Compact course notes

Pure Math 450/650, Winter 2012

Lebesgue integration and Fourier analysis

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April 11, 2012

Contents

Note: Not all presented theorems are proved.

1 Toward the Lebesgue integral

1.1 Review of real analysis

Proposition 1.1.1. If $\varphi : X \to Y$ is a continuous bijection for X, Y compact, then φ is a homeomorphism.

Definition 1.1.2. A Banach space is a space over $\mathbb{R}$ or $\mathbb{C}$ equipped with a norm $\| \cdot \|$, that is

$\|x\| \geqslant 0$ for all $x \in X$	non-negativity
$\|x\| = 0 \iff x = 0$	non-degeneracy
$\|x + y\| \leqslant \|x\| + \|y\|$ for $x, y \in X$	sub-additivity
$\|\lambda x\| = \|\lambda\| \|x\|$ for $x \in X$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$	homogeneity

such that the metric space $(X, \| \, \|)$ is complete (Cauchy sequences have limits).

Definition 1.1.3. A metric space X is termed separable if it has a subset D which has a countable number of points and is dense in X.

Definition 1.1.4. The power set of a set A is defined as $\mathbb{P}(A) := \{S \mid S \subset A\}$.

Definition 1.1.5. A set A is a set of first category iff it can be realized as a countable union of sets whose closures are nowhere dense.

Definition 1.1.6. Let $a < b$ in $\mathbb{R}$ and $f : [a, b] \to X$ a function. Then:
· a partition of $[a, b]$ is a collection of points $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$
· a Riemann sum for f over P is any sum of the following form:

$$S(f, P) = \sum_{i=1}^{n} \underbrace{f(t_i^*)}_{\text{vector}} \underbrace{(t_i - t_{i-1})}_{\text{scalar}} \quad \text{for} \quad t_i^* \in [t_{i-1}, t_i] \text{ and } i \in \{1, 2, \ldots, n\}$$

Definition 1.1.7. We say that $f : [a, b] \to X$ is Riemann integrable if there is a point $x \in X$ such that for every $\varepsilon > 0$, there is a partition P_ε of $[a, b]$ such that for every refinement $P \supset P_\varepsilon$ and every associated Riemann sum $S(f, P)$, we have that $\|x - S(f, P)\| < \varepsilon$.

Definition 1.1.8. Note that if x as above exists, it is unique, and is termed the Riemann integral of f over $[a, b]$ and denoted

$$x = \int_a^b f = \int_a^b f(t)dt$$

Theorem 1.1.9. [Cauchy criterion for Riemann integrability]
Let $a < b$ in $\mathbb{R}$ and X a Banach space with $f : [a, b] \to X$. Then equivalently:
1. f is Riemann integrable on $[a, b]$
2. for every $\varepsilon > 0$, there is a partition Q_ε such that for any pair of refinements $P, Q \supset Q_\varepsilon$ and any associated Riemann sums, $\|S(f, P) - S(f, Q)\| < \varepsilon$.

Proof: ($1 \Rightarrow 2$) Easy exercise.

($1 \Leftarrow 2$) For each $n \in \mathbb{N}$, let Q_n be a partition of $[a, b]$ such that for any refinements $P, Q \supset Q_n$ and any associated Riemann sums, we have $\|S(f, Q) - S(f, P)\| < \frac{1}{2^n}$.
Now let $P_1 = Q_1$, $P_2 = Q_1 \cup Q_2, \ldots, P_n = \sum_{i=1}^{n} Q_i$, and let $X_n = S_n(f, P_n)$ be a fixed Riemann sum.
Note that $P_n \supset Q_n$ and $P_1 \subset P_2 \subset \cdots$

If $n > m$, then we have

$$\begin{aligned}
\|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \cdots - x_{m+1} + x_{m+1} - x_m\| \\
&\leqslant \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\
&= \|S_n(f, P_n) - S_{n-1}(f, P_{n-1})\| + \cdots \|D_{m+1}(f, P_{m+1}) - S_m(f, P_m)\| \\
&\leqslant \frac{1}{s^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m} \\
&= \frac{1}{2^{m-1}} \left(\frac{1}{2^{n-m}} + \cdots + \frac{1}{2} \right) \\
&< \frac{1}{2^{m-1}}
\end{aligned}$$

If $\varepsilon > 0$ is given, choose m such that $\frac{1}{2^{m-1}} < \varepsilon$, and then $(x_n)_{n=1}^{\infty}$ will be Cauchy in X.
Since X is a Banach space, there exists a limit $x \in X$ of $(x_n)_{n=1}^{\infty}$.
Now we must show that $x = \int_a^b f$, or that f satisfies the condition of Riemann integrability.
Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be such that $\frac{1}{2^{n-1}} < \frac{\varepsilon}{2}$.
If P_n is as above and $P \supset P_n$, then for any Riemann sum $S(f, P)$ we have that

$$\begin{aligned}
\|S(f, P) - x\| &\leqslant \|S(f, P) - x_{n+1}\| + \|x_{n+1} - x\| \\
&= \|S(f, P) - S_{n+1}(f, P_{n+1})\| + \lim_{m \to \infty} [\|x_{n+1} - x_m\|] \\
&< \frac{1}{2^n} + \frac{1}{2^n} \\
&= \frac{1}{2^{n-1}} \\
&< \varepsilon
\end{aligned}$$

■

1.2 Outer measure

Definition 1.2.1. For $E \subset \mathbb{R}$, a sequence $\{I_n\}_{n=1}^{\infty}$ of open intervals is termed a <u>cover</u> of E if $E \subset \bigcup_{n=1}^{\infty} I_n$.

Definition 1.2.2. The <u>length</u> of an interval $E \subset \mathbb{R}$ is defined naively, with the added condition that $\ell((-\infty, a)) = \ell((b, \infty)) = \infty$.

Definition 1.2.3. For $E \subset \mathbb{R}$, define the <u>outer measure</u> of E to be

$$\lambda^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \mid \{I_n\}_{n=1}^{\infty} \text{ is a cover of } E \text{ by open intervals} \right\}$$

Proposition 1.2.4. The outer measure is a function $\lambda^* : \mathbb{P}(\mathbb{R}) \to [0, \infty]$ with the properties:

1. $\lambda^*(\emptyset) = 0$

2. $\lambda^*(E) \geqslant 0$ for $E \subset P(\mathbb{R})$ — non-negativity

3. $E \subset F \subset \mathbb{R} \implies \lambda^*(E) \leqslant \lambda^*(F)$ — increasing

4. i. $\lambda^*(\bigcup_i E_i) \leqslant \sum_i \lambda^*(E_i)$ — σ-subadditivity

ii. $\lambda^*(\dot{\bigcup}_i E_i) = \sum_i \lambda^*(E_i)$

<u>*Proof:*</u> (**4.i.**) If $\sum \lambda^*(E_n) = \infty$, then we are done.
If not, then we can cover E_n by intervals so that $\sum_{i=1}^{\infty} \ell(I_{i,n}) < \lambda^*(E_n) + \frac{\varepsilon}{2^n}$.

Now consider $(I_{i,n})_{i=1}^{\infty}{}_{n=1}^{\infty}$ which covers $\bigcup_n E_n$, such that

$$\begin{aligned}\lambda^*(E) &= \sum_n \sum_i \ell(I_{i,n}) \\ &\leqslant \sum_n \left(\lambda^*(E_n) + \tfrac{\varepsilon}{2^n}\right) \\ &= \sum \lambda^*(E_n) + \sum \tfrac{\varepsilon}{2^n} \\ &= \sum \lambda^*(E_n) + \varepsilon\end{aligned}$$

■

Remark 1.2.5. The notation $\dot{\bigcup}_i E_i$ denotes a union of sets that is disjoint, i.e. $E_i \cap E_j = \emptyset \iff i \neq j$.

Proposition 1.2.6. Let $a < b \in \mathbb{R}$. Then $\lambda^*(J)$ is constant for any $J \in \{[a,b],[a,b),(a,b],(a,b)\}$.

Proof: Let $\varepsilon > 0$.
Then $(a-\varepsilon, b+\varepsilon)$ covers J, with $\lambda^*(J) \leqslant \ell((a-\varepsilon,b+\varepsilon)) = b-a+2\varepsilon$.
Since $\varepsilon > 0$, $\lambda^*(J) \leqslant b-a$.
Fix $J = [a,b)$.
Let $\varepsilon > 0$ such that $\varepsilon < b-a$, so that $K = [a, b-\varepsilon]$ is compact, and $((c_i,d_i))_{i=1}^{\infty}$ cover J by open intervals.
So it also covers K with a finite subcover, up to some $n \in \mathbb{N}$.
Arrange the is so that $c_1 < a$, $d_n > b-\varepsilon$, and $d_i < c_{i+1}$ for all $i = 1, \ldots, n$.
Then

$$\begin{aligned}\sum_{i=1}^{\infty} \ell((c_i,d_i)) &\geqslant \sum_{i=1}^{n} \ell((c_i,d_i)) \\ &= \sum_{i=1}^{n} (d_i - c_i) \\ &> c_1 - d_n \\ &= b - a - \varepsilon\end{aligned}$$

The other intervals are done in a similar fashion. ■

1.3 Lebesgue measure

Definition 1.3.1. [Caratheodory]
A set $A \subset \mathbb{R}$ is termed (Lebesgue) measurable if for any $E \subset \mathbb{R}$,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Note that the statement $\lambda^*(E) \leqslant \lambda^*(E \cap A) + \lambda^*(E \setminus A)$ always holds by σ-subadditivity, so only the opposite direction must be shown for measurability. Furthermore, we define

$$\mathcal{L}(\mathbb{R}) := \{A \subset \mathbb{R} \mid A \text{ is measurable}\}$$

Theorem 1.3.2.
1. $\emptyset, \mathbb{R} \in \mathcal{L}(\mathbb{R})$
2. $A \in (\mathbb{R}) \implies \mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$
3. $A_1, A_2, \ldots . \in \mathcal{L}(\mathbb{R}) \implies \bigcup_i A_i \in \mathcal{L}(\mathbb{R})$

Proof: (**2.**) Suppose that $A \in \mathcal{L}(\mathbb{R})$, so then

$$\lambda^*(E \cap (\mathbb{R} \setminus A)) + \lambda^*(E \setminus (\mathbb{R} \setminus A)) = \lambda^*(E \setminus A) + \lambda^*(E \cap A) = \lambda^*(E)$$

Therefore $\mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$.

(**3.**) Let $A_1, A_2, \cdots \in \mathcal{L}(\mathbb{R})$ with $E \subset \mathbb{R}$ and $A = \bigcup_i A_i$, so that

$$\begin{aligned} E \cap A &= \textstyle\bigcup_i (E \cap A_i) \\ &= (E \cap A_1) \cup (E \cap A_2) \cup (E \cap A_3) \cup \cdots \\ &= (E \cap A_1) \uplus ((E \setminus A_1) \cap A_2) \uplus ((E \setminus (A_1 \cup A_2)) \cap A_3) \uplus \cdots \end{aligned}$$

Note that by σ-subadditivity,

$$\lambda^*(E) \leqslant \lambda^*(E \cap A) + \lambda^*(E \setminus A) \leqslant \sum_{i=1}^{\infty} \lambda^* \left(\left(E \setminus \bigcup_{k=1}^{i-1} A_k \right) \cap A_i \right) + \lambda^*(E \setminus A)$$

Since $A_i \in \mathcal{L}(\mathbb{R})$ for all i,

$$\begin{aligned} \lambda^*(E) &= \lambda^*(E \cap A_1) + \lambda^*(E \setminus A_1) \\ &= \lambda^*(E \cap A_1) + \lambda^*((E \setminus A_1) \cap A_2) + \lambda^*((E \setminus A_1) \setminus A_2) \\ &= \sum_{i=1}^{n} \left(\lambda^* \left(\left(E \setminus \bigcup_{k=1}^{i-1} A_n \right) \cap A_i \right) \right) + \lambda^* \left(E \setminus \bigcup_{i=1}^{n} A_n \right) \\ &\geqslant \sum_{i=1}^{n} \left(\lambda^* \left(\left(E \setminus \bigcup_{k=1}^{i-1} A_n \right) \cap A_i \right) \right) + \lambda^*(E \setminus A) \end{aligned}$$

This also holds in the limit as $n \to \infty$.
Combining the above with σ-subadditivity, we get equality, so that $\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$.
Thus $A \in \mathcal{L}(\mathbb{R})$. ■

Definition 1.3.3. Define Lebesgue measure through outer measure, by

$$\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})} : \mathcal{L}(\mathbb{R}) \to [0, \infty]$$

Theorem 1.3.4. Lebesgue measure satisfies:

1. $\lambda(\emptyset) = 0$
2. $\lambda(A) \geqslant 0$ for $A \in \mathcal{L}(\mathbb{R})$
3. $A, B \in \mathcal{L}(\mathbb{R})$ and $A \subset B \implies \lambda(A) \leqslant \lambda(B)$
4. **i.** $\lambda(\bigcup_i A_i) \leqslant \sum_i \lambda(A_i)$
ii. $\lambda(\biguplus_i A_i) = \sum_i \lambda(E_i)$

Lemma 1.3.5. If $a < b \in \mathbb{R}$, then $(a, b) \in \mathcal{L}(\mathbb{R})$.

Proof: If $\lambda^*(E) = \infty$, we are done.
Suppose $\lambda^*(E) < \infty$.
Let $\varepsilon > 0$.
Then we can find for E a cover $(I_n)_{n=1}^{\infty}$ of open intervals such that $\sum_{n=1}^{\infty} \ell(I_n) < \lambda^*(E) + \frac{\varepsilon}{2}$.
For each $n \in \mathbb{N}$, let

$$J_n = I_n \cap (a, b) \qquad L_n = I_n \cap (-\infty, a) \qquad R_n = I_n \cap (b, \infty)$$

Then $(J_n)_{n=1}^{\infty}$ covers $E \cap (a,b)$ and $K_n = \{L_n, R_n, (a - \frac{\varepsilon}{8}, a + \frac{\varepsilon}{8}), (b - \frac{\varepsilon}{8}, b + \frac{\varepsilon}{8})\}$ covers $E \setminus (a,b)$, and

$$\sum_{n=1}^{\infty} \ell(K_n) = \sum_{n=1}^{\infty} (\ell(L_n) + \ell(R_n)) + \frac{\varepsilon}{2}$$

By the definition of λ^*, we have

$$\begin{aligned}
\lambda^*(E \cap (a,b)) + \lambda^*(E \setminus (a,b)) &\leqslant \sum_{n=1}^{\infty} \ell(J_n) + \sum_{n=1}^{\infty} \ell(K_n) \\
&= \sum (\ell(J_n) + \ell(R_n) + \ell(L_n)) + \frac{\varepsilon}{2} \\
&= \sum_{n=1}^{\infty} \ell(I_n) + \frac{\varepsilon}{2} \\
&< \lambda^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \lambda^*(E) + \varepsilon
\end{aligned}$$

Since ε was arbitrary, $\lambda^*(E \cap (a,b)) + \lambda^*(E \setminus (a,b)) \leqslant \lambda^*(E)$.
Becauce of σ-additivity, equality holds. ■

Corollary 1.3.6. Any open set $G \in \mathcal{L}(\mathbb{R})$.

Proof: By Assignment 1, question 4, we can find (a_n, b_n) for all $n \in \mathbb{N}$ such that $G = \dot{\bigcup}_{n=1}^{\infty} (a_n, b_n)$.

Apply the above lemma to all these intervals. ■

1.4 Algebras and measurability

Definition 1.4.1. Let X be a set. An algebra of subsets of X is any family of subsets $\mathcal{M} \subset \mathbb{P}(X)$ such that

1. $\emptyset, X \in \mathcal{M}$

2. $A \in \mathcal{M} \implies X \setminus A \in \mathcal{M}$

3. $A_1, A_2, \ldots, A_n \in \mathcal{M} \implies \bigcup_{i=1}^{n} \in \mathcal{M}$

Further, $\mathcal{M}$ is termed a σ-algebra if it satisfies **1.** and **2.** above, and **3.** for countable unions.

Example 1.4.2. These are some of the most common σ-algebras:

· Trivial σ-algebra $\mathcal{M} = \{\emptyset, X\}$

· Power set $\mathcal{M} = \mathbb{P}(X)$

· Lebesgue measurable sets $\mathcal{M} = \mathcal{L}(\mathbb{R})$

· For $\{\mathcal{M}_\beta \mid \beta \in B\} \subset \mathbb{P}(X)$ a family of σ-algebras, $\mathcal{M} = \bigcap_{\beta \in B} \mathcal{M}_\beta = \{A \subset X \mid A \in \mathcal{M}_\beta \ \forall \ \beta \in B\}$

· Borel σ-algebra $B(\mathbb{R}) = \bigcap \{\mathcal{M} \mid \mathcal{M} \subset \mathbb{P}(\mathbb{R})$ is a σ-algebra containing all open sets in $X\} \subset \mathcal{L}(\mathbb{R})$

Remark 1.4.3. Heuristically speaking, a Borel set is a subset of $\mathbb{R}$ that may be formed by taking countable intersections and/or unions of open subsets of $\mathbb{R}$.

Definition 1.4.4. Let $A \subset \mathbb{P}(X)$ be a family of sets such that $\emptyset, X \in A$. Then we define

$$A_\sigma := \left\{ \bigcup_{n=1}^{\infty} A_n \mid A_1, A_2, \dots \in A \right\} \qquad A_\delta := \left\{ \bigcap_{n=1}^{\infty} A_n \mid A_1, A_2, \dots \in A \right\}$$

Proposition 1.4.5. If $\mathcal{M}$ is a σ-algebra and $A_1, A_2, \dots \in \mathcal{M}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

Proof: Suppose $\mathcal{M} \subset \mathbb{P}(X)$.
Then each $X \setminus A_n \in M$, and hence $\bigcup_{n=1}^{\infty}(X \setminus A_n) \in M$, which leads to

$$\bigcap_{n=1}^{\infty} A_n = X \setminus \left(X \setminus \bigcap_{n=1}^{\infty} A_n \right) = X \setminus \left(\bigcup_{n=1}^{\infty} (X \setminus A_n) \right) \in M$$

■

Definition 1.4.6. Define

$$\mathcal{G} := \{\text{open sets in } \mathbb{R}\} \qquad \mathcal{F} := \{\text{closed sets in } \mathbb{R}\}$$

Then we have that $\mathcal{G}_\delta = \mathcal{G}$ and $\mathcal{F}_\sigma = \mathcal{F}$.

Definition 1.4.7. A set $A \subset \mathbb{R}$ is a G_δ-set if A may be expressed as a countable intersection of open subets of $\mathbb{R}$. Similarly, A is an F_σ-set if it may be expressed as a countable union of closed subets of $\mathbb{R}$.

Proposition 1.4.8. For the sets as defined above, $\mathcal{G} \subset \mathcal{F}_\sigma$ and $\mathcal{F} \subset \mathcal{G}_\delta$.

