^2_{22} - (v')^3 \Gamma^1_{22}$$

Lemma 4.2.2. Suppose that a parametrization X is orthogonal, i.e. $X_u \cdot X_v = 0$. Then

$$\begin{aligned} \Gamma^2_{11} g_{22} + \Gamma^1_{12} g_{11} = 0 \\ \Gamma^2_{12} g_{22} + \Gamma^1_{22} g_{11} = 0 \end{aligned}$$

Theorem 4.2.3. [Liouville]
Let S be a regular surface with $X : U \to S$ a parametrization with $X_u \cdot X_v = 0$. Consider a curve $\gamma(t) = X(u(t), v(t))$ with arc-length parametrization. Let $\theta(t)$ be the angle between $\gamma'(t)$ and $X_u(u(t), v(t))$. Let $\kappa_{(u)}$ and $\kappa_{(v)}$ be geodesic curvatures of u-curves and v-curves, respectively. Then the geodesic curvature is given by

$$\kappa_g = \frac{\partial \theta}{\partial t} + \kappa_{(u)} \cos(\theta) + \kappa_{(v)} \sin(\theta)$$

The proof involves the previous two lemmas and some other facts, including

$$\kappa_{(u)} = \frac{\sqrt{\det(g)}\, \Gamma^2_{11}}{g_{11}^{3/2}} \qquad \kappa_{(v)} = -\frac{\sqrt{\det(g)}\, \Gamma^1_{22}}{g_{22}^{3/2}}$$

$$\cos(\theta) = \sqrt{g_{11}}\, u' \qquad \sin(\theta) = \sqrt{g_{22}}\, v'$$

Theorem 4.2.4. [Green]
Let C be a positively oriented closed piecewise regular plane curve with interior R. Then

$$\iint_R (P\ dx + Q\ dy) = \iint_R \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\ dy$$

Lemma 4.2.5. Suppose that $X : U \to S$ is an orthogonal parametrization. Let $R \subset U$ be closed and bounded with $X(R) = S'$, such that ∂R is continuous and piecewise smooth, parametrized by $\gamma : [a, b] \to S'$

with $\|\gamma'\| = 1$, having discontinuities at $\{t_1, \ldots, t_k\} \subset [a, b]$. Then

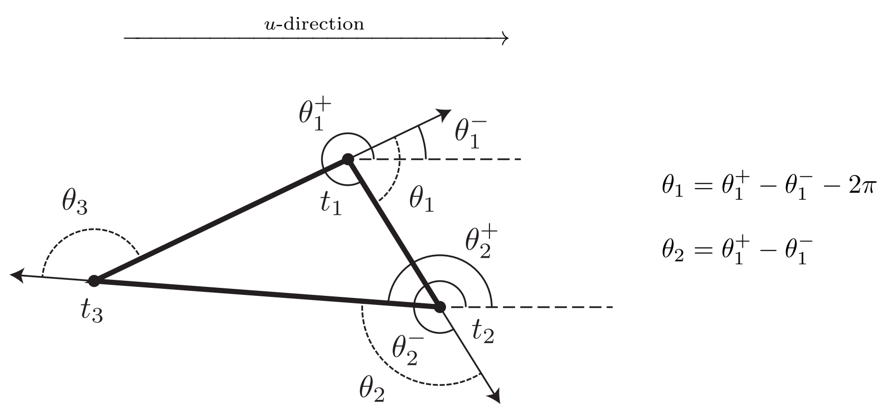
$$\begin{aligned}
\int_{\partial S'} \kappa_g \, ds &= \int_a^b \kappa_g(t) \|\gamma'(t)\| \, dt \\
&= \int_a^b \left(\frac{d\theta}{dt} + \frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u} \frac{dv}{dt} - \frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{11}}}{\partial v} \frac{dv}{dt} \right) dt \\
&= \sum_{i=1}^{k} \left(\theta_i^+ - \theta_i^- \right) + \iint_{R'} \frac{1}{\sqrt{\det(g)}} \left(\frac{\partial}{\partial u} \left(\frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{11}}}{\partial v} \right) \right) \sqrt{\det(g)} \, du \, dv \\
&= \pm 2\pi + \sum_{i=1}^{k} \theta_i - \iint_{R'} k \sqrt{\det(g)} \, du \, dv
\end{aligned}$$

Where

$$\begin{aligned}
\theta_i^{\pm} &= \text{the angle between } \gamma'(t_i^{\pm}) = \lim_{t \to t_i^{\pm}} \left[\frac{\gamma(t) - \gamma(t_i)}{t - t_i} \right] \text{ and } X_u \\
\theta_i &= \theta_i^+ - \theta_i^- - \left(\tfrac{1}{2} + \tfrac{(-1)^{p_i}}{2} \right) 2\pi \\
p_i &= \begin{cases} 0 & \text{if } \gamma'(t_i^-) \text{ is above } X_u \\ 1 & \text{if } \gamma'(t_i^-) \text{ is below } X_u \end{cases}
\end{aligned}$$

Definition 4.2.6. Given a piecewise smooth continuous curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha'(0^+) \neq \alpha 1(0^-)$, we say that the curve has a <u>vertex</u> at 0, and the <u>exterior angle</u> at 0 is the angle $\theta \in [-\pi, \pi]$ swept from $\alpha'(0^-)$ to $\alpha'(0^+)$.