Proof: Let $G \in \mathcal{G}$.
By Assignment 1, question 4, $G = \dot{\bigcup}_{n=1}^{\infty}(a_n, b_n)$
For each $k \in \mathbb{N}$, let $F_k = \bigcup_{n=1}^{k}[a_n + \frac{1}{n}, b_n - \frac{1}{n}]$ for $[c, d] = \emptyset$ if $d < c$, and $[a, \infty - \frac{1}{n}] = [a, \infty]$.
Then each F_k, being a finite union of closed sets, is closed.
Moreover, $G = \bigcup_{k=1}^{\infty} F_k$.
Therefore $\mathcal{G} \subset \mathcal{F}_\sigma$.
If $F \in \mathcal{F}$, then $G = \mathbb{R} \setminus F \in \mathcal{G}$.
Hence $\mathbb{R} \setminus F = \bigcup_{k=1}^{\infty} F_k$ for each $F_k \in \mathcal{F}$.
Then

$$F = \mathbb{R} \setminus (\mathbb{R} \setminus F) = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} F_k = \bigcap_{k=1}^{\infty} (\mathbb{R} \setminus F_k) \in \mathcal{G}_\delta$$

■

Definition 1.4.9. Define the Cantor set in the following manner.

$$\begin{aligned}
C_0 &= [0, 1] & & \\
C_1 &= [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] & &= C_0 \setminus I_{11} \\
C_2 &= [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1] & &= C_1 \setminus (I_{21} \cup I_{22}) \\
&\vdots & & \\
C_n &= C_{n=1} \setminus (I_{n1} \cup \dots \cup C_{n2^{n-1}}) & &
\end{aligned}$$

Then $C = \bigcap_{n=1}^{\infty} C_n$ is termed the Cantor set. We note that C is compact and non-empty.

Proposition 1.4.10. For C the Cantor set as above,
1. C is nowhere dense in $\mathbb{R}$
2. $\lambda(C) = 0$
3. $|C| = |\mathbb{R}|$

Note that the Cantor set has Jordan content.

Proof: (**3.**) For $x \in [0,1]$, we may express x in ternary expansion,

$$x = 0.t_1t_2t_3\cdots = \sum_{i=1}^{\infty} \frac{t_i}{3} \quad \text{for } t_i \in \{0,1,2\}$$

However, this expansion is not unique, as $0.10000\cdots = 0.02222\ldots$.
We claim that C is the set of points with ternary expansions not containing 1s.
To see this, observe that for $I_{n,k}$ as in the definition of the Cantor set,

$$\begin{aligned}
I_{11} &= (\tfrac{1}{3},\tfrac{2}{3}) = \{x = 0.1t_2t_3\ldots \mid t_i \in \{0,1,2\}, t_i \neq 2 \text{ for some } i \geqslant 2 \text{ and } t_i \neq 0 \text{ for some } i \geqslant 2\} \\
I_{21} &= (\tfrac{1}{9},\tfrac{2}{9}) = \{x = 0.01t_3t_4\ldots \mid t_i \in \{0,1,2\}, t_i \neq 2 \text{ for some } i \geqslant 3 \text{ and } t_i \neq 0 \text{ for some } i \geqslant 3\} \\
I_{22} &= (\tfrac{7}{9},\tfrac{8}{9}) = \{x = 0.21t_3t_4\ldots \mid t_i \in \{0,1,2\}, t_i \neq 2 \text{ for some } i \geqslant 3 \text{ and } t_i \neq 0 \text{ for some } i \geqslant 3\} \\
&\vdots \\
I_{n,k} &= \{x = 0.t_1t_2\ldots t_{n-1}1t_{n+1}t_{n+2}\ldots \mid t_i \in \{0,1,2\}, t_i \neq 2 \text{ for some } i \geqslant n+1 \text{ and } t_i \neq 0 \text{ for some } i \geqslant n+1 \\
&\qquad \text{and } t_i \neq 1 \ \forall\, i \in \{1,\ldots,n-1\}
\end{aligned}$$

Further we note that

$$C = \bigcap_{n=1}^{\infty} C_n = [0,1] \setminus \bigcup_{n=1}^{\infty} \underbrace{\bigcup_{k=1}^{2^{n-1}} I_{n,k}}_{\text{set of points with } t_n = 1}$$

This induces a bijection

$$\begin{aligned}
\varphi : \quad C &\to \{0,2\}^{\mathbb{N}} \\
0.t_1t_2\ldots &\mapsto (t_i)_{i=1}^{\infty}
\end{aligned}$$

Since we know that $|\mathbb{R}| = |\{0,1\}^{\mathbb{N}}|$, this completes the proof. ∎

Proposition 1.4.11. If $x \in \mathbb{R}$ and $E \subset \mathbb{R}$, define the translate of E by x by

$$x + E := \{x + y \mid y \in E\}$$

The translate has the following properties:
1. $\lambda^*(E) = \lambda^*(x+E)$
2. $E \in \mathcal{L}(\mathbb{R}) \implies (x+E) \in \mathcal{L}(\mathbb{R})$
From these two we conclude that $\lambda(E) = \lambda(x+E)$ for all $x \in \mathbb{R}$.

Proof: (**1.**) For $G \subset \mathbb{R}$ open, we know $G = \dot\bigcup_{n=1}^{\infty}(a_n,b_n)$ with $\lambda(G) = \sum_{n=1}^{\infty}\lambda((a_n,b_n)) = \sum_{n=1}^{\infty}(b_n - a_n)$.
Then for fixed $x \in \mathbb{R}$, $x + G = \dot\bigcup_{n=1}^{\infty}(x+a_n, x+b_n)$ with $\lambda(x+G) = \sum_{n=1}^{\infty}((x+b_n)-(x+a_n)) = \lambda(G)$.
Now note that $E \subset G$ iff $x + E \subset x + G$.
Hence $\lambda^*(E) = \inf\{\lambda(G) \mid E \subset G, G \text{ open}\} = \inf\{\lambda(G) \mid x+E \subset x+G, G \text{ open}\} = \lambda^*(x+E)$.

(**2.**) Let $E \subset \mathcal{L}(\mathbb{R})$ and $A \subset \mathbb{R}$. Then

$$\begin{aligned}
\lambda^*(A \cap (x+E)) + \lambda^*(A \setminus (x+E)) &= \lambda^*(x + ((-x+A)\cap E)) + \lambda^*(x + ((-x+A)\setminus E)) \\
&= \lambda^*((-x+A)\cap E) + \lambda^*((-x+A)\setminus E) \\
&= \lambda^*(-x+A) \\
&= \lambda^*(A)
\end{aligned}$$

∎

Theorem 1.4.12. There exists a subset $E \subset \mathbb{R}$ that is not Lebesgue measurable.

Proof: Fix $a > 0$.
On $(-a, a)$, define a relation $x \sim y$ iff $x - y \in \mathbb{Q}$.
For $x \in [-a, a]$, define the equivalence class $[x] = (x + \mathbb{Q}) \cap (-a, a)$.
Let $E \subset (-a, a)$ be such that:
1. if $x, y \in E$, then $x \neq y$ implies $x \nsim y$
2. $(-a, a) = \dot{\bigcup}_{x \in E} [x]$
Note that E contains exactly one point from each equivalence class, which requires the axiom of choice.
Now enumerate $(-2a, 2a) \cap \mathbb{Q} = (r_k)_{k=1}^{\infty}$
We claim that $(-a, a) \subset \dot{\bigcup}_{k=1}^{\infty} (r_k + E) \subset (-3a, 3a)$.
Note that $(r_k + E) \cap (r_j + E) = \emptyset \iff i \neq j$, since $x = r_k + y - r_j + z$ implies $y - z = r_j - r_k \in \mathbb{Q}$.

To see the first inclusion, note if $x \in (-a, a)$, then $x \in [y]$ for some $y \in E$.
Thus $x - y \in \mathbb{Q}$ and $|x - y| < 2a$, so $x - y = r_k$ for some k.
To see the second inclusion, note that $|r_k + x| < 3a$ for any $x \in (-a, a)$ and $r_k \in (-2a, 2a)$.

Now we will show that $E \notin \mathcal{L}(\mathbb{R})$.
Suppose $E \in \mathcal{L}(\mathbb{R})$, so either $\lambda(E) = 0$ or $\lambda(E) = \alpha > 0$.
If $\lambda(E) = 0$, then $\lambda(k + E) = 0$ for all k.
Then by the increasing and σ-additivity properties,

$$2a = \lambda((-a, a)) \leqslant \lambda\left(\dot{\bigcup_{k=1}^{\infty}} (E_k + r_k)\right) = \sum_{k=1}^{\infty} \lambda(E + r_k) = 0$$

Hence its measure must be non-zero.
Then by the increasing, σ-additivity, and translation invariance properties for $n \in \mathbb{N}$,

$$n\alpha = \sum_{k=1}^{n} \lambda(r_k + E) = \lambda\left(\dot{\bigcup_{k=1}^{n}} (r_k + E)\right) \leqslant \lambda\left(\dot{\bigcup_{k=1}^{\infty}} (r_k + E)\right) \leqslant \lambda((-3a, 3a)) = 6a$$

Clearly this cannot hold for $n > \frac{6a}{\alpha}$.
Therefore $E \notin \mathcal{L}(\mathbb{R})$. ■

Remark 1.4.13. If the axiom of choice is weakened to only allow countable choice, then $\mathbb{P}(\mathbb{R}) = \mathcal{L}(\mathbb{R}) = B(\mathbb{R})$. This makes irrelevant most of the previous work, so we acept the axiom of choice.

Remark 1.4.14. For the non-measurable set E above, we have
i. $0 < \lambda^*(E) \leqslant 2a$
ii. $\lambda_*(E) = 0$
We will learn about the Lebesgue inner measure λ_* in the current Assignment.

Definition 1.4.15. A subset $N \subset \mathbb{R}$ is termed a (Lebesgue) null set if $\lambda^*(N) = 0$. Note that null sets are measurable, as

$$\lambda^*(\underbrace{E \cap N}_{\subset N}) + \lambda^*(\underbrace{E \setminus N}_{\subset E}) \leqslant \lambda^*(N) + \lambda^*(E) = \lambda^*(E)$$

2 The Lebesgue integral

2.1 Function measurability

Definition 2.1.1. Define the (characteristic) indicator function $\chi_A : X \to \{0, 1\}$ for a set $A \subset X$ by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Definition 2.1.2. A function $f : \mathbb{R} \to \mathbb{R}$ is termed measurable if $f^{-1}((\alpha, \infty)) = \{x \mid x \in \mathbb{R}, f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

We say that f is Borel measurable if $f^{-1}((\alpha, \infty)) \in B(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.

Proposition 2.1.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equivalent.

1. f is measurable
2. $f^{-1}((-\infty, \alpha]) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$
3. $f^{-1}((-\infty, \alpha)) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$
4. $f^{-1}((\alpha, -\infty)) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$

Proof: (**1.**$\Leftrightarrow$**2.**) First note that

$$f^{-1}((-\infty, \alpha]) = \{x \in \mathbb{R} \mid f(x) \leqslant \alpha\} = \mathbb{R} \setminus \{x \in \mathbb{R} \mid f(x) > \alpha\}$$

Now recall that for $A \subset \mathbb{R}$, we had that $A \in \mathcal{L}(\mathbb{R})$ iff $\mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$.

(**2.**$\Rightarrow$**3.**) Observe that

$$f^{-1}((-\infty, \alpha)) = f^{-1}\left(\bigcup_{n=1}^{\infty} (-\infty, \alpha - \tfrac{1}{n}]\right) = \bigcup_{n=1}^{\infty} \left(f^{-1}\left((-\infty, \alpha - \tfrac{1}{n}]\right)\right)$$

(**3.**$\Leftrightarrow$**4.**) as (**1.**$\Leftrightarrow$**3.**)
(**4.**$\Rightarrow$**2.**) as (**2.**$\Rightarrow$**3.**) ■

Corollary 2.1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is measurable iff $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ for any $B \in B(\mathbb{R})$.

Proof: ($\Rightarrow$) Let $G \subset \mathbb{R}$ be open, so that $G = \dot\bigcup_{n=1}^{\infty}(a_n, b_n)$, and

$$f^{-1}(G) = f^{-1}\left(\dot\bigcup_{n=1}^{\infty}(a_n, b_n)\right) = \dot\bigcup_{n=1}^{\infty}\left(\underbrace{f^{-1}((-\infty, b_n))}_{\in \mathcal{L}(\mathbb{R})} \cap \underbrace{f^{-1}((a_n, \infty))}_{\in \mathcal{L}(\mathbb{R})}\right) \in \mathcal{L}(\mathbb{R})$$

($\Leftarrow$) Let $M_f = \{M \in \mathbb{R} \mid f^{-1}(M) \in \mathcal{L}(\mathbb{R})\}$.
Note that $f^{-1}(\mathbb{R}) = \mathbb{R}$ implies $\mathbb{R} \in M_f$.
Also, $M_1, M_2, \dots \in M_f$ implies $f^{-1}\left(\bigcup_{i=1}^{\infty} M_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(M_i) \in \mathcal{L}(\mathbb{R})$.
Further, if $M \in M_f$, then $f^{-1}(\mathbb{R} \setminus M) = \mathbb{R} \setminus f^{-1}(M) \in \mathcal{L}(\mathbb{R})$.
Therefore M_f is a σ-algebra.
From above we have that $\mathcal{G} \subset M_f$.
Since $B(\mathbb{R})$ is the smallest σ-algebra containing $\mathcal{G}$, we must have $B(\mathbb{R}) \subset M_f$. ■

Proposition 2.1.5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable and $c \in \mathbb{R}$ fixed with $\varphi : \mathbb{R} \to \mathbb{R}$ continuous. Then

$$cf \qquad f + g \qquad \varphi \circ f \qquad fg$$

are all measurable functions.

Proof: (**1.**) Let $\alpha \in \mathbb{R}$. Then

$$(cf)^{-1}((\alpha, \infty)) = \begin{cases} f^{-1}((\frac{\alpha}{c}, \infty)) & c > 0 \\ \mathbb{R} & c = 0, \alpha < 0 \\ \emptyset & c = 0, \alpha \geqslant 0 \\ f^{-1}((-\infty, \frac{\alpha}{c})) & c > 0 \end{cases}$$

Since all of the results are in $\mathcal{L}(\mathbb{R})$ by the assumption, cf is Lebesgue measurable.

(**2.**) Enumerate the rationals by $(r_k)_{k=1}^{\infty}$, so that

$$\begin{aligned}(f+g)^{-1}((\alpha,\infty)) &= \{x\in\mathbb{R} \mid f(x)+g(x)>\alpha\}\\ &= \{x\in\mathbb{R} \mid f(x)>\alpha-g(x)\}\\ &= \bigcup_{k=1}^{\infty}\{x\in\mathbb{R} \mid f(x)>r_k \text{ and } r_k>\alpha-g(x)\}\\ &= \bigcup_{k=1}^{\infty}\left(\{x\in\mathbb{R} \mid f(x)>r_k\}\cap\{x\in\mathbb{R} \mid g(x)>\alpha-r_k\}\right)\\ &= \bigcup_{k=1}^{\infty}\left(f^{-1}((r_k,\infty))\cap g^{-1}((\alpha-r_k,\infty))\right)\\ &\in \mathcal{L}(\mathbb{R})\end{aligned}$$

(**3.**) Let $\alpha\in\mathbb{R}$. Then

$$(\varphi\circ f)^{-1}((\alpha,\infty)) = f^{-1}(\underbrace{\varphi^{-1}(\underbrace{(\alpha,\infty)}_{\text{open set}})}_{\text{open set}})\in\mathcal{L}(\mathbb{R})$$

(**4.**) Merely observe that, for m signifying a measurable function,

$$fg = \underbrace{\tfrac{1}{4}\left(\underbrace{(\underbrace{f+g}_{m})^2}_{m}-\underbrace{(\underbrace{f-g}_{m})^2}_{m}\right)}_{m}$$

■

Definition 2.1.6. Let $f:\mathbb{R}\to\mathbb{R}$. Then we define non-negative functions

$$\begin{aligned}|f|(x) &= |f(x)|\\ f^{-}(x) &= \max\{f(x),0\}\\ f^{+}(x) &= \max\{-f(x),0\}\end{aligned}$$

Corollary 2.1.7. Let $f:\mathbb{R}\to\mathbb{R}$. Then $|f|, f^{+}, f^{-}$ are all measurable.

· To see this, note that $x\mapsto|x|$ is continuous, and $f^{\pm}=\frac{1}{2}(|f|\pm f)$, and use a previous result.

Definition 2.1.8. Let $A\in\mathcal{L}(\mathbb{R})$. Then define $M(A)=\{f:A\to\mathbb{R}\mid f \text{ is measurable}\}$, which is an algebra of functions.

· We extend f to $\tilde{f}:\mathbb{R}\to\mathbb{R}$ by letting $\tilde{f}(x)=\left\{\begin{smallmatrix} f(x) & x\in A\\ 0 & x\notin A\end{smallmatrix}\right.$ so then f is measurable iff $\tilde{f}$ is measurable.

Definition 2.1.9. Define the <u>extended real numbers</u> to be $\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,\infty\}$.

Remark 2.1.10. A function $f:\mathbb{R}\to\overline{\mathbb{R}}$ is termed <u>extended real valued</u>. Then f is measurable if $f^{-1}(B)\in\mathcal{L}(\mathbb{R})$ for each $B\in B(\mathbb{R})$, and $f^{-1}(\{\pm\infty\})\in\mathcal{L}(\mathbb{R})$.

Proposition 2.1.11. Let $f_n:\mathbb{R}\to\overline{\mathbb{R}}$ be measurable for all n. Then

$$\sup_n\{f_n\} \qquad \inf_n\{f_n\} \qquad \limsup_{n\to\infty}\{f_n\} \qquad \liminf_{n\to\infty}\{f_n\}$$

are all measurable functions.

Proof: (**1.**) Fix $\alpha \in \mathbb{R}$. Then

$$\left(\sup_n\{f_n\}\right)^{-1}([-\infty,\alpha]) = \{x \in \mathbb{R} \mid \sup_n\{f_n\} \leqslant \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}([-\infty,\alpha])}_{\in\mathcal{L}(\mathbb{R})}$$

(**2.**) Similarly as above, show that $(\inf_n\{f_n\})^{-1}([-\infty,\alpha]) \in \mathcal{L}(\mathbb{R})$.

(**3.**) Observe that

$$\limsup_{n\to\infty}[f_n] = \lim_{n\to\infty}\left[\sup_{k\geqslant n}\{f_k\}\right] = \inf_n\left\{\sup_{k\geqslant n}\{f_k\}\right\}$$

Since $\sup_{k\geqslant n}\{f_k\}$ is measurable by **1.**, the result follows.

(**4.**) Same as (**3.**), ■

Corollary 2.1.12. If $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable for all n and $\lim_{n\to\infty}[f(x)] \in \overline{\mathbb{R}}$ for all x, then $\lim_{n\to\infty}[f(x)]$ is measurable.

Proof: The following sets are all measurable:

$$\lim_{n\to\infty}[f_n] = \limsup_{n\to\infty}[f_n] = \liminf_{n\to\infty}[f_n]$$

■

2.2 Simple function integration

Definition 2.2.1. Let $A \in \mathcal{L}(\mathbb{R})$. A function $f : A \to \mathbb{R}$ is termed simple if $f(A) = \{a_1, \ldots, a_n\}$ is finite. Then we express f in standard form as $f(A) = \{a_0 < \cdots < a_n\}$, and for $E_i = f^{-1}(\{a_i\})$, then $f = \sum_{i=1}^{n} a_i \chi_{E_i}$.

Proposition 2.2.2. A simple function $f : A \to \mathbb{R}$ is measurable iff when written in standard form as above, each E_i is measurable.

Definition 2.2.3. Define

$$S(A) := \{\varphi : A \to \mathbb{R} \mid \varphi \text{ is simple and measurable}\}$$
$$S^+(A) := \{\varphi \in S(A) \mid \varphi \geqslant 0\}$$

Then we define the proto-integral for $\varphi \in S^+(A)$ in standard form as

$$I_A(\varphi) := \sum_{i=1}^{n} a_i \lambda(E_i)$$

which may be infinite, and with the condition that $0 \cdot \infty = \infty \cdot 0 = 0$.

Proposition 2.2.4. Let $\varphi, \psi \in S^+(A)$ and $c \geqslant 0$. Then

1. $I_A(c\varphi) = cI_A(\varphi)$
2. $I_A(\varphi + \psi) = I_A(\varphi) + I_A(\psi)$
3. $\varphi \leqslant \psi \implies I_A(\varphi) \leqslant I_A(\psi)$

<u>*Proof:*</u> (**2.**) Let $\varphi(A) = \{a_1 < \cdots < a_n\}$, $\psi(A) = \{b_1 < \cdots < b_m\}$ and $E_i = \varphi^{-1}(\{a_i\})$, $F_j = \psi^{-1}(\{b_j\})$.
Consider $\{a_i + b_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\} = \{c_1 < \cdots < c_\ell\}$.
For $1 \leqslant k \leqslant \ell$, let $D_k = \dot\bigcup_{a_i+b_j=c_k}(E_i \cap F_j)$, so then

$$\begin{aligned}\varphi + \psi &= \sum_{o=1}^{n} a_i \chi_{E_i} + \sum_{j=1}^{m} b_j \chi_{F_j} \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{E_i \cap F_j} + \sum_{j=1} b_j \sum_{i=1}^{n} \chi_{E_i \cap F_j} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) \chi_{E_i \cap F_j} \\ &= \sum_{k=1}^{\ell} c_k \chi_{D_k}\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}I_A(\varphi) + I_A(\psi) &= \sum_{i=1}^{n} a_i \lambda(E_i) + \sum_{j=1}^{m} b_j \lambda(F_j) \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \lambda(E_i \cap F_j) + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \lambda(E_i \cap F_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) \lambda(E_i \cap F_j) \\ &= \sum_{k=1}^{\ell} c_k \lambda(D_k) \\ &= I_A(\varphi + \psi)\end{aligned}$$

(**3.**) Let a_i, b_j, E_i, F_j as above.
Then $E_i \cap F_j \neq \emptyset$ implies $a_i \leqslant b_j$, as $\varphi \leqslant \psi$, so then

$$I_A(\varphi) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \lambda(E_i \cap F_j) \leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \lambda(E_i \cap F_j) = I_A(\psi)$$

■

Definition 2.2.5. Given $A \in \mathcal{L}(\mathbb{R})$, define

$$\overline{M}^+(A) := \{f : A \to [0, \infty] \mid f \text{ is measurable}\}$$

Definition 2.2.6. If $A \in \mathcal{L}(\mathbb{R})$ and $f \in \overline{M}^+(A)$, then define

$$\begin{aligned}S_f^+(A) &:= \{\varphi \in S^+(A) \mid \varphi \leqslant f\} \\ \int_A f &:= \sup\{I_A(\varphi) \mid \varphi \in S_f^+(A)\}\end{aligned}$$

The latter is termed the <u>Lebesgue integral</u> of f.

Proposition 2.2.7. Let $A \in \mathcal{L}(\mathbb{R}) \setminus \{\emptyset\}$ and $f, g \in \overline{M}^+(A)$. Then

1. $f \leqslant g$ on $A \implies \int_A f \leqslant \int_A g$

2. $\emptyset \subsetneq B \subset A$ and $B \in \mathcal{L}(\mathbb{R}) \implies \int_B f = \int_A f\chi_B$

3. $\varphi \in S^+(A) \implies \int_A \varphi = I_A(\varphi)$

Proof: (**1.**) If $S_f^+(A) \subset S_g^+(A)$, then

$$\int_A f = \sup_{\varphi \in S_f^+(A)} \{I_A(\varphi)\} \leqslant \sup_{\psi \in S_g^+(A)} \{I_A(\psi)\} = \int_A g$$

(**2.**) For $\varphi \in S_f^+(B)$, define $\tilde{\varphi}$ on A by $\tilde{\varphi}(x) = \left\{ \begin{smallmatrix} \varphi(x) & x \in B \\ 0 & x \in A \setminus B \end{smallmatrix} \right.$.

We note that $\tilde{\varphi}$ is simple and measurable.

So then $\tilde{\varphi} \in S^+f(A)$ and $\{\tilde{\varphi} \mid \varphi \in S_f^+(B)\} = S_{f\chi_B}^+(A)$, so

$$\int_A f\chi_B = \sup_{\varphi \in S_{f\chi_B}^+(A)} \{I_A(\varphi)\} = \sup_{\varphi \in S_f^+(B)} \{I_A(\tilde{\varphi})\} = \sup_{\varphi \in S_f^+(B)} \{I_B(\varphi)\}$$

(**3.**) First we note if $\psi \in S_\varphi^+(A)$, then $I_A(\psi) \leqslant I_A(\varphi)$, and hence

$$\int_A \varphi = \sup_{\psi \in S_\varphi^+(A)} \{I_A(\psi)\} \leqslant I_A(\varphi)$$

On the other hand, $\varphi \in S_\varphi^+(A)$, so $I_A(\varphi) \leqslant \int_A \varphi$. ■

Lemma 2.2.8. If $A_1 \subset A_2 \subset \cdots \in \mathcal{L}(\mathbb{R})$, then $\lambda\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} [\lambda(A_n)]$.

Proof: Let $C_1 = A_1$ and $C_n = A_n \setminus A_{n-1}$ for $n \geqslant 2$.

Since $A_1 \subset A_2 \subset \cdots$, we have $C_i \cap C_j = \emptyset$ iff $i \neq j$. Then

$$\lambda\left(\bigcup_{n=1}^\infty A_n\right) = \lambda\left(\dot{\bigcup_{n=1}^\infty} C_n\right) = \sum_{n=1}^\infty \lambda(C_n) = \lim_{N\to\infty}\left[\sum_{n=1}^N \lambda(C_n)\right] = \lim_{N\to\infty}[\lambda(A_N)]$$

■

Theorem 2.2.9. [Monotone convergence theorem]

Let $(f_n)_{n=1}^\infty \subset \overline{M}^+(A)$ with $f_1 \leqslant f_2 \leqslant \cdots$ pointwise, and let $f = \lim_{n\to\infty}[f_n]$. Then

$$\int_A f = \lim_{n\to\infty}\left[\int_A f_n\right]$$

In particular, if $\sup_{n\in\mathbb{N}}\left[\int_A f_n\right] < \infty$, then $\int_A f < \infty$.

Proof: Since $f_1 \leqslant f_2 \leqslant \cdots$, we have that $\int_A f_1 \leqslant \int_A f_2 \leqslant \cdots$.

Hence $\lim_{n\to\infty}\left[\int_a f_n\right] = \sup_{n\in\mathbb{N}}\left[\int_A f_n\right]$.

Also, $f \in \overline{M}^+(A)$ by a previous result.

Since $f_n \leqslant f$ for each n, we have that $\int_A f_n \leqslant \int_A f$ which implies $\lim_{n\to\infty}\left[\int_a f_n\right] \leqslant \int_a f$.

Thus it remains to establish that $\lim_{n\to\infty}\left[\int_A f_n\right] \geqslant \int_A f_n$.

Let $\varphi \in S_f^+(A)$ and $0 < \eta < 1$.
Let $A_n = \{x \in A \mid f_n(x) \geqslant \eta\varphi(x)\}$, so then:
· $A_1 \subset A_2 \subset \cdots$
· $\bigcup_{i=1}^\infty A_i = A$
Now let $\eta\varphi(A) = \{a_1 < a_2 < \cdots < a_m\}$ and $E_i = (\eta\varphi)^{-1}(\{a_i\}) \subset A$ for all $i = 1, \ldots, m$.
Then, for each n,

$$\int_A f_n \geqslant \int_A f_n \chi_{A_n} = \int_{A_n} f_n \geqslant \int_{A_n} \eta\varphi = \sum_{i=1}^m a_i \chi_{(E_i \cap A_i)}$$

By a previous lemma, since $E_i = E_i \cap A = \bigcup_{n=1}^\infty (E_i \cap A_n)$, taking the limit as $n \to \infty$, we get

$$\sum_{i=1}^m a_i \lambda(E_i) = \int_A \eta\varphi = \eta \int_A \varphi$$

Therefore $\lim\limits_{n\to\infty} [\int_A f_n] \geqslant \eta \int_A \varphi$.
Since this holds for all $0 < \eta < 1$, we have that $\lim\limits_{n\to\infty} [\int_A f_n] \geqslant \lim\limits_{\eta\to 1} [\eta \int_A \varphi] = \int_A \varphi$.
Thus, since $\varphi \in S_f^+(A)$

$$\lim_{n\to\infty} \left[\int_A f_n\right] \geqslant \sup_{\varphi \in S_f^+(A)} \left[\int_A \varphi\right] = \int_A f$$

■

Lemma 2.2.10. Let $f : A \to [0, \infty]$ for $A \in \mathcal{L}(\mathbb{R}) \setminus \{\emptyset\}$. Then

$$f \in \overline{M}^+(A) \iff \text{there exists a sequence } (\varphi_n)_{n=1}^\infty \subset S^+(A) \text{ with } \lim_{n\to\infty} [\varphi_n] = f \text{ pointwise}$$

<u>*Proof:*</u> ($\Leftarrow$) A limit of a sequence of measurable functions is measurable.

($\Rightarrow$) For each $k \in \mathbb{N}$, let $F_k = f^{-1}([k, \infty])$.
For each $i = 1, 2, \ldots, k2^k$, let $E_{k,i} = f^{-1}([\frac{i-1}{2^k}, \frac{i}{2^k}])$.
Then we have that

$$A = F_k \cup \dot{\bigcup_{i=1}^{k2^k}} E_{k,i}$$

$$\varphi_k = k\chi_{F_k} + \sum_{i=1}^{k2^k} \frac{i-1}{2^k} \chi_{E_{k,i}}$$

Then clearly $\varphi_1 \leqslant \varphi_2 \leqslant \cdots$ and $\lim\limits_{k\to\infty} [\varphi_k] = f$. ■

Corollary 2.2.11. Let $A \in \mathcal{L}(\mathbb{R}) \setminus \{\emptyset\}$. Then

1. if $f, g \in \overline{M}^+(A), c \geqslant 0$, then $\int_A cf = c\int_A f$ and $\int_A (f+g) = \int_A f + \int_A g$

2. if $(f_n)_{n=1}^\infty \subset \overline{M}^+(A)$, then $\int_A \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int_A f_n$

3. if $A_i \in \mathcal{L}(\mathbb{R})$ for all $i \in \mathbb{N}$ with $A = \dot{\bigcup}_{i\in\mathbb{N}} A_i$, then $\int_A f = \sum_{i\in\mathbb{N}} \int_{A_i} f$

<u>*Proof:*</u> **(1.)** Let $(\varphi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty \subset S^+(A)$ nondecreasing with $\lim\limits_{n\to\infty} [\varphi_n] = f$ and $\lim\limits_{n\to\infty} [\psi_n] = g$.
Then $\varphi_1 + \psi_1 \leqslant \varphi_2 + \psi_2 \leqslant \cdots$ and $\lim\limits_{n\to\infty} [\varphi_n + \psi_n] = f + g$.

Using MCT, we have that

$$\begin{aligned}\int_A (f+g) &= \lim_{n\to\infty}\left[\int_A (\varphi_n+\psi_n)\right]\\ &= \lim_{n\to\infty}\left[\int_A \varphi_n + \int_A \psi_n\right]\\ &= \lim_{n\to\infty}\left[\int_A \varphi_n\right] + \lim_{n\to\infty}\left[\int_A \psi_n\right]\\ &= \int_A f + \int_A g\end{aligned}$$

(**2.**) Let $g_n = \sum_{k=1}^{\infty} f_k \in \overline{M}^+(A)$ with $g_1 \leqslant g_2 \leqslant \cdots$ such that $\lim\limits_{n\to\infty}[g_n] = \sum\limits_{k=1}^{\infty} f_k$.
Use (**1.**) to see that $\int_A g_n = \sum_{k=1}^{n}\int_A f_k$ and MCT for

$$\sum_{k=1}^{\infty}\int_A f_k = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\int_A f_n\right] = \lim_{n\to\infty}\left[\int_A g_n\right] = \int_A \sum_{k=1}^{\infty} f_k$$

(**3.**) Let $f_n = \chi_{A_n}$, and note $f\chi_{A_n} \in \overline{M}^+(A)$ with $f = \sum_{n=1}^{\infty} f\chi_{A_n}$
Use (**2.**) above and the fact that $\int_A f\chi_{A_n} = \int_{A_n} f$ to get the result. ■

Definition 2.2.12. Define $\overline{M}(A) := \{f : A \to \bar{\mathbb{R}} \mid f \text{ is measurable}\}$.

Definition 2.2.13. Let $A \in \mathcal{L}(\mathbb{R})\backslash\{\emptyset\}$. Then $f \in \overline{M}(A)$ is termed Lebesgue integrable if $\int_A f^+, \int_A f^- < \infty$. For such f, we define the Lebesgue integral of f to be

$$\int_A f = \int_A f^+ - \int_A f^-$$

We introduce new sets:

$$\begin{aligned}\overline{L}(A) &:= \{f : A \to \overline{\mathbb{R}} \mid f \in \overline{M}(A), f \text{ is Lebesgue integrable}\}\\ L(A) &:= \{f : A \to \mathbb{R} \mid f \in M(A), f \text{ is Lebesgue integrable}\}\end{aligned}$$

Lemma 2.2.14.
1. If $f \in \overline{L}(A)$, then $\lambda(f^{-1}(\{-\infty,\infty\})) = 0$
2. If $f \in \overline{M}(A)$, then $\left(\int_A |f| = 0 \iff \lambda(f^{-1}([-\infty,0)) \cup f^{-1}((0,\infty])) = 0\right)$

Proof: (**1.**) Let $E^+ = f^{-1}(\{\infty\})$.
Then for any $n \in \mathbb{N}$, $n\chi_{E^+} \leqslant f^+$, and

$$n\lambda(E^+) = \int_A n\chi_{E^+} \leqslant \int_A f^+ < \infty$$

Rearranging, $\lambda(E^+) \leqslant \frac{1}{n}\int_A f^+$, and so $\lambda(E^+) = 0$.
Doing a similair approach for the pullback of $\{-\infty\}$, we see that the statement holds. ■

Corollary 2.2.15. Let $f \in \overline{L}(A)$. Then there exists $f_0 \in L(A)$ such that $f(x) = f_0(x)$ for all $x \in A \setminus N$ with $\lambda(N) = 0$.

This relationship is defined as $f = f_0$ almost everywhere, expressed $f = f_0$ (a.e.).

Proof: We know that since $f \in \overline{L}(A)$, $\lambda(f^{-1}(-\infty) \cup f^{-1}(\infty)) = 0$.
Define $f_0 : A \to \mathbb{R}$ by

$$f_0(x) = \begin{cases} f(x) & x \notin f^{-1}(-\infty) \cup f^{-1}(\infty) \\ 0 & \text{else} \end{cases}$$

■

Theorem 2.2.16. [Properties of the integral]
Let $f, g \in L(A)$ and $c \in \mathbb{R}$. Then

$$\begin{array}{rcl} (cf) \in L(A) & \text{and} & \int_A cf = c\int_A f \\ (f+g) \in L(A) & \text{and} & \int_A (f+g) = \int_A f + \int_A g \\ |f| \in L(A) & \text{and} & \int_A |f| \leqslant \int_A f \end{array}$$

Moreover, if $f : A \to \mathbb{R}$, then $f \in L(A)$ iff $|f| \in L(A)$ and $f \in M(A)$.

Proof: (**2.**) First note that $(f+g) = (f+g)^+ - (f+g)^-$, and $(f+g)^\pm \leqslant f^\pm + g^\pm$.
Hence $\int_A (f+g)^\pm \leqslant \int_A (f^\pm + g^\pm) = \int_A f^\pm + \int_A g^\pm$
Therefore $f + g \in L(A)$.
Claim: For $h, k, \varphi, \psi \in L^+(A)$ and $h - k = \varphi - \psi$, we have that $\int_A h - \int_A k = \int_A \varphi - \int_A \psi$.
By a previous corollary, $\int_A + \int_A \psi = \int_A (h+\psi) = \int_A (\varphi + k) = \int_A \varphi + \int_A k$ for finite integrals.
Now we have $(f+g)^+ + (f+g)^- = f + g = f^+ - f^- + g^+ - g^- = (f^+ + g^+) - (f^- + g^-)$.
Since all these functions are Lebesgue integrable,

$$\begin{aligned} \int_A (f+g) &= \int_A (f+g)^+ - \int_A (f+g)^- \\ &= \int_A (f^+ + g^+) - \int_A (f^- + g^-) \\ &= \int_A f^+ + \int_A g^+ - \int_A f^- - \int_A g^- \\ &= \int_A f + \int_A g \end{aligned}$$

(**3.**) First we recall that $|f| = f^+ + f^-$, so then

$$\begin{aligned} \left| \int_A f \right| &= \left| \int_A f^+ + \int_A f^- \right| \\ &\leqslant \left| \int_A f^+ \right| + \left| \int_A f^- \right| \\ &= \int_A f^+ + \int_A f^- \\ &= \int_A (f^+ + f^-) \\ &= \int_A |f| \end{aligned}$$

Since all the terms are finite, the result is finite.
If $|f| \in L(A)$ and $f \in M(A)$, then we get that $f^+, f^- \in M(A)$.
Then the first assumption gives us that

$$\int_A f^+, \int_A f^- \leqslant \int_A f^+ + \int_A f^- = \int_a (f^+ + f^-) = \int_A |f| < \infty$$

■

Lemma 2.2.17. [Fatou]
Let $(f_n)_{n=1}^{\infty} \subset \overline{M}^+(A)$. Then

$$\int_A \liminf_{n\to\infty} [f_n] \leqslant \liminf_{n\to\infty} \left[\int_A f_n\right]$$

Proof: Let $g_n = \inf_{k \geqslant n} \{f_k\}$.