The following example demonstrates this definition. Here the curve α is a polygon, and is directed in a clockwise direction. The vertices are at t_1, t_2, t_3, with corresponding exterior angles $\theta_1, \theta_2, \theta_3$.



Lemma 4.2.7. It is always possible to find an orthogonal parametrization for any surface S.

Theorem 4.2.8. [Gauss, Bonnet - local version]
Given a simply connected surface S with boundary ∂S positively oriented with a fuinite number of vertices with exterior angles,

$$\iint_S k \, ds = -\int_{\partial S} \kappa_g \, ds + \sum_{i=1}^{k} \theta_i - 2\pi$$

Definition 4.2.9. Given a decomposition D in polygons of a closed surface S, the <u>Euler characteristic</u> is defined as

$$\chi(S, D) = V - E + F$$

where the decomposition D has V vertices, E edges, and F faces.

Theorem 4.2.10. The Euler characteristic is independent of the decomposition D.

Theorem 4.2.11. [Gauss, Bonnet]
Suppose S is a closed compact orientable surface. Then

$$\iint_S k\ ds = 2\pi\chi(S)$$

Proof: Decompose S into polygons appropriate for orthogonal parametrization, say $S = \bigcup S_i$.

Label positively oriented (counter-clockwise) edges C_{ij} and exterior angles θ_{ij} of polygons S_i, so then

$$\iint_S k\ ds = \sum_i \iint_{S_i} k\ ds = -\sum_{i,j} \int_{C_{ij}} \kappa_g\ ds + \sum_i \left(2\pi - \sum_j \theta_{ij}\right)$$

Since each edge C_{ij} appears twice in the first term, with opposite orientation, the first term vanishes. As for the second term, we have that

$$\sum_i 2\pi = 2\pi F$$

$$-\sum_{ij} \theta_{ij} = \sum_{ij} \left(\pi - \theta'_{ij}\right) = 2\pi V - \pi \sum_{\text{vertices}} \deg(\text{vertex}) = 2\pi V - 2\pi E$$

Here $\theta'_{ij} = \pi - \theta_{ij}$ are the oppositely directed internal angles.
Combining the terms gives the desired result. ■

Corollary 4.2.12. It is impossible to have a metric on $\mathbb{S}^2$ with $k \leqslant 0$.

Theorem 4.2.13. By embedding a surface with g handles in $\mathbb{R}^n$ for $n \in \mathbb{N}$, it is possible to get

$$k \text{ equal to } \begin{cases} 1 & \text{with } g = 0 \\ 0 & \text{with } g = 1 \\ -1 & \text{with } g > 1 \end{cases}$$

5 Handy tables

5.1 Common parametrized curves

These are all maps $\alpha : I \to \mathbb{R}^3$ for $I = [0, 1] \subset \mathbb{R}$.

Curve	Parametrization	Curvature	Torsion
circle of radius r	$(r\sin(t), r\cos(t), 0)$	$\frac{1}{r}$	
helix with radii a, b	$(a\cos(t), a\sin(t), bt)$	$\frac{a}{a^2+b^2}$	$\frac{-b}{\sqrt{a^2+b^2}}$
trefoil knot	$((3+\cos(3t))\cos(2t), (3+\cos(3t))\sin(2t), \sin(3t))$		

5.2 Common parametrized surfaces

These are all maps $X : U \to \mathbb{R}^3$ for $U \subset \mathbb{R}^2$.

Surface	Parametrization	Induced metric	κ
plane through p spanned by a, b	$(x, y, ax + by + p)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
sphere of radius r	$(r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi))$	$\begin{bmatrix} \sin^2(\varphi) & 0 \\ 0 & 1 \end{bmatrix}$	$\frac{1}{r^2}$
cylinder of radius a	$(a\cos(u), a\sin(u), v)$	$\begin{bmatrix} a^2 & 0 \\ 0 & 1 \end{bmatrix}$	
ellipsoid of radii a, b			$\frac{1}{ab}$
torus of radii a, b	$(a\cos(u), (b + a\sin(u))\cos(v), (b + a\sin(u))\sin(v))$		$\frac{\sin(u)}{b+a\sin(v)}$