Then $0 \leqslant g_1 \leqslant g_2 \leqslant \cdots$, and $\lim_{n\to\infty} [g_n] = \liminf_{n\to\infty} [f_n]$ by definition.

Applying MCT, we get $\int_A \liminf_{n\to\infty} [f_n] = \lim_{n\to\infty} [\int_A g_n]$.

Since $g_n \leqslant f_k$ for all $k \geqslant n$, we have both $\int_A g_n \leqslant \int_A f_k$ and $\int_A g_n \leqslant \liminf_{k\to\infty} [\int_A f_k]$.

Combining the above equations we get the desired result. ■

· An example of a sequence where strict inequality holds for the above lemma is $f_n = n\chi_{(0,1/n)}$.

· Note that both Fatou and MCT still hold if we replace $f = \liminf_{n\to\infty} [f_n]$ pointwise by $f = \liminf_{n\to\infty} [f_n]$ (a.e.).

Theorem 2.2.18. [Dominated convergence theorem - Lebesgue]
Let $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$ and $(f_n)_{n=1}^{\infty} \subset M(A)$ and $g \in L^+(A)$ such that

1. there exists $f : A \to \mathbb{R}$ such that $f = \lim_{n\to\infty} [f_n]$ (a.e.)

2. for each n, $|f_n| \leqslant g$ where g is termed the integrable majorant

Then $f \in L(A)$ and $\int_A f = \lim_{n\to\infty} \left[\int_A f_n\right]$.

Proof: Let $N = \bigcup_{n=1}^{\infty} \{x \in A \mid |f_n(x)| > g(x)\} \cup \{x \in A \mid \lim_{n\to\infty} [f_n(x)] \neq f(x) \text{ or DNE}\}$, which is a null set.

Note that $\int_N f_n = 0 = \int_A g$.

Thus let $A \leftarrow A \setminus N$.

Note that $f = \lim_{n\to\infty} [f_n]$ pointwise is measurable and that $|f| = \lim_{n\to\infty} [|f_n|] \leqslant g$.

Since $\int_A |f| \leqslant \int_a g < \infty$, we have that f is integrable.

Next note that $g + f_n \geqslant 0$ and $g + f = \lim_{n\to\infty} [g + f_n] = \liminf_{n\to\infty} [g + f_n]$ pointwise.

Then Fatou gives us that $\int_A (g + f) \leqslant \liminf_{n\to\infty} [\int_A (g + f_n)]$, and so

$$\int_A g + \int_A f = \int_A (g + f) \leqslant \liminf_{n\to\infty} \left[\int_A (g + f_n)\right] = \liminf_{n\to\infty} \left[\int_A g + \int_A f_n\right] = \int_A g + \liminf_{n\to\infty} \left[\int_A f_n\right]$$

This shows that $\int_A f \leqslant \liminf_{n\to\infty} [\int_A f_n]$.

Further, note that $g - f_n \geqslant 0$ and $g - f = \lim_{n\to\infty} [g - f_n] = \liminf_{n\to\infty} [g - f_n]$.

As above, from Fatou we get that

$$\int_A g - \int_A f = \int_A (g - f) \leqslant \liminf_{n\to\infty} \left[\int_A (g - f_n)\right] = \liminf_{n\to\infty} \left[\int_A g - \int_A f_n\right] = \int_A g - \liminf_{n\to\infty} \left[\int_A f_n\right]$$

This shows that $\int_A f \geqslant \liminf_{n\to\infty} [\int_A f_n]$.

Combining, we have that $\limsup_{n\to\infty} [\int_A f_n] \leqslant \int_A f_n \leqslant \liminf_{n\to\infty} [\int_A f_n]$.

Therefore $\int_A f = \lim_{n\to\infty} [\int_A f_n]$. ■

3 L_p spaces

3.1 Construction

Proposition 3.1.1. For $f \in L(A)$, let $\|f\|_1 = \int_A f$. Then

$$\begin{array}{lll} \|cf\|_1 = |c|\|f\|_1 & \text{for } c \in \mathbb{R} & |\cdot|\text{-homogeneity} \\ \|f+g\|_1 \leqslant \|f\|_1 + \|g\|_1 & \text{for } g \in L(A) & \text{subadditivity} \end{array}$$

Definition 3.1.2. Define an equivalence relation $\sim$ on $L(A)$ by:

$$f \sim g \quad \text{iff} \quad f = g \text{ (a.e.)}$$

Definition 3.1.3. Define the L_1-space by $L_1(A) = L(A)/\sim$.

· Note that elements in this vector space are sets of equivalence classes. However, we treat them as functions with the proviso that $f = f_1$ in $L_1(A)$ iff $f = f_1$ (a.e.).

Remark 3.1.4. We say that $f = \lim_{n\to\infty} [f_n]$ in L_1 iff $\lim_{n\to\infty} [\|f - f_n\|_1] = 0$.

Definition 3.1.5. Let $1 < p < \infty$ and $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$. Define the L_p-space and its norm by

$$L_p(A) = \left\{ f : A \to \mathbb{R} \mid \int_A |f|^p < \infty \right\} \Big/ \sim_{\text{(a.e.)}}$$

$$\|f\|_p = \left(\int_A |f|^p \right)^{1/p}$$

where $f \in L_p(A)$. It will be proved below that $\|\cdot\|_p$ is indeed a norm.

Definition 3.1.6. If $1 < p < \infty$ is fixed, define the conjugate index q by $\frac{1}{p} + \frac{1}{q} = 1$, so $q = \frac{p}{p-1}$.

Lemma 3.1.7. For p and q as above, if $ab \geqslant 0$, then $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$ with equality iff $a^p = b^q$.

Theorem 3.1.8. [Holder]
Let $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$ and $1 < p < \infty$ with q the conjugate index. If $f \in L_p(A)$ and $g \in L_q(A)$, then

$$fg \in L_1(A)$$
$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q$$

with equality holding iff $\|g\|_q |f|^p = \|f\|_p |g|^q$ (a.e.).

Theorem 3.1.9. [Minkowski]
Let $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$ and $1 < p < \infty$. If $f, g \in L_p(A)$, then

$$f + g \in L_p(A)$$
$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$$

with equality holding iff there exist $c_1, c_2 \in \mathbb{R}_{\geqslant 0}$ with $c_1 + c_2 > 0$ such that $c_1 f = c_2 g$ (a.e.).

Lemma 3.1.10. Let $(X, \|\cdot\|)$ be a normed vector space. Then X is complete with respect to $\|\cdot\|$ iff for every sequence $(x_n)_{n=1}^\infty \subset X$ with $\sum_{n=1}^\infty \|x_n\| < \infty$ we have

$$\sum_{n=1}^\infty x_n = \lim_{N\to\infty} \left[\sum_{n=1}^N x_n \right]$$

Proof: (⇒) Here we employ the abstract Weierstrass test.

Let $(x_n)_{n=1}^{\infty} \subset X$ with $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Let $S_n = \sum_{k=1}^{n} x_k$, so if $m < n$, then

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \leqslant \sum_{k=n+1}^{m} \|x_k\|$$

Since $\sum_{n=1}^{\infty} \|x_n\| < \infty$, we can make $\|S_m - S_n\|$ small so that $(S_n)_{n=1}^{\infty}$ is Cauchy in X.

Since X is complete, $(S_n)_{n=1}^{\infty}$ converges in $S \in X$ for

$$S = \lim_{n\to\infty} [S_n] = \lim_{n\to\infty} \left[\sum_{k=1}^{n} x_k \right] = \sum_{k=1}^{\infty} x_k$$

(⇐) Let $(x_n)_{n=1}^{\infty} \subset X$ be Cauchy.

Pick

$$\begin{aligned} n_1 \in \mathbb{N} \text{ such that } n, m \geqslant n_1 \text{ implies } \|x_n - x_m\| < \tfrac{1}{2} \\ n_2 \in \mathbb{N} \text{ such that } n_2 \geqslant n_1 \text{ and } n, m \geqslant n_2 \text{ implies } \|x_n - x_m\| < \tfrac{1}{2^2} \\ \vdots \\ n_k \in \mathbb{N} \text{ such that } n_k \geqslant n_{k-1} \text{ and } n, m \geqslant n_k \text{ implies } \|x_n - x_m\| < \tfrac{1}{2^k} \end{aligned}$$

In this manner we get a sequence $(x_{n_k})_{k=1}^{\infty}$.

For each k, let $y_k = x_{n_{k+1}} - x_{n_k}$.

Then we have that

$$\sum_{j=1}^{k} \|y_j\| = \sum_{j=1}^{k} \|x_{n_j} - x_{n_{j-1}}\| < \sum_{j=1}^{k} \frac{1}{2^j}$$

And for the infinite sum,

$$\sum_{j=1}^{\infty} \|y_j\| = \lim_{N\to\infty} \left[\sum_{j=1}^{N} \|y_j\| \right] \leqslant \lim_{N\to\infty} \left[\sum_{j=1}^{N} \frac{1}{2^j} \right] = 1$$

By the hypothesis, $x = \lim_{j\to\infty} \left[\sum_{k=1}^{j} x_k \right]$ exists.

We observe that since we are dealing with a telescoping sum, we have

$$\sum_{k=1}^{j} y_k = \sum_{k=1}^{j} (x_{n_k} - x_{n_{k-1}}) = x_{n_{k+1}} - x_{n_1}$$

Therefore $x + x_{n_1} = \lim_{j\to\infty} [x_{n_{j+1}}]$ exists.

Now we have that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a Cauchy subsequence.

Hence $(x_n)_{n=1}^{\infty}$ also converges in X. ■

Theorem 3.1.11. Let $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$. Then $L_p(A)$ is complete.

Proof: We use the previous lemma, and let $(f_n)_{n=1}^{\infty} \subset L_p(A)$ with $M = \sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

We consider each f_n as a measurable function on A with $\int_A |f|^p < \infty$.

Let $g_n = \sum_{k=1}^{n} |f_k|$ (so $g_1 \leqslant g_2 \leqslant \cdots$) and for each x, let $g(x) = \lim_{n\to\infty} [g_n(x)]$ pointwise.

Observe that

$$\|g_n\|_p \leqslant \sum_{k=1}^{n} \||f_k|\|_p = \sum_{k=1}^{n} \|f_k\|_p \leqslant \sum_{k=1}^{\infty} \|f_k\|_p < \infty$$

So by MCT, we have that

$$\int_A g^p = \lim_{n\to\infty}\left[\int_A g_n^p\right] = \lim_{n\to\infty}\left[\|g_n\|_p^p\right] \leqslant M^p < \infty$$

Since g^p is integrable, $g(x) < \infty$ (a.e.) on A, and so g represents an element of $L_p(A)$.
Then for $x \in A$ (a.e.), $\sum_{k=1}^n |f_k(x)| = g_n(x) \leqslant g(x)$, and so $\sum_{k=1}^\infty |f_k(x)| < \infty$.
Then there is some f such that $f(x) = \lim_{n\to\infty}[\sum_{k=1}^n f_k(x)]$, and so

$$|f|^p = \left|\lim_{n\to\infty}\left[\sum_{k=1}^n f_k\right]\right|^p \leqslant \lim_{n\to\infty}\left[\left(\sum_{k=1}^n |f_k|\right)^p\right] = \lim_{n\to\infty}[g_n^p] = g^p \text{ (a.e.)}$$

Therefore $\int_A |f|^p \leqslant \int_A g^p < \infty$, and f represents an element in $L_p(A)$.
Now we have left to show that $\|f - \sum_{k=1}^n f_k\|_p \xrightarrow{n\to\infty} 0$.
Observe that

$$\left|f - \sum_{k=1}^n f_k\right|^p \leqslant \left(|f| + \left|\sum_{k=1}^n f_k\right|\right)^p \leqslant (g+g)^p = 2^p g^p \quad \text{and} \quad \int_A 2^p g^p < \infty$$

Note that from the definition of f, we have $\lim_{n\to\infty}[|f - \sum_{k=1}^n f_k|] = 0$ (a.e.).
So by LDCT we have that

$$\lim_{n\to\infty}\left[\left\|f - \sum_{k=1}^n f_n\right\|_p^p\right] = \lim_{n\to\infty}\left[\int_A \left|f - \sum_{k=1}^\infty f_k\right|^p\right] = \int_A 0 = 0$$

Hence $\sum_{k=1}^\infty f_k = f$ is in $(L_p(A), \|\cdot\|_p)$.
Thus $L_p(A)$ is complete. ■

Definition 3.1.12. If we allow $p = \infty$, then we may construct the L_∞ space. Let $f \in M(A)$ for $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$. Then define

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in A}\{|f(x)|\} = \inf\{c > 0 \mid \lambda(\{x \in A \mid |f(x)| > c\}) = 0\}$$

Proposition 3.1.13. The function $\|\cdot\|_\infty$ is a norm on $L_\infty(A)$.

Proof: If $f \in L_\infty(A)$, then $\|f\|_\infty \geqslant 0$.
If $\|f\|_\infty = 0$, then $\lambda(\{x \in A \mid |f(x)| > \frac{1}{n}\}) = 0$.
So $\{x \in A \mid f(x) \neq 0\} = \{x \in A \mid |f(x)| > 0\} = \bigcup_{n=1}^\infty \{x \in A \mid |f(x)| > \frac{1}{n}\}$.
And a countable union of null sets is still null, so $f = 0$.

Let $f, g \in L_\infty(A)$.
First note that $\{x \in A \mid |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^\infty \{x \in A \mid |f(x)| > \|f\|_\infty > \frac{1}{n}\}$.
Thus $\lambda(\{x \in A \mid |f(x)| > \|f\|_\infty\}) = 0$.

Consider

$$\begin{aligned}\{x \in A \mid |f(x)+g(x)| > \|f\|_\infty + \|g\|_\infty\} &\subset \{x \in A \mid |f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty\} \\ &\subset \{x \in A \mid |f(x) > \|f\|_\infty\} \cup \{x \in A \mid |g(x)| > \|g(x)\|_\infty\}\end{aligned}$$

And the above union has measure zero.
Hence it follows that

$$\|f+g\|_\infty = \inf\{c > 0 \mid \lambda(\{x \in A \mid |f(x)+g(x)| > c\}) = 0\} \leqslant \|f\|_\infty + \|g\|_\infty$$

■

Theorem 3.1.14. The metric space $(L_\infty(A), \|\cdot\|_\infty)$ is complete and hence a Banach space.

Proof: Let $(f_k)_{k=1}^\infty \subset L_\infty(A)$.
Suppose that $\sum_{k=1}^\infty \|f_k\|_\infty < \infty$.
We need to show that $\sum_{k=1}^\infty f_k$ defines an element of $L_\infty(A)$.
Let $E_k = \{x \in A \mid |f(x)| > \|f_k\|_\infty\}$ which is a null set.
Hence $E = \bigcup_{k=1}^\infty E_k$ is also a null set.
So then if $x \in A \setminus E$, by absolute convergence $|\sum_{k=1}^\infty f_k(x)| \leqslant \sum_{k=1}^\infty \|f_k\|_\infty$.
Therefore $\sum_{k=1}^\infty f_k$ pointwise defines an element of $L_\infty(A)$. ■

Remark 3.1.15. For $(f_k)_{n=1}^\infty$ such that $f \in M(A)$, we have notions for several types of convergence,
- $\lim_{k\to\infty} [f_k] = f$ pointwise
- $\lim_{k\to\infty} [f_k] = f$ a.e.
- $\lim_{k\to\infty} [f_k] = f$ in L_p

3.2 Inclusion properties

Theorem 3.2.1. Let $[a, b]$ be a compact interval with $a < b$. Then for $1 \leqslant p < r < \infty$ and $f \in L_r([a, b])$,

$$L_r([a,b]) \subset L_p([a,b]) \quad \text{and} \quad \|f\|_p \leqslant (b-a)^{\frac{r-p}{pr}} \|f\|_r$$

Proof: Let $f \in L_r([a, b])$.
Then $|f|^p \in L_{r/p}([a, b])$, i.e. $\int_{[a,b]} (|f|^p)^{r/p} = \int_{[a,b]} |f|^r < \infty$.
Let q be the conjugate index to r, so $q = \frac{r}{r-p}$.
Then applying Holder's inequality, we have

$$\begin{aligned} \|f\|_p^p &= \int_{[a,b]} |f|^p \\ &= \int_{[a,b]} |f|^p \cdot 1 \\ &\leqslant \left(\int_{[a,b]} (|f|^p)^{r/p}\right)^{p/r} \left(\int_{[a,b]} |1|^q\right)^{1/q} \\ &= \left(\left(\int_{[a,b]} f^r\right)^{1/r}\right)^p (b-a)^{1/q} \\ &= \|f\|_r^p (b-a)^{1/q} \end{aligned}$$

■

- We note some containment relations, for $p \in (1, \infty)$:

$$C([a,b]) \subsetneq L_\infty([a,b]) \subsetneq L_p([a,b]) \subsetneq L_1([a,b])$$

Remark 3.2.2. Observe that $L_\infty([a, b]) \subset L_p([a, b])$ for $1 \leqslant p < \infty$, and there is $k > 0$ that is a function of p, a, b such that $\|f\|_p \leqslant k\|f\|_\infty$.

Proposition 3.2.3. Let $1 \leqslant p < r < \infty$. Then $L_p([a, b]) \not\subset L_r([a, b])$.

Proof: Let $[a,b]=[0,1]$.

Let $f(x)=\frac{1}{x^{p/r}}$ (a.e.).

Now compute

$$\int_{[0,1]}|f(x)|^p dx=\int_{[0,1]}\frac{1}{x^{p/r}}dx=\lim_{a\to\infty}\left[\int_0^1 x^{-p/r}dx\right]=\lim_{a\to\infty}\left[\frac{x^{1-p/r}}{1-p/r}\Bigg|_0^1\right]=\frac{r}{r-p}<\infty$$

Thus $f\in L_p([0,1])$.

It is easy to check that $\int_{[0,1]}|f|^r=\infty$. ■

Theorem 3.2.4. If $a<b$ in $\mathbb{R}$ and $1\leqslant p<\infty$, then $L_p([a,b])$ is separable.

Proof: By Assignment 4, we know $C([a,b])\subset L_p([a,b])$ with $\|f\|_p\leqslant k\|f\|_\infty$ for $f\in C([a,b])$ with fixed constant $k>0$.

Also we know that $C([a,b])$ is dense in $L_p([a,b])$.

Recall that $(C([a,b]),\|\cdot\|_\infty)$ is separable.

Let $\mathbb{R}[x]$ denote then space of polynomial functions in $[a,b]$.

By Stone, Weierstrass, we have that $\overline{\mathbb{R}[x]}^{\|\cdot\|_\infty}=C([a,b])$.

Now note the facts that

· $\mathbb{Q}[x]\subset\mathbb{R}[x]$ is countable, call it $(d_n)_{n=1}^\infty$

· for each $p\in\mathbb{R}[x]$ and $\varepsilon>0$, there is $d\in\mathbb{Q}[x]$ such that $\|p-d\|_\infty<\varepsilon$.

So if $f\in L_p([a,b])$ and $\varepsilon>0$, we first find $h\in C([a,b])$ such that $\|f-h\|_p<\frac{\varepsilon}{2}$.

Then we find $g\in\mathbb{R}[x]$ such that $\|h-g\|_p<\frac{\varepsilon}{4k}$ and $n\in\mathbb{N}$ such that $\|g-d_n\|_\infty<\frac{\varepsilon}{4k}$.

This all gives us

$$\begin{aligned}\|f-d_n\|_p&\leqslant\|f-h\|_p+\|h-d_n\|_p\\&<\tfrac{\varepsilon}{2}+k\|h-d_n\|_\infty\\&\leqslant\tfrac{\varepsilon}{2}+k\left(\|h-g\|_\infty+\|g-d_n\|_\infty\right)\\&<\tfrac{\varepsilon}{2}+k\tfrac{\varepsilon}{2k}\\&=\varepsilon\end{aligned}$$

■

Theorem 3.2.5. $L_\infty([0,1])$ is not separable.

Proof: For each $a=\{a_0,a_1,\dots\}\subset\{0,1\}^{\mathbb{N}}$ for $a_i\in\{0,1\}$, let $f_a=\sum_{n=1}^\infty a_n\chi_{[\frac{1}{n+1},\frac{1}{n}]}$.

We observe that if $a,b\in\{0,1\}^{\mathbb{N}}$, then

$$\|f_a-f_b\|_\infty=\left\|\sum_{n=1}^\infty(a_n-b_n)\chi_{[\frac{1}{n+1},\frac{1}{n}]}\right\|_\infty=\sup_{n\in\mathbb{N}}\left[\|a_n-b_n\|\right]$$

Thus if $a\neq b$, then $\|f_a-f_b\|_\infty=1$.

If there was a dense subset $(d_n)_{n=1}^\infty$ of $L_\infty[0,1]$, then for each $a\in\{0,1\}^{\mathbb{N}}$ there would be a unique $n=n(a)$ such that $\|f_a-f_{n(a)}\|_\infty<\frac{1}{2}$.

Note that $n(a)\neq n(b)$ for $a\neq b$, since otherwise we would have

$$\|f_a-f_b\|_\infty=\|f_a-d_{n(a)}+d_{n(b)}-f_b\|_\infty\leqslant\|f_a-d_{n(a)}\|_\infty+\|d_{n(a)}-f_b\|_\infty<1$$

This would contradict the above.

Then we would have $a\mapsto n(a):\{0,1\}^{\mathbb{N}}\to\mathbb{N}$ is injective, implying $|\{0,1\}^{\mathbb{N}}|\leqslant|\mathbb{N}|$, which is absurd. ■

Remark 3.2.6. For $a<b$ in $\mathbb{R}$, if $f\in L_\infty([a,b])$ then $\lim_{p\to\infty}[\|f\|_p]=\|f\|_\infty$.

3.3 Operators and functionals

Definition 3.3.1. Let X, Y be Banach spaces. A linear operator $T : X \to Y$ is termed bounded iff

$$\|T\|_* = |||T|||_X = \sup\{\|Tx\|_Y \mid x \in X, \|x\|_X < 1\} < \infty$$

Proposition 3.3.2. Let X, Y be Banach spaces with $T : X \to Y$ linear. Then equivalently:

1. T is continuous
2. T is bounded
3. T is Lipschitz

If T is Lipschitz, then $\|Tx - Tx'\|_Y \leqslant |||T||| \cdot \|x - x'\|_X$ where $|||T|||$ is the smallest Lipschitz constant.

Proof: ($1 \Rightarrow 2$) Let $B(y) = \{y \in Y \mid \|y\| < 1\}$ which is an open neighborhood of $0y$ for 0 the zero operator.
Since T is continuous and $T0x = 0y$, there is some $\delta > 0$ such that $\|x - 0\|_X < \delta$ implies $\|Tx - 0y\|_y < 1$.
Equivalently, the above may be stated as $\|x\|_X < \delta \implies \|Tx\|_Y < 1$.
Now suppose $x \in X$ and $\|x\|_X < 1$.
Then $\|\delta x\|_X = \delta\|x\|_X < \delta$.
Thus $\delta\|Tx\|_Y = \|\delta Tx\|_Y < 1$.
So $|||T||| = \sup\limits_{\substack{x \in X \\ \|x\|_X < 1}} \{\|Tx\|_Y\} \leqslant \dfrac{1}{\delta}$.

($2 \Rightarrow 3$) For $x \in X$ and $\varepsilon > 0$ we have that

$$\left\| \frac{1}{\|x\|_X + \varepsilon} \cdot x \right\| = \frac{1}{\|x\|_X + \varepsilon} \cdot \|x\|_X < 1$$

Hence

$$\frac{1}{\|x\|_X + \varepsilon} \cdot \|Tx\|_Y = \left\| T\left(\frac{1}{\|x\|_X + \varepsilon} \cdot x \right) \right\|_Y < |||T|||$$

So then we have that $\|Tx\|_Y \leqslant |||T|||(\|x\|_X + \varepsilon)$, which reduces to $\|Tx\|_Y \leqslant |||T||| \cdot \|x\|_X$.
And if $x, x' \in X$, then $\|Tx - Tx'\|_Y = \|T(x - x')\|_Y \leqslant |||T||| \cdot \|x - x'\|_Y$.
Finally, if $0 < c < |||T|||$, then there exists $x \in B(x)$ such that $\|Tx\|_Y > c > c\|x\|_X$.
That is, $\|Tx - T0\|_Y > c\|x - 0\|_X$, so c is not a Lipschitz estimate.

($3 \Rightarrow 1$). Note Lipschitz implies uniform continuity implies continuity. ■

Theorem 3.3.3. Let $1 < p < \infty$ with q the conjugate index. If $g \in L_q(A)$, then the functional $\Gamma_g : L_p(A) \to \mathbb{R}$ given by $\Gamma_g(f) = \int_A gf$ is a bounded linear functional with $\|\Gamma g\|_* = \|g\|_q$.

Proof: First, if $g \in L_q(A)$ and $f \in L_p(A)$, then by Holder $gf \in L_1(A)$, and

$$|\Gamma g(f)| = |\int_A gf| \leqslant \int_A |gf| \leqslant \|g\|_q \|f\|_p$$

From above we know that $\|\Gamma g\|_*$ is the smallest $c > 0$ such that $|\Gamma g(f)| \leqslant c\|f\|_P$ and thus $\|\Gamma g\|_* \leqslant \|g\|_q$.
It is easy to verify that Γg is linear.
To get the converse equality, observe that $\int_A |fg| = \|f\|_p \|g\|_q$ provided $|f|^p = c|g|^q$.
Now define $\mathrm{sgn} : \mathbb{R} \to \{1, -1\}$ by $x \mapsto \left\{ \begin{smallmatrix} 1 & x \geqslant 0 \\ -1 & x < 0 \end{smallmatrix} \right.$ which is Borel measurable.

Further, let $f = c|g|^{q/p}\mathrm{sgn} \circ g$ for c defined below, so that $\|f\|_p = 1$, and

$$\begin{aligned}\int_A |f|^p &= \int_A \left| c|g|^{q/p}\mathrm{sgn} \circ g \right|^p \\ &= c^p \int_A \left(|g|^{q/p}\right)^p |\mathrm{sgn} \circ g|^p \\ &= c^p \int_A |g|^q \\ &= c^p \|g\|_q^q\end{aligned}$$

Thus $\|f\|_p^p = c^p\|g\|_q^q$ and so $\|f\|_p = c\|g\|_q^{q/p}$.
Now we let $c = \frac{1}{\|g\|_q^{q/p}}$ to get $\|f\|_p = 1$.
Finally we compute

$$\begin{aligned}\|\Gamma g\|_* &= \sup\{|\Gamma g(f)| \mid f \in L_p(A), \|f\|_p \leqslant 1\} \\ &\geqslant \left| \Gamma g \left(\frac{1}{\|g\|_q^{q/p}} |g|^{q/p}\mathrm{sgn} \circ g \right) \right| \\ &= \left| \int_A g \frac{1}{\|g\|_q^{q/p}} |g|^{q/p}\mathrm{sgn} \circ g \right| \\ &= \frac{1}{\|g\|_q^{p/q}} \int_A |g|^{q/p+1} \\ &= \frac{1}{\|g\|_q^{p/q}} \int_A |g|^q \\ &= \|g\|_q^{q-q/p} \\ &= \|g\|_q\end{aligned}$$

Therefore $\|\Gamma g\|_* = \|g\|_q$. ■

Remark 3.3.4. If $\Gamma : L_p(A) \to \mathbb{R}$ is a bounded linear functional, then there is $g \in L_q(A)$ such that $\Gamma = \Gamma g$.

Theorem 3.3.5. If $\varphi \in L_\infty(A)$, let $\Gamma_\varphi : L_1(A) \to \mathbb{R}$ by $\Gamma_\varphi(f) = \int_A \varphi f$. Then $\Gamma_\varphi(f)$ is a bounded linear functional with $\|\Gamma_\varphi\|_* = \|\varphi\|_\infty$.

Proof: First observe that for $f \in L_1(A)$, we have $|\varphi f| \leqslant \|\varphi\|_\infty |f|$ (a.e.), and

$$\int_A |\varphi f| \leqslant \int_A \|\varphi\|_\infty |f| = \|\varphi\|_\infty \int_A |f| = \|\varphi\|_\infty \|f\|_1$$

Hence $\varphi f \in L_1(A)$ and

$$|\Gamma_\varphi(f)| = \left| \int_A \varphi f \right| \leqslant \int_A |\varphi f| \leqslant \|\varphi\|_\infty \|f\|_1$$

It remains to verify that $\|\Gamma_\varphi\|_* \geqslant \|\varphi\|_\infty$.
For $\varepsilon > 0$, let $A_\varepsilon = \{x \in A \mid \|\varphi\|_\infty - \varepsilon \leqslant \|\varphi(x)\|\}$
Then $\lambda(A_\varepsilon) > 0$ by definition, but it may be that the measure is ∞.
So we find $A'_\varepsilon \subset A_\varepsilon$ such that $0 < \lambda(A'_\varepsilon) < \infty$.
Let $f_\varepsilon = \frac{1}{\lambda(A'_\varepsilon)} \chi_{A'_\varepsilon} \mathrm{sgn} \circ \varphi$ so that

$$\|f_\varepsilon\|_1 = \int_A \left| \frac{1}{\lambda(A'_\varepsilon)} \chi_{A'_\varepsilon} \right| = \frac{1}{\lambda(A'_\varepsilon)} \int_A \chi_{A'_\varepsilon} = \frac{\lambda(A'_\varepsilon)}{\lambda(A'_\varepsilon)} = 1$$

Moreover, we have that

$$\|\Gamma_\varphi\|_\infty \geqslant |\Gamma_\varphi(f_\varepsilon)| = \left|\int_A \varphi \frac{1}{\lambda(A'_\varepsilon)}\chi_{A'_\varepsilon}\text{sgn}\circ\varphi\right| = \frac{1}{\lambda(A'_\varepsilon)}\int_A |\varphi|\chi_{A'_\varepsilon} \geqslant \frac{\|\varphi\|_\infty - \varepsilon}{\lambda(A'_\varepsilon)}\int_A \chi_{A'_\varepsilon} = \|\varphi\|_\infty - \varepsilon$$

Therefore $\|\Gamma_\varphi\|_* = \|\varphi\|_\infty$. ■

Theorem 3.3.6. Let $a < b$ in $\mathbb{R}$ and for $f \in L_1[a,b]$, let $\Gamma_f : L_\infty[a,b] \to \mathbb{R}$ be given by $\Gamma_f(\varphi) = \int_{(a,b)} f\varphi$.

i. the functional Γ_f is linear and bounded, with $\|\Gamma_f\|_* = \|f\|_1$

ii. for $\Gamma_f : C[a,b] \to \mathbb{R}$, we have $\|\Gamma_f\|_* = \sup\{|\Gamma_f(h)| \mid h \in C[a,b], \|h\|_\infty = \|f\|_1\}$

<u>*Proof:*</u> (**i.**) Recall from above we have that $\int_{(a,b)} |\varphi f| \leqslant \|\varphi\|_\infty \|f\|_1$.

This tells us that $\|\Gamma_f\|_* \leqslant \|f\|_1$ (it is clear that Γ_f is linear).

Let $\varphi = \text{sgn}\circ f$, so that $\|\varphi\| \leqslant 1$, and

$$\begin{aligned}\|\Gamma_f\|_* &= \sup\left\{\left|\int_{[a,b]} f\varphi\right| \mid \varphi \in L_\infty[a,b], \|\varphi\|_\infty \leqslant 1\right\} \\ &\geqslant \left|\int_{[a,b]} f\text{sgn}\circ f\right| \\ &= \int_{[a,b]} |f| \\ &= \|f\|_1\end{aligned}$$

(**ii.**) From the proof of Assignment 1, question 4, we know that there exists $(h_n)_{n=1}^\infty \subset C[a,b]$ such that

- $\|h_n\|_\infty \leqslant 1$ for all n
- $\lim\limits_{n\to\infty}[h_n] = \text{sgn}\circ f$ (a.e.)

Note that $|fh_n| \leqslant |f||h_n| \leqslant |f|$, so $|f|$ is an integrable majorant of $(fh_n)_{n=1}^\infty$.

Then

$$\int_{[a,b]} fh_n \xrightarrow{n\to\infty} \int_{[a,b]} f\text{sgn}\circ f = \int_{[a,b]} |f| = \|f\|_1$$

Thus as a functional on $C[a,b]$, $\|\Gamma_f\|_* \geqslant \sup\limits_{n\in\mathbb{N}}\{|\int_{[a,b]} fh_n|\} \geqslant \lim\limits_{n\to\infty}[|\int_{[a,b]} fh_n|] = \|f\|_1$.

But also we have that

$$\sup\left\{\left|\int_{[a,b]} fh_n\right| \mid h \in C[a,b], \|h\|_\infty \leqslant 1\right\} \leqslant \sup\left\{\left|\int_{[a,b]} f\varphi\right| \mid \varphi \in L_\infty[a,b], \|\varphi\|_\infty \leqslant 1\right\} = \|f\|_1$$

■

4 Fourier Analysis

4.1 Foundations

Definition 4.1.1. A function $f : [a,b] \to \mathbb{C}$ is termed <u>measurable</u> provided $\Re(f), \Im(f) : [a,b] \to \mathbb{R}$ are measurable. If both of these functions are also integrable, then f is integrable, and $\int_a^b f = \int_a^b \Re(f) + i\int_a^b \Im(f)$.

Definition 4.1.2. For $n \in \mathbb{Z}$, we let $e^{int} = e^n(t)$.

Definition 4.1.3. Define the trigonometric polynomials to be members of the set

$$\mathrm{Trig}(\mathbb{T}) = \mathrm{span}\{e^n \mid n \in \mathbb{Z}\} = \left\{ \sum_{n=-N}^{N} c_n e^n \mid N \in \mathbb{N}, c_n \in \mathbb{C} \right\}$$

A series of the form $\sum_{n=-\infty}^{\infty} c_n e^n$ for $c_n \in \mathbb{C}$ is termed a formal Fourier series, with variants as below.

$$\begin{aligned} &\text{Dirichlet kernel of order } n && D_n := \sum_{k=-n}^{n} e^k \\ &k\text{th Fourier coefficient of } f && c_k(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-iks}\ ds \\ &n\text{th Fourier sum of } f && s_n(f,t) := \sum_{k=-n}^{n} c_k(f)e^{ikt} \\ & && \qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(t-s)\ ds \end{aligned}$$

Moreover, we let

$$L_1(\mathbb{T}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \ \text{ is measurable, } 2\pi\text{-periodic, } \int_{-\pi}^{\pi} |f| < \infty \right\} \Big/ \sim_{\text{(a.e.)}}$$

Definition 4.1.4. Define the translation function $* : X \times \{f : X \to Y\} \to \{f : X \to Y\}$ by

$$*(s,f)(x) = (s * f)(x) = s * f(x) = f(x-s) \ \ \text{(a.e.)}$$

4.2 Homogeneous subspaces

Definition 4.2.1. A subspace $B \subset L_1(\mathbb{T})$ is termed a homogeneous Banach space over $\mathbb{T}$ if it is equipped with a norm $\|\cdot\|_B$ under which B is Banach and for which

1. $\mathrm{Trig}(\mathbb{T}) \subset B$

2. $s * f \in B$ for $s \in \mathbb{R}$, $f \in B$

- **i.** $\|s * f\|_B = \|f\|_B$
- **ii.** the function $\mathbb{R} \to B$ given by $s \mapsto s * f$ (for fixed f) is continuous for each f in B

Example 4.2.2. These are some examples of homogeneous Banach spaces:

i. $C(\mathbb{T}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is } 2\pi\text{-periodic and continuous}\}$

ii. $L_p(\mathbb{T}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable, } 2\pi\text{-periodic, } \int_{-\pi}^{\pi} |f|^p < \infty\}/\sim_{\text{(a.e.)}}$ for $1 \leqslant p < \infty$

iii. $L_\infty(\mathbb{T})$ is not a homogeneous Banach space

Definition 4.2.3. Let B be a homogeneous Banach space over $\mathbb{T}$ and let $h \in C(\mathbb{T})$. Define the convolution of h and f by the vector-valued integral

$$h * f := \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)(s * f)\ ds$$

Observe that our assumptions on h provide that $s \mapsto h(s)(s * f)$ is continuous.

Note that for $t \in \mathbb{R}$ (a.e.),

$$\begin{aligned}(h * f)(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)f(t-s)\ ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t+s)f(-s)\ ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t-s)f(s)\ ds \\ &= (f * h)(t)\end{aligned}$$

Therefore the convolution operator is symmetric in its arguments.

Remark 4.2.4. We note that $s_n(f,t) = D_n * f(t)$.

Proposition 4.2.5. If B is a homogeneous Banach space over $\mathbb{T}$ with $h \in C(\mathbb{T})$, then the convolution operator $C(h)(f) := h * f$ is linear and bounded, with $|||C(h)|||_B \leqslant \|h\|_1$.

Proof: The linearity of $C(h)$ is a consequence of linearity of Riemann integration.

As for the inequality, note that for $f \in B$, we have

$$\begin{aligned}\|C(h)(f)\|_B &= \|h * f\|_B \\ &= \left\| \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)(s * f)\ ds \right\|_B \\ &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \|h(s)(s * f)\|_B\ ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(s)|\|s * f\|_B\ ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(s)|\|f\|_B\ ds \\ &\leqslant \|h\|_1\|f\|_B\end{aligned}$$

■

Theorem 4.2.6. Let $h \in C(\mathbb{T})$. Then

1. $|||C(h)|||_{C(\mathbb{T})} = \|h\|_1$

2. $|||C(h)|||_{L_1(\mathbb{T})} = \|h\|_1$

Proof: (**1.**) Let $f \in C(\mathbb{T})$.

Then if $\check{h}(s) = h(-s)$, we have

$$h * f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)f(0-s)\ ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(-s)f(s)\ ds = \Gamma_{\check{h}}(f)$$

Hence we have that $\|C(h)(f)\|_\infty \geqslant |h * f(0)| = |\Gamma_{\check{h}}(f)|$.

Therefore

$$\begin{aligned}|||C(h)|||_{C(\mathbb{T})} &= \sup_f\{\|C(h)(f)\|_\infty \mid f \in C(\mathbb{T}), \|f\|_\infty \leqslant 1\} \\ &\geqslant \sup_f\{|\Gamma_{\check{h}}(f)| \mid f \in C(\mathbb{T}), \|f\|_\infty \leqslant 1\} \\ &= \|\Gamma_{\check{h}}\|_* \\ &= \|\check{h}\|_1 \\ &= \|h\|_1\end{aligned}$$

Then using the previously proved inequality, the desired equality follows.

(**2.**) For $n \in \mathbb{N}$, let $f_n = \pi n \chi_{[0,2/n]}$, so $\|f_n\|_1 = 1$.
Now, for $t \in \mathbb{R}$ (a.e.),

$$\begin{aligned} h * f_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f_n(t-s)\ ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s+t) f_n(-s)\ ds \\ &= \frac{n}{2} \int_{-\pi}^{\pi} h(s+t)\ ds \end{aligned}$$

Given $\varepsilon > 0$, there is $\delta > 0$ such that $|h(t) - h(s+t)| < \varepsilon$ for $|s - 0| = |s| < \delta$.
Then for $n > \frac{1}{\delta}$, we have

$$\|h - (h * f_n)\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(t) - (h * f(t))|\ dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| h(t) - \frac{n}{2} \int_{-1/n}^{1/n} h(s+t)\ ds \right|\ dt$$

Then

$$\begin{aligned} h(t) &= \frac{n}{2} \int_{-1/n}^{1/n} h(t)\ ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{n}{2} \left| \int_{-1/n}^{1/n} (h(t) - h(t+s))\ ds \right|\ dt \\ &\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{n}{2} \int_{-1/n}^{1/n} |h(t) - h(t+s)| ds\ dt \\ &\leqslant \varepsilon \end{aligned}$$

Hence $\|h - (h * f_n)\|_1$ is also bounded above by ε, and

$$\lim_{n\to\infty} [\|h\|_1 - \|(h * f_n)\|_1] \leqslant \lim_{n\to\infty} [\|h - (h * f_n)\|_1] = 0$$

This in turn implies

$$\begin{aligned} \|C(h)\|_{L_p(\mathbb{T})} &= \sup_f \{\|C(h)(f)\| \mid f \in L_1(\mathbb{T}), \|f\|_1 \leqslant 1\} \\ &\geqslant \sup_{n\in\mathbb{N}} \{\|C(h)(f_n)\|_1\} \\ &\geqslant \lim_{n\to\infty} [\|C(h)(f_n)\|_1] \\ &= \|h\|_1 \end{aligned}$$

■

Theorem 4.2.7. [Properties of the Dirichlet kernel]
The Dirichlet kernel of order n, D_n, satisfies the following:

1. D_n is real-valued, 2π-periodic and even
2. $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n = 1$
3. for $t \in [-\pi, \pi]$, we have $D_n(t) = \begin{cases} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)} & t \neq 0 \\ 2n+1 & t = 0 \end{cases}$

4. $\lim_{n\to\infty} [\|D_n\|_1] = \infty$

We often call $\|D_n\|_1 = L_n$ the nth Lebesgue constant.

Definition 4.2.8. Let X be a metric space. A set $F \subset X$ is of first category (or meager) if $F \subset \bigcup_{n=1}^{\infty} F_n$ for every F_n closed and nowhere dense. A set $U \subset X$ is of second category (or non-meager) if it is not meager.

· Recall that the Baire category theorem states that if a metric space X is complete, then it is non-meager.

Theorem 4.2.9. [Banach, Steinhaus]
Let B, X be Banach spaces with $\mathcal{F}$ a family of bounded linear maps from B to X.
If $\sup_f \{\|Tf\|_X \mid T \in \mathcal{F}\} < \infty$ for each f in a non-meager set $U \subset B$, then $\sup_{T\in\mathcal{F}} \{|||T|||\} < \infty$.

Proof: For each $n \in \mathbb{N}$, let

$$F_n := \{f \in B \mid \|Tf\|_X \leqslant n \ \forall\, T \in \mathcal{F}\} = \bigcap_{T\in\mathcal{F}} \{f \in B \mid \|Tf\|_X \leqslant n\}$$

Note that $\{f \in B \mid \|Tf\|_X \leqslant n\}$ is closed, as for $g_T(f) = \|Tf\|_X$, so is $g_T^{-1}(\{z \in \mathbb{C} \mid |z| \leqslant n\})$.
Then for the specified U non-meager, we have $U \subset \bigcup_{n=1}^{\infty} F_n$ with at least some $F_{n_0}^{\circ} \neq \emptyset$.
Hence there exists $f_0 \in B$ and $r > 0$ such that $B_r(f_0) = \{f \in B \mid \|f - f_0\|_B < r\} \subset F_{n_0}$.
Note that if $f \in B_r(f_0) \subset F_{n_0}$, then $\|Tf\|_X \leqslant n_0$ for $T \in \mathcal{F}$.
Fix $f \in B$ with $\|f\|_B \leqslant 1$, so for such f we have that $f_0 + \frac{r}{2}f$ and $f_0 - \frac{r}{2}f$ are both in $B_r(f_0)$.
So then for $T \in B_r(f_0)$, we have

$$\begin{aligned}
\|Tf\|_X &= \left\| T\left(\tfrac{1}{r}\left(f_0 + \tfrac{r}{2} - (f_0 - \tfrac{r}{2}f)\right)\right)\right\|_q^r \\
&= \tfrac{1}{r}\left\| T\left(f_0 + \tfrac{r}{2}f\right) - T\left(f_0 - \tfrac{r}{2}f\right)\right\|_X \\
&\leqslant \tfrac{1}{r}\left(\left\| T\left(f_0 + \tfrac{r}{2}f\right)\right\|_X - \left\| T\left(f_0 - \tfrac{r}{2}f\right)\right\|_X\right) \\
&\leqslant \tfrac{2n_0}{r} \\
&< \infty
\end{aligned}$$

Hence $|||T||| = \sup_T\{\|Tf\|_X \mid f \in B \text{ and } \|f\|_B \leqslant 1\} < \infty$ for all $T \in \mathcal{F}$. ■

Corollary 4.2.10. Let B, X be Banach spaces and for $n \in \mathbb{N}$, let $T_n : B \to X$ be a bounded linear operator. Suppose $\sup_{n\in\mathbb{N}}\{|||T_n|||\} = \infty$. Then there exists $U \subset B$ with $B \setminus U$ meager such that $\sup\{\|T_n f\|_X\} = \infty$ for all $f \in U$.

Proof: Let $\mathcal{F} = \{f \in B \mid \sup_{n\in\mathbb{N}}\{\|T_n(f)\|_X\} < \infty\}$.
If $\mathcal{F}$ were non-meager, then the Banach-Steinhaus theorem would show that $\sup\{|||T|||\} < \infty$.
This would be a contradiction, so $\mathcal{F}$ is meager.
Therefole $U = B \setminus \mathcal{F}$. ■

Theorem 4.2.11.

1. The set of $f \in C(\mathbb{T})$ for which $\sup_{n\in\mathbb{N}} \{\|S_n(f)\|_\infty\} < \infty$ is a meager subset of $C(\mathbb{T})$

2. The set of $f \in L_1(\mathbb{T})$ for which $\sup_{n\in\mathbb{N}} \{\|S_n(f)\|_1\} < \infty$ is a meager subset of $L_1(\mathbb{T})$

Proof: (**1.**) We have seen the following facts:
$s_n(f) = D_n * f = C(D_n)(f)$
$|||C(D_n)|||_{C(\mathbb{T})} = \|D_n\|_1$

$\|D_n\|_1 = L_n \xrightarrow{n\to\infty} \infty$

Hence by the Banach-Steinhaus theorem, $\sup_{n\in\mathbb{N}}\{\|s_n(f)\|_\infty\} = \infty$ for all $f \in C(\mathbb{T}) \setminus F$ for F meager.

(**2.**) Similarly. ■

4.3 Averaging and kernels

Definition 4.3.1. Let $(x_n)_{n=1}^\infty$ be a sequence in a Banach space X. Then the expression

$$\sigma_n := \tfrac{1}{n}(x_1 + \cdots + x_n)$$

is termed the nth Cesaro sum of the given sequence. Further, if $f \in L_1(\mathbb{T})$, then the nth Cesaro sum of f is defined to be

$$\begin{aligned}\sigma_n(f) &:= \tfrac{1}{n+1}(s_0(f) + \cdots + n_n(f)) \\ &= \tfrac{1}{n+1}(D_0 * f + \cdots D_n * f) \\ &= \tfrac{1}{n+1}(D_0 + \cdots + D_n) * f \\ &= K_n * f\end{aligned}$$

With respect to the above, K_n is termed the Fejer kernel.

Theorem 4.3.2. [Properties of the Fejer kernel]

1. K_n is real-valued, 2π-periodic, and even

2. $K_n(t) = \begin{cases} \frac{1}{n+1}\left(\frac{\sin\left(\frac{1}{2}(n+1)t\right)}{\sin\left(\frac{1}{2}t\right)}\right)^2 & t \neq 0 \\ n+1 & t = 0\end{cases}$ for $t \in [0, 2\pi]$

3. $\|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1$

4. if $0 < |t| \leqslant 2\pi$, then $0 \leqslant K_n(t) \leqslant \frac{\pi^2}{t^2(n+1)}$

Definition 4.3.3. A summability kernel is a sequence $(k_n)_{n=1}^\infty$ of 2π-periodic, bounded and piecewise continuous functions such that

1. $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n = 1$

2. $\sup_{n\in\mathbb{N}}\{\|k_n\|_1\} < \infty$

3. for any $0 < \delta \leqslant \pi$, $\lim_{n\to\infty}\left[\int_{-\pi}^{-\delta} |k_n| + \int_{\delta}^{\pi} |k_n|\right] = 0$

Proposition 4.3.4. The Fejer kernel $(K_n)_{n=1}^\infty$ is a summability kernel.

Theorem 4.3.5. [Abstract summability kernel theorem]

Let B be a homogeneous Banach space on $\mathbb{T}$ and $(k_n)_{n=1}^\infty$ be a summability kernel. Then for $f \in B$,

$$\lim_{k\to\infty}[\|k_n * f - f\|_B] = 0$$

Proof: Fix $f \in B$.

Let $F : \mathbb{R} \to B$ be given by $F(s) = s * f$ for $s \in \mathbb{R}$ (a.e.).

The axioms of a homogeneous Banach space tell us that F is continuous and 2π-periodic, and that

$$\|F(s)\|_B = \|s * f\|_B = \|f\|_B \qquad \text{and} \qquad 0 * f = f = F(0)$$

Now we compute

$$\begin{aligned}
\|(k_n * f) - f\|_B &= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s)(s * f)\ ds - f \right\|_B \\
&= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s)F(s)\ ds - F(0) \right\|_B \\
&= \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(s)(F(s) - F(0))\ ds \right\|_B \\
&\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(s)| \|F(s) - F(0)\|_B\ ds
\end{aligned}$$

Given $\varepsilon > 0$, we can find $\delta > 0$ such that $\|F(s) - F(0)\|_B < \frac{\varepsilon}{2M}$ for $0 \leqslant |s| < \delta$ and $M = \sup_{n \in \mathbb{N}} \{\|k_n\|_1\} < \infty$.
Then choose $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$\frac{1}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) |k_n| < \frac{\varepsilon}{4\|f\|_B}$$

We may safely assume that $\|f\|_B > 0$.
Then for all $n \geqslant N$, we have that

$$\begin{aligned}
\|(k_n * f) - f\|_B &\leqslant \frac{1}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) |k_n(s)| \|F(s) - F(0)\|_B\ ds + \frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(s)| \|F(s) - F(0)\|_B\ ds \\
&\leqslant \frac{1}{2\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) |k_n(s)| (2\|f\|_B) ds + \frac{1}{2\pi} \int_{-\delta}^{\delta} |k_n(s)| \left(\frac{\varepsilon}{2M} \right)\ ds \\
&= \frac{\|f\|_B}{\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) |k_n(s)| ds + \frac{\varepsilon}{4M\pi} \int_{-\delta}^{\delta} |k_n(s)|\ ds \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4M\pi} \int_{-\pi}^{\pi} |k_n(s)|\ ds \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} \|k_n(s)\|_1 \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

■

Corollary 4.3.6.

1. For $f \in C(\mathbb{T})$, we have $\lim_{n \to \infty} [\|\sigma_n(f) - f\|_\infty] = 0$

2. If $1 \leqslant p < \infty$, then for $f \in L_p(\mathbb{T})$ we have $\lim_{n \to \infty} [\|\sigma_n(f) - f\|_p] = 0$

Corollary 4.3.7. Suppose $f, g \in L_1(\mathbb{T})$, and $c_k(f) = c_k(g)$ for each $k \in \mathbb{Z}$, Then $f = g$ (a.e.).

Proof: Recall that

$$\sigma_n(f, t) = \frac{1}{n+1} \sum_{j=0}^{n} s_j(f, t) = \frac{1}{n+1} \sum_{j=0}^{n} \sum_{k=-j}^{j} c_k(f) e^{ikt}$$

Since $c_k(f) = c_k(g)$ for all k, we have that for all $n \in \mathbb{N}$,

$$\|f-g\|_1 = \|f - \sigma_n(f) + \sigma_n(g) - g\|_1 \leqslant \|f - \sigma_n(f)\|_1 + \|\sigma_n(g) - g\|_1 \xrightarrow{n\to\infty} 0$$

Therefore $\|f-g\|_1 = 0$, or $f = g$ a.e. ■

Definition 4.3.8. Let $f \in L(\mathbb{T})$ and $s \in \mathbb{R}$. Then the average value of f at s is defined to be

$$\omega_f(s) := \frac{1}{2} \lim_{h\to 0^+} [f(s-h) + f(s+h)]$$

provided that it exists.

Theorem 4.3.9. [Fejer]

1. If $f \in L(\mathbb{T})$ and $x \in [-\pi, \pi]$ such that $\omega_f(x)$ exists, then $\lim_{n\to\infty} [\sigma_n(f,x)] = \omega_f(x)$

2. If I is an open interval on which f is continuous, then for any closed subinterval J of I,

$$\lim_{n\to\infty} \left[\sup_{t\in J} \{|\sigma_n(f,t) - f(t)|\} \right] = 0$$

Proof: (**1.**) Suppose that $\omega_f(x)$ is finite.

Given $\varepsilon > 0$, let $\delta > 0$ be such that $0 \leqslant |s| \leqslant \delta$ implies $|\omega_f(x) - \frac{1}{2}(f(x-s) + f(x+s))| < \varepsilon$.

Then we have that for our choice of δ

$$\begin{aligned}
|\sigma_n(f,x) - \omega_f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) f(x-s)\ ds - \omega_f(x) \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) f(x-s)\ ds - \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) \omega_f(x)\ ds \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s)(f(x-s) - \omega_f(x))\ ds \right| \\
&\leqslant \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\ ds \right| + \left| \frac{1}{2\pi} \left(\int_{\pi}^{-\delta} + \int_{\delta}^{\pi} \right) K_n(s)(f(x-s) - \omega_f(x))\ ds \right|
\end{aligned}$$

Now note that for every $n \in \mathbb{N}$, since K_n is even,

$$\left| \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\ ds \right| = \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(-s)(f(x-s) - \omega_f(x))\ ds \right|$$

Then we have

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\ ds \right| &= \left| \frac{1}{4\pi} \int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\ ds + \frac{1}{4\pi} \int_{-\delta}^{\delta} K_n(s)(f(x+s) - \omega_f(x))\ ds \right| \\
&= \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(s) \left(\tfrac{1}{2}(f(x-s) + f(x+s)) - \omega_f(x) \right)\ ds \right| \\
&\leqslant \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(s) \left| \tfrac{1}{2}(f(x-s) + f(x+s)) - \omega_f(x) \right|\ ds \\
&\leqslant \frac{\varepsilon}{2\pi} \int_{-\delta}^{\delta} K_n(s)\ ds \\
&\leqslant \frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} K_n(s)\ ds \\
&= \varepsilon
\end{aligned}$$

As for the other part of the original integral, we have that

$$\begin{aligned}
\frac{1}{2\pi}\left|\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)K_n(s)(f(x-s)-\omega_f(x))\ ds\right| &\leqslant \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)K_n(s)|f(x-s)-\omega_f(x)|\ ds \\
&\leqslant \frac{\pi^2}{2(n+1)s^2}\cdot\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)|f(x-s)-\omega_f(x)|\ ds \\
&\leqslant \frac{\pi^2}{2(n+1)\delta^2}\cdot\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)|\check{f}(s-x)-\omega_f(x)|\ ds \\
&\leqslant \frac{\pi^2}{2(n+1)\delta^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}|\check{f}(s-x)-\omega_f(x)|\ ds \\
&\leqslant \frac{\pi^2}{2(n+1)\delta^2}\cdot\|(x*\check{f})-\omega_f(x)\|_1 \\
&\xrightarrow{n\to\infty} 0
\end{aligned}$$

Hence we conclude that $\limsup\limits_{n\to\infty}\left[|\sigma_n(f,x)-\omega_f(x)|\right]\leqslant\varepsilon$.
Then we conclude that the limit exists and vanishes.

(**2.**) Note that all the estimates performed were done uniformly over x for the choice of δ. ■

Corollary 4.3.10. Suppose that $f\in L(\mathbb{T})$, $x\in[-\pi,\pi]$ such that $\omega_f(x)$ exists, as does $\lim\limits_{n\to\infty}[S_n(f,x)]$. Then

$$\lim_{n\to\infty}[S_n(f,x)]=\omega_f(x)$$

Proof: If $\lim\limits_{n\to\infty}[S_n(f,x)]$ exists, then

$$\lim_{n\to\infty}[\sigma_n(f,x)]=\lim_{n\to\infty}\left[\frac{1}{n+1}\sum_{j=0}^{n}S_j(f)\right]=\lim_{n\to\infty}[S_n(f,x)]$$

Since Fejer gives that $\lim\limits_{n\to\infty}[\sigma_n(f,x)]=\omega_f(x)$, the result follows. ■

Definition 4.3.11. If $f\in L[a,b]$, then a point $x\in(a,b)$ is termed a Lebesgue point of f iff

$$\lim_{n\to0^+}\left[\frac{1}{n}\int_0^n\left|\frac{f(x-s)+f(x+s)}{2}-f(x)\right|ds\right]=0$$

Proposition 4.3.12. If $\omega_f(x)$ exists, then x is a Lebesgue point of f.

Theorem 4.3.13. [Lebesgue, Fejer]
If $x\in[-\pi,\pi]$ is a Lebesgue point for $f\in L(\mathbb{T})$, then $f(x)=\lim\limits_{n\to\infty}[\sigma_n(f,x)]$.

Lemma 4.3.14. If $f\in L_1(\mathbb{T})$, then for all $k\in\mathbb{Z}$, $|c_k(f)|\leqslant\|f\|_1$.

Proof:

$$|c_k(f)|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-ikt}\ dt\right|\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)||e^{-ikt}|\ dt=\|f\|_1$$

■

Theorem 4.3.15. [Riemann, Lebesgue]
If $f \in L_1(\mathbb{T})$, then $\lim_{|k|\to\infty} [|c_k(f)|] = 0$.

Proof: Let $\varepsilon > 0$.
By the abstract summability kernel theorem, we can find $n_0 \in \mathbb{N}$ such that $\|\sigma_{n_0}(f) - f\|_1 < \varepsilon$.
We note that

$$\sigma_{n_0}(f) = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j} c_k(f)e^k = \sum_{j=-n_0}^{n_0} \frac{n_0+1-|j|}{n_0+1} c_j(f)e^j = \sum_{j=-n_0}^{n_0} b_j e^j$$

Thus if $|k| \geqslant n_0$, then we have

$$\begin{aligned} c_k(\sigma_{n_0}(f) - f) &= c_k(\sigma_{n_0}(f)) - c_k(f) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{j=-n_0}^{n_0} b_j e^{j-k} - c_k(f) \\ &= \frac{1}{2\pi}\sum_{j=-n_0}^{n_0} \underbrace{\int_{-\pi}^{\pi} b_j \underbrace{e^{j-k}}_{\neq 1}}_{=0} - c_k(f) \\ &= -c_k(f) \end{aligned}$$

We then have that $|c_k(f)| = |c_k(\sigma_{n_0}(f) - f)| \leqslant \|\sigma_{n_0}(f) - f\|_1 < \varepsilon$. ■

Corollary 4.3.16.

$$\lim_{n\to\infty}\left[\int_{-\pi}^{\pi} f(t)\cos(nt)\ dt\right] = 0 \qquad \lim_{n\to\infty}\left[\int_{-\pi}^{\pi} f(t)\sin(nt)\ dt\right] = 0$$

Definition 4.3.17. Define the space $\mathcal{C}_0(\mathbb{Z})$ as

$$\mathcal{C}_0(\mathbb{Z}) = \left\{(c_k)_{k=1}^{\infty} \mid c_k \in \mathbb{C}, \lim_{|k|\to\infty}[c_k] = 0\right\}$$

This turns out to be a Banach space.

Theorem 4.3.18. [Open mapping theorem]
If X, Y are Banach spaces and $T : X \to Y$ is a surjective bounded linear operator, then T is open, i.e. $U \subset X$ is open implies that $T(U) \subset Y$ is open.

Corollary 4.3.19. [Inverse Mapping theorem]
Let X, Y be Banach spaces, with $T : X \to Y$ a bounded linear operator. If T is bijective, then $T^{-1} : Y \to X$ is bounded.

Corollary 4.3.20. There exist elements of $\mathcal{C}_0(\mathbb{Z})$ which are not arising from Fourier transforms. That is, there exist sequences $(c_k)_{k=1}^{\infty}$ for which there is no $f \in L_1(\mathbb{T})$ such that $c_k = c_k(f)$ for each k.

Proof: Let $T : L_1(\mathbb{T}) \to \mathcal{C}_0(\mathbb{Z})$ be given by $Tf = (c_k(f))_{k=-\infty}^{\infty}$.
Then T is linear with range$(T) \subset \mathcal{C}_0(\mathbb{Z})$ (by Riemann-Lebesgue), and T is bounded with $|||T||| \leqslant 1$.
Also, T is injective by the corollary to the abstract summability kernel theorem.
If it were the case that T were bijective, then $T^{-1} : \mathcal{C}_0(\mathbb{Z}) \to L_1(\mathbb{T})$ would be bounded.
However, consider

$$d_n = (\ldots, 0, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, 0, \ldots) \in \mathcal{C}_0(\mathbb{Z})$$

So the sequence has nonzero values from the $-n$th to the nth positions only.
Then $\|d_n\|_\infty = \frac{1}{2}$, and $T^{-1}(d_n) = D_n$, the Dirichlet kernel of order n.
But this leads us to

$$|||T^{-1}||| \geqslant \sup_{n\in\mathbb{N}}\{\|T^{-1}(d_n)\|_1\} = \sup_{n\in\mathbb{N}}\{\|D_n\|_1\} = \infty$$

This contradicts the inverse mapping theorem.
Thus we have a contradiction, and T is not bijective, so not surjective. ∎

Remark 4.3.21. The map introduced in the proof above is termed the Fourier transform, and is given by

$$\begin{array}{rrcl} T: & L_1(\mathbb{T}) & \to & \mathcal{C}_0(\mathbb{Z}) \\ & f & \mapsto & (c_k(f))_{k=-\infty}^{\infty} \end{array}$$

4.4 Localisation

Lemma 4.4.1. If $f \in L(\mathbb{T})$ with $\int_{-\pi}^{\pi} \left| \frac{f(t)}{t} \right| dt < \infty$, then $\lim_{n\to\infty} [s_n(f,0)] = 0$.

Proof: Recall that $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
Therefore we have that

$$\begin{aligned} D_n(s) &= \frac{\sin\left(\left(n+\frac{1}{2}\right)s\right)}{\sin\left(\frac{1}{2}s\right)} = \frac{\sin(ns)\cos\left(\frac{1}{2}s\right)}{\sin\left(\frac{1}{2}s\right)} + \cos(ns) \\ s_n(f,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(s)\ ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(ns)\cos\left(\frac{1}{2}s\right)}{\sin\left(\frac{1}{2}s\right)} f(s)\ ds + \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(ns)f(s)\ ds \end{aligned}$$

Note that for $0 < |\theta| < \frac{\pi}{2}$ we have that $|\sin(\theta)| \geqslant \frac{2}{\pi}|\theta|$.
Thus we have that $\left|\sin\left(\frac{1}{2}t\right)\right| \geqslant \frac{1}{\pi}|t|$ for $t \in [-\pi,\pi]$, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\cos\left(\tfrac{1}{2}s\right)\right| \left| \frac{f(s)}{\sin\left(\frac{1}{2}s\right)} \right| ds \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f(s)|}{\frac{1}{\pi}|s|}\ ds = \frac{1}{2}\int_{-\pi}^{\pi} \left| \frac{f(s)}{s} \right| ds < \infty$$

So the map $s \mapsto \frac{\cos\left(\frac{1}{2}s\right)f(s)}{\sin\left(\frac{1}{2}s\right)}$ (for a.e. $s \in [-\pi,\pi]$ extended periodically) defines an element in $L(\mathbb{T})$.
Then we apply the corollary to the Riemann-Lebesgue lemma to see that

$$s_n(f,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(ns)\cos\left(\frac{1}{2}s\right)}{\sin\left(\frac{1}{2}s\right)} f(s)\ ds + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(ns)f(s)\ ds \xrightarrow{n\to\infty} 0$$

∎

Theorem 4.4.2. [Localisation Principle]
If $f \in L(\mathbb{T})$ and $f(t) = 0$ for a.e. t on an open interval I, then for $t \in I$ we have $\lim_{n\to\infty}[s_n(f,t)] = 0$.

Proof: Let $g \in L(\mathbb{T})$ be given by $g(s) = f(t-s) = \check{f}(s-t)$ so $g = t * \check{f}$ when $t \in I$ is fixed.

Then $g(s) = 0$ for a.e. s in neighborhood of 0, say for $s \in (-\delta, \delta)$, and so

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \frac{g(s)}{s} \right| ds &= \int_{-\delta}^{\delta} \left| \frac{0}{s} \right| ds + \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) \left| \frac{g(s)}{s} \right| ds \\
&\leqslant \int_{-\pi}^{\pi} \frac{1}{\delta} |g(s)| \ ds \\
&= \frac{1}{\delta} \int_{-\pi}^{\pi} |t * \check{f}| \ ds \\
&= \frac{1}{\delta} 2\pi \|t * \check{f}\|_1 \\
&= \frac{2\pi}{\delta} \|f\|_1 \\
&< \infty
\end{aligned}$$

Hence by the lemma, $\lim_{n\to\infty} [s_n(g, 0)] = 0$, and by translation and inversion invariance we find that

$$s_n(g, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) g(s - 0) \ ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) f(t - s) \ ds = s_n(f, t)$$

Therefore we have that $\lim_{n\to\infty} [s_n(f, t)] = \lim_{n\to\infty} [s_n(g, t)] = 0$. ■

Corollary 4.4.3. If $f, g \in L_1(\mathbb{T})$ and $f(t) = g(t)$ for a.e. t on an open interval I, then for $t \in I$ we have that $\lim_{n\to\infty} [s_n(f, t)]$ exists iff $\lim_{n\to\infty} [s_n(g, t)]$ exists, and then two limits coincide.

Proof: Observe that

$$\lim_{n\to\infty} [s_n(f - g, t)] = \lim_{n\to\infty} [s_n(f, t) - s_n(g, t)] = \lim_{n\to\infty} [s_n(f, t)] - \lim_{n\to\infty} [s_n(g, t)] = 0$$

■

Theorem 4.4.4. [Dini]
If $f \in L_1(\mathbb{T})$ and f is differentiable at $t \in [-\pi, \pi]$, then for such t we have $\lim_{n\to\infty} [s_n(f, t)] = f(t)$.

Proof: Given $\varepsilon > 0$, there is $\delta > 0$ such that $|s| < \delta$ implies $\left| \frac{f(t-s) - f(t)}{s} - f'(t) \right| < \varepsilon$.

Hence the map $s \mapsto \frac{f(t-s) - f(t)}{s}$ is bounded on $(-\delta, \delta)$.

Let $g = t * \check{f} - f(t)$, so $g(s) = f(t - s) - f(t)$ and

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \frac{g(s)}{s} \right| ds &= \int_{-\delta}^{\delta} \left| \frac{g(s)}{s} \right| ds + \int_{[-\delta, -\pi] \cup [\delta, \pi]} \left| \frac{g(s)}{s} \right| ds \\
&\leqslant \int_{-\delta}^{\delta} (|f'(t)| + \varepsilon) \ ds + \frac{1}{\delta} \int_{-\pi}^{\pi} |g(s)| \ ds \\
&= 2\delta(|f'(t)| + \varepsilon) + \frac{1}{\delta} \|t * \check{f} - f(t)\|_1 \\
&< \infty
\end{aligned}$$

Applying the lemma, we have that $\lim_{n\to\infty} [s_n(g, 0)] = 0$.

As before, we conclude that

$$s_n(g, 0) = s_n(t * \check{f} - f(t), 0) = s_n(t * \check{f}, 0) - s_n(f(t), 0) = s_n(f, t) - f(t)$$

■

Theorem 4.4.5. [Dini]
If $f \in L_1(\mathbb{T})$ and f is Lipschitz on an interval I, then for $t \in I$ we have $\lim_{n\to\infty}[s_n(f,t)] = f(t)$.

Proof: Fix $t \in I$, an open interval.
Then $(t-\delta, t+\delta) \subset I$ for some $\delta > 0$, so for $s \in (-\delta, \delta)$, let $g(s) = f(t-s) - f(t)$.
For such s we will then have that for some constant M

$$\left| \frac{g(s)}{s} \right| = \left| \frac{f(t-s) - f(t)}{(t-s) - t} \right| \leqslant M$$

As before, we partition $[-\pi, \pi] = [-\delta, \delta] \cup [-\pi, -\delta] \cup [\delta, \pi]$.
By the same calculations as above, we find that

$$\int_{-\pi}^{\pi} \left| \frac{g(s)}{s} \right| ds < \infty$$

Therefore $\lim_{n\to\infty}[s_n(g,0)] = 0$ and we conclude, as above, that $\lim_{n\to\infty}[s_n(f,t)] = f(t)$. ■

5 Hilbert spaces

5.1 The inner product

Definition 5.1.1. Let X be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). An inner product on X is a map

$$(\cdot,\cdot) : \quad X \times X \quad \to \quad \mathbb{C}$$

satisfying the following properties.
1. $(f,f) \geqslant 0$
2. $(f,f) = 0$ iff $f = 0$
3. $(f,g) = \overline{(g,f)}$
4. $(\alpha f, g) = \alpha(f,g)$
5. $(f+g,h) = (f,h) + (g,h)$
We also define a norm (to be proved that it is actually a norm) for $f \in X$ by $\|f\| = \sqrt{(f,f)}$.

Theorem 5.1.2. [Cauchy, Schwarz]
Given a vector space X with an inner product $(\cdot,\cdot)$, for $f, g \in X$,

$$|(f,g)| \leqslant \|f\|\|g\|$$

with equality holding iff $t_1 f = t_2 g$ for $|t_1| + |t_2| > 0$.

Proof: Substitute $(f,g)g$ for g so then $(f,(f,g)g) = \overline{(f,g)}(f,g) \geqslant 0$, and we assume $|(f,g) \geqslant 0$.
If $t \in \mathbb{R}$, then $\bar{t} = t$, and

$$\begin{aligned} 0 &\leqslant (tf+g, tf+g) \\ &= t^2(f,f) + t(f,g) + t(g,f) + (g,g) \\ &= t^2\|f\|^2 + 2t\mathrm{Re}((f,g)) + \|g\|^2 \\ &= p(t) \end{aligned}$$

So p is a polynomial in t, and our assumptions imply that p has at most 1 root, so

$$\begin{aligned} 4\mathrm{Re}^2((f,g)) - 4\|f\|^2\|g\|^2 &\leqslant 0 \\ |(f,g)| &\leqslant \|f\|\|g\| \end{aligned}$$

■

Proposition 5.1.3. $\|\cdot\|$ is a norm.

Proof: First, note that $\|\alpha f\| = |\alpha|\|f\|$ (this is a straightforward exercise).

Also note that

$$\begin{aligned}\|f+g\|^2 &= (f+g, f+g) \\ &= \|f\|^2 + 2\mathrm{Re}((f,g)) + \|g\|^2 \\ &\leqslant \|f\|^2 + 2|\mathrm{Re}((f,g))| + \|g\|^2 \\ &\leqslant \|f\|^2 + 2|(f,g)| + \|g\|^2 \\ &\leqslant \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2\end{aligned}$$

■

Definition 5.1.4. A vector space associated with an inner product is termed an inner product space. An inner product space that is complete with respect to $\|f\| = (f,f)^{1/2}$ is termed a Hilbert space.

These are some examples of inner product spaces:

- The Hilbert space $\mathbb{C}^n$ with $((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum x_i\overline{y_i}$
- The Hilbert space $L_2(A)$ with $(f,g) = \int_A f\overline{g}$
- The space $C[a,b]$ with $(f,g) = \int_a^b f\overline{g}$
- The Hilbert space ℓ_2 with $(x,y) = \sum x_i\overline{y_i}$

Definition 5.1.5. Let $(X, (\ ,\ ))$ be an inner product space. For some indexing set I, a set $\{e_i\}_{i\in I} \subset X$ is termed orthogonal iff $e_i \neq 0$ for all $i \in I$, and $(e_i, e_j) = 0 \iff i \neq j$. The same set is termed orthonormal iff $(e_i, e_j) = \delta_{ij}$.

Proposition 5.1.6. [Pythagoras]
If $\{f_1,\ldots,f_n\}$ is orthogonal in an inner product space $(X, (\ ,\ ))$, then

$$\|f_1 + \cdots + f_n\|^2 = \|f_1\|^2 + \cdots + \|f_n\|^2$$

Proof: Let $n = 2$, so that

$$\|f_1 + f_2\|^2 = (f_1 + f_2, f_2 + f_2) = \|f_1\|^2 + 2\mathrm{Re}((f_1, f_2)) + \|f_2\|^2 = \|f_1\|^2 + \|f_2\|^2$$

Use induction, noting that $(f_1 + \cdots + f_{n-1}, f_n) = 0$. ■

Theorem 5.1.7. [Linear approximation lemma]
Let $\{e_1 \ldots, e_n\}$ be an orthonormal set in an inner product space $(X, (\ ,\ ))$ with $E = \mathrm{span}\{e_1,\ldots,e_n\}$. Define for $f \in X$ the function

$$\mathrm{dist}(f,E) := \inf\{\|f-g\| \mid g \in E\}$$

Then

$$\mathrm{dist}(f,E)^2 = \left\| f - \sum_{i=1}^n (f,e_i)e_i \right\|^2 = \|f\|^2 - \sum_{i=1}^n |(f,e_i)|^2$$

Moreover, $\sum_{i=1}^n (f,e_i)e_i$ is the unique vector $g \in E$ such that $\|f-g\| = \mathrm{dist}(f,E)$.

Proof: Let $g = \sum_{i=1}^{n} \alpha_i e_i$ be an arbitrary element of E, so then

$$\begin{aligned}
\|f-g\|^2 &= (f-g, f-g) \\
&= \|f\|^2 - 2\mathrm{Re}((f,g)) + \|g\|^2 \\
&= \|f\|^2 - 2\mathrm{Re}\left(\sum_{i=1}^{n} \overline{\alpha_i}(f,e_i)\right) + \sum_{i=1}^{n} |\alpha_i|^2 \\
&\geqslant \|f\|^2 - 2\sum_{i=1}^{n} |\alpha_i||(f,e_i)| + \sum_{i=1}^{n} |\alpha_i|^2 \\
&= \|f\|^2 - \sum_{i=1}^{n} |(f,e_i)|^2 + \sum_{i=1}^{n} |(f,e_i)|^2 - 2\sum_{i=1}^{n} |\alpha_i||(f,e_i)| + \sum_{i=1}^{n} |\alpha_i|^2 \\
&= \|f\|^2 - \sum_{i=1}^{n} |(f,e_i)|^2 + \sum_{i=1}^{n} (|(f,e_i)| - |\alpha_i|)^2 \\
&\geqslant \|f\|^2 - \sum_{i=1}^{n} |(f,e_i)|^2
\end{aligned}$$

Notice that both inequalities become equalities exactly when $\alpha_i = (f, e_i)$.
Moreover, if $g = \sum_{i=1}^{n} (f, e_i)e_i$, then the third line turns into the last line (above).
Hence this vector g corresponds exactly to $\inf\{\|f-h\| \mid h \in E\} = \|f-g\|$. ■

Proposition 5.1.8. Let $(X, (\ ,\ ))$ be an inner product space with $g \in X$. Then the functional

$$\begin{array}{rccc} \Gamma_g : & X & \to & \mathbb{C} \\ & f & \mapsto & (f,g) \end{array}$$

is linear and bounded, with $\|\Gamma_g\|_* = \|g\|$.

Proof: Linearity follows from the properties of the inner product.
By Cauchy and Schwarz, $\|\Gamma_g\|_* < \|g\|$ comes from the equalities

$$|\Gamma_g(g)| = |(f,g)| \leqslant \|f\|\|g\|$$

Further, if $g \neq 0$, then

$$\Gamma\left(\frac{1}{\|g\|} g\right) = \left(\frac{1}{\|g\|} g, g\right) = \frac{1}{\|g\|}(g,g) = \|g\|$$

Therefore $\Gamma_g\|_* \geqslant \|g\|$. ■

Remark 5.1.9. By the Riesz representation theorem, every bounded linear functional in a Hilbert space H of the form $\Gamma : H \to \mathbb{C}$ is of the form Γ_g for $g \in H$.

5.2 Orthonormal bases

Theorem 5.2.1. [Orthonormal basis theorem]
Let $(X, (\ ,\ ))$ be an inner product space with $(e_i)_{i=1}^{\infty}$ an orthonormal sequence. Then equivalently

1. The set span$\{e_i\}$ is dense in X
2. *Bessel's equality:* For every $f \in X$, $\|f\|^2 = \sum_{n=1}^{\infty} \|(f, e_i)\|^2$
3. For every $f \in X$, we have $f = \lim_{n\to\infty} [\sum_{i=1}^{n} (f, e_i)e_i] = \sum_{i=1}^{\infty} (f, e_i)e_i$
4. *Parseval's identity:* For every $f, g \in X$ we have $(f, g) = \sum_{i=1}^{\infty} (f, e_i)(e_i, g)$

Proof: (**1.**$\Leftrightarrow$**3.**) Let $E_n = \text{span}\{e_1, \ldots, e_n\}$.
Then $E_n \subset E_{n+1}$ for all n.
Then for $f \in X$, $\text{dist}(f, E_n) \geqslant \text{dist}(f, E_{n+1})$.
Then by the linear approximation lemma,

$$\left(\begin{array}{c} \text{span}(e_n)_{n=1}^{\infty} = \bigcup_{n=1}^{\infty} E_n \\ \text{is dense in } X \end{array} \right) \iff \left(\begin{array}{c} \text{for each } f \in X, \\ \left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\| = \text{dist}(f, E_n) \xrightarrow{n\to\infty} 0 \end{array} \right)$$

(**2.**$\Leftrightarrow$**3.**) By the linear approximation lemma,

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|^2 = \|f\|^2 - \sum_{i=1}^{n} |(f, e_i)|^2$$

Which implies that

$$\|f\|^2 = \lim_{n\to\infty} \left[\sum_{i=1}^{n} |(f, e_i)|^2 \right] \quad \text{iff} \quad \lim_{n\to\infty} \left[\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|^2 \right] = 0$$

(**3.**$\Rightarrow$**4.**) Let $g \in X$.
By above, the functional $\Gamma_g : X \to \mathbb{C}$ for $\Gamma_g(f) = (f, g)$ is continuous, so

$$\begin{aligned} (f, g) &= \Gamma_g(f) \\ &= \Gamma_g \left(\lim_{n\to\infty} \left[\sum_{i=1}^{n} (f, e_i) e_i \right] \right) \\ &= \lim_{n\to\infty} \left[\Gamma_g \left(\sum_{i=1}^{n} (f, e_i) e_i \right) \right] \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} (f, e_i) \Gamma_g(e_i) \right] \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} (f, e_i)(e_i, g) \right] \end{aligned}$$

(**4.**$\Rightarrow$**2.**) For $f = g$, note that

$$(f, e_1)(e_i, f) = (f, e_i)\overline{(f, e_i)} = |(f, e_i)|^2$$

■

Proposition 5.2.2. [Bessel]
For $(X, (\ ,\ )) \ni f$ an inner product space with $(e_i)_{i=1}^{\infty}$ an orthonormal sequence,

$$\|f\|^2 \geqslant \sum_{i=1}^{\infty} |(f, e_i)|^2$$

Theorem 5.2.3. [Planchard]
Let $(X, (\ ,\ ))$ be an inner product space and $(e_i)_{i=1}^{\infty}$ an orthonormal basis for X. Then the following operator defines an isometry:

$$\begin{array}{rccc} U: & X & \to & \ell_2 \\ & f & \mapsto & ((f, e_i))_{i=1}^{\infty} \end{array}$$

That is, $\|U(f)\|_2 = \|U(f)\|_{\ell_2} = \|f\|_X$ and $(U(f), U(g))_{\ell_2} = (f, g)_X$.

Proof: By Bessel, for $f \in X$

$$\|U(f)\|_2^2 = \sum_{i=1}^{n} |(f, e_i)|^2 \leqslant \|f\|^2$$

Therefore U is linear, and so applying Parseval,

$$(U(f), U(g)) = (((f, e_i))_{i=1}^{\infty}, ((g, e_i))_{i=1}^{\infty}) = \sum_{i=1}^{\infty} (f, e_i)(e_i, g) = (f, g)$$

The first equality is justified by the fact that

$$U((e_i)_{i=1}^{\infty}) = \{U(e_i) \mid i \in \mathbb{N}\} = \{(0, \dots, 0, \underbrace{1}_{\text{position } i}, 0, \dots, 0, \dots) \mid i \in \mathbb{N}\}$$

As $(U(e_i))_j = 1$ iff $i = j$ and 0 otherwise, we have that

$$(U(f), U(e_i))(U(e_i), U(g)) = (f, e_i)(e_i, g)$$

By letting $f = g$ we get the result. ■

Theorem 5.2.4.
1. $\{e^k \mid k \in \mathbb{Z}\}$ is an orthonormal basis for $L_2(\mathbb{T})$.
2. $\{e^k \mid k \in \mathbb{Z}\}$ is an orthonormal basis for $C(\mathbb{T})$ with inner product $(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\overline{g}$

Proof: Since $C(\mathbb{T})$ is dense in $L_2(\mathbb{T})$, the proofs are the same, but are given for elucidation.
(**1.**) It remains to verify that $\{e^k \mid k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{T})$, as we know it is orthonormal.
Note that $\sigma_n(f) \in \text{span}\{e^k \mid -n \leqslant k \leqslant n\}$ and by the abstract summability kernel theorem,

$$\text{dist}(f,\ \text{span}\{e^k \mid -n \leqslant k \leqslant n) \leqslant \|f - \sigma_n(f)\|_2 \xrightarrow{n \to \infty} 0$$

This satisfies condition (**3.**) of the OBT.
Now (**2.**) follows by estimating $\|\cdot\|_\infty$ with $\|\cdot\|_2$.

(**2.**) Observe that $\text{Trig}(\mathbb{T}) = \text{span}\{e^k \mid k \in \mathbb{Z}\}$ is an algebra of functions on $\mathbb{T}/\sim_{-\pi=\pi}$ which is point separating and conjugation closed.
By Stone-Weierstrass, $\text{Trig}(\mathbb{T})$ is dense in $C(\mathbb{T})$ under the norm $\|\cdot\|_\infty$.
So for any $\varepsilon > 0$ and $f \in C(\mathbb{T})$, we can find $h \in \text{Trig}(\mathbb{T})$ s.t. $\|f - h\|_\infty < \varepsilon$, and since the 2-norm is bounded above by the sup-norm, $\|f - h\|_2 < \varepsilon$. ■

Corollary 5.2.5. For $f \in L_2(\mathbb{T})$,

$$\lim_{n \to \infty} [\|f - s_n(f)\|_2] = 0$$

Proof: First note that

$$s_n(f) = \sum_{k=-n}^{n} c_k(f) e^k = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\ dt \right) = \sum_{k=-n}^{n} (f, e^k) e^k$$

And by the OBT,

$$\lim_{n \to \infty} \left[\left\| f - \sum_{k=-n}^{n} (f, e^k) e^k \right\|_2 \right] = 0$$

■

Theorem 5.2.6. [Riesz, Fischer]
For $f \in L_1(\mathbb{T})$,

$$f \in L_2(\mathbb{T}) \quad \text{iff} \quad \sum_{k=-\infty}^{\infty} |c_k(f)|^2 < \infty$$

Proof: ($\Rightarrow$) Since $c_k(f) = (f, e^k)$, by Bessel we have that $\|f\|_2^2 \geqslant \sum_{k=-n}^{n} |c_k(f)|^2$.
Moreover,

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \sup_{n\in\mathbb{N}} \left\{ \sum_{k=-n}^{n} |c_k(f)| \right\} \leqslant \|f\|_2^2 < \infty$$

($\Leftarrow$) Let $f_n = \sum_{k=-n}^{n} c_k(f)e^k$ with $m < n$, so by Pythagoras,

$$\|f_n - f_m\|_2^2 = \sum_{k=-n}^{-(m+1)} |c_k(f)|^2 + \sum_{k=m+1}^{n} |c_k(f)|^2 \xrightarrow{n\to\infty} 0$$

It follows that $(f_n)_{n=1}^{\infty}$ is Cauchy in $L_2(\mathbb{T})$.
Since $L_2(\mathbb{T})$ is complete, there is $\tilde{f} \in L_2(\mathbb{T})$ such that

$$\left\| \tilde{f} - \sum_{k=-n}^{n} c_k(f)e^k \right\|_2 \xrightarrow{n\to\infty} 0$$

Applying Γ_{e^k} we see that $c_k(f) = c_k(\tilde{f})$ for $k \in \mathbb{Z}$.
Therefore $f = \tilde{f}$ a.e. and $f = \tilde{f}$ in $L_2(\mathbb{T})$. ■

Remark 5.2.7. Recall that for $1 \leqslant p < \infty$, the ℓ_p-space is defined as the set of sequences $a = (a_n)_{n=0}^{\infty}$ for each $a_n \in \mathbb{R}$, such that

$$\|a\|_p = \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{1/p} < \infty$$

Theorem 5.2.8. [Plancherel]
The following operator defines a surjective isometry:

$$\begin{array}{rccc} U: & L_2(\mathbb{T}) & \to & \ell_2(\mathbb{Z}) \\ & f & \mapsto & (c_n(f))_{n=-\infty}^{\infty} \end{array}$$

That is, $(U(f), U(g))_{\ell_2} = (f, g)_{L_2}$.

Proof: This is a near restatment of Riesz-Fischer.
However, if $(c_n)_{n=-\infty}^{\infty} \in \ell_2(\mathbb{Z})$, we need to show that there is $f \in L_2(\mathbb{T})$ with $c_n(f) = c_n$ for all n.
Define $f_n = \sum_{k=-n}^{n} c_k e^k$.
Verify that $(f_n)_{n=1}^{\infty}$ is Cauchy in $L_2(\mathbb{T})$, and hence converges to $f \in L_2(\mathbb{T})$.
Moreover, $c_n(f) = c_n$ for each n.
That U is an isometry is a result of Bessel and Parseval from the OBT. ■

Corollary 5.2.9. The space $\ell_2(\mathbb{Z})$ is complete.

Proof: Suppose that $((c_k^{(n)})_{k=-\infty}^{\infty})_{n=1}^{\infty} \subset \ell_2(\mathbb{Z})$ is Cauchy.
Then for each $(c_k^{(n)})_{k=-\infty}^{\infty}$, there is $f \in L_2(\mathbb{T})$ such that $c_k(f_n) = c_k^{(n)}$ for each k and each n.
Moreover,

$$\|f_n - f_m\|_{L_2} = \|Uf_n - Uf_m\|_{\ell_2} = \left\| (c_k^{(n)})_{k=-\infty}^{\infty} - (c_k^{(m)})_{k=-\infty}^{\infty} \right\|_{\ell_2}$$

Therefore $(f_n)_{n=1}^{\infty} \subset L_2(\mathbb{T})$ is Cauchy.
Hence $f = \lim_{n\to\infty} [f_n]$ exists.
Also, $(c_k(f))_{k=-\infty}^{\infty}$ is the limit of $(c_k(f_n))_{k=-\infty}^{\infty} = (c_k^{(n)})_{k=-\infty}^{\infty}$. ■

Remark 5.2.10. For $f \in L(\mathbb{T})$ with $\int_{-\pi}^{\pi} |f|^p < \infty$ for $1 < p < \infty$, we have that for a.e. x

$$\lim_{n\to\infty} [s_n(f,x)] = f(x)$$

Lemma 5.2.11. Let X be a Banach space with $(a_k)_{k=-\infty}^{\infty} \subset X$. Let $s_n = \sum_{k=-n}^{n} a_k$ and $\sigma_n = \frac{1}{n+1}\sum_{j=0}^{n} s_j$. If $\lim_{n\to\infty}[\sigma_n]$ exists and $\sup_{n\in\mathbb{N}} \{|k|\|a_k\|\} < \infty$, then $\lim_{n\to\infty}[s_n] = \lim_{n\to\infty}[\sigma_n]$ and both limits exist.

Theorem 5.2.12. [Tauberian theorem - Hardy]
1. If $f \in L(\mathbb{T})$ and $\sup_k\{|kc_k(f)|\} < \infty$, then for any $t \in [-\pi,\pi]$ for which $\lim_{n\to\infty}[\sigma_n(f,t)]$ exists, $\lim_{n\to\infty}[s_n(f,t)]$ exists as well.
2. Let $B \ni f$ be a homogeneous Banach space. If $\|e^k\|_B \leqslant C$ for some fixed constant C for all k, and $\sup_k\{|kc_k(f)|\} < \infty$, then $\lim_{n\to\infty}[\|s_n(f) - f\|_B] = 0$.

Proof: (**1.**) In the context of the previous lemma, we let $X = \mathbb{C}$.
Then for $t \in [-\pi,\pi]$, we have that $|kc_k(f)e^{ikt}| = |kc_k(f)|$, the supremum of which over k is finite.
Hence the conditions of the lemma are satisfied.
Apply Fejer's theorem to get the result.

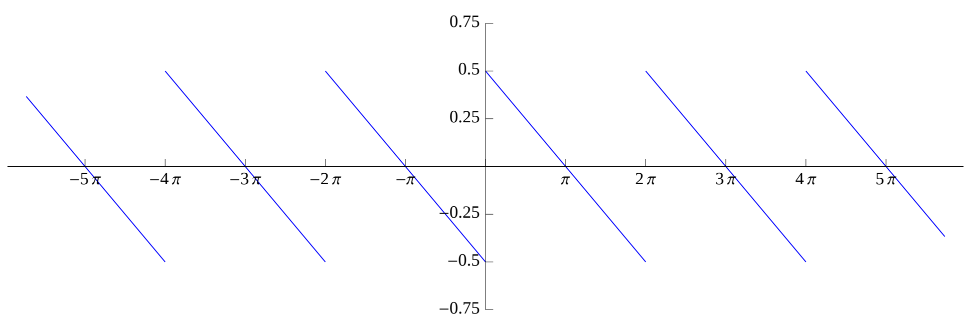
(**2.**) Let $X = B$, so then we have that for some constant C,

$$\|kc_k(f)e^k\|_B = |kc_k(f)|\|e^k\|_B \leqslant |kc_k(f)| \cdot C$$

Finally note that $\lim_{n\to\infty}[\sigma_n(f)] = f$ in B by virtue of the ASKT. ■

5.3 The Gibbs phenomenon

In this section we will use the periodic function $F(t) = \frac{1}{2} - \frac{t}{2\pi}$ on the interval $[-\pi,\pi]$ repeated across $\mathbb{R}$.



Proposition 5.3.1. For the function F as defined above,

$$c_0(F) = 0 \qquad s_n(F,t) = \sum_{k=1}^{n} \frac{\sin(kt)}{\pi k}$$

$$c_k(F) = \frac{1}{2\pi i k} \quad , \quad k > 0$$

In particular, we have that

$$\lim_{n\to\infty} [s_n(F,0)] = 0 = \omega_F(0) \qquad\qquad \lim_{n\to\infty} [s_n(F,t)] = F(t) \quad \text{for } t \in [-\pi,\pi] \setminus \{0\}$$

Lemma 5.3.2. For the function F as defined above,

$$\lim_{n\to\infty} \left[s_n\left(F, \frac{\pi}{n}\right)\right] = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(x)}{x}\, dx \quad \approx \quad 0.58949$$

<u>*Proof:*</u> Recall that $s_n(F,t) = \sum_{k=1}^{n} \frac{\sin(kt)}{k\pi}$, so then

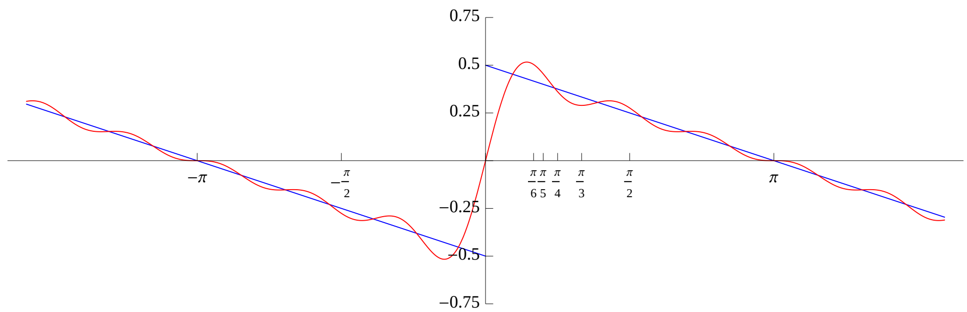
$$s_n\left(F, \tfrac{\pi}{n}\right) = \sum_{k=1}^{n} \frac{\sin\left(\frac{\pi k}{n}\right)}{\frac{\pi k}{n}} \cdot \frac{1}{n} = \frac{1}{\pi}\sum_{k=1}^{n} \frac{\sin\left(\frac{\pi k}{n}\right)}{\frac{\pi k}{n}} \left(\frac{k\pi}{n} - \frac{(k-1)\pi}{n}\right) \xrightarrow{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin(x)}{x}\, dx$$

The rest is numerical estimation. ■

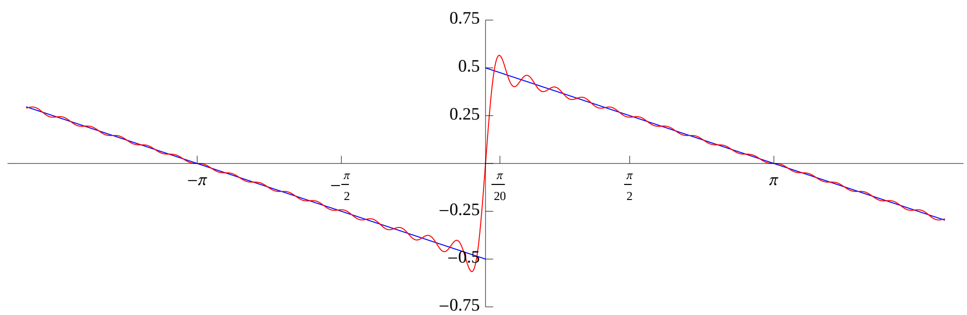
Definition 5.3.3. The <u>Gibbs constant</u> is defined as

$$G_s = \lim_{n\to\infty}\left[s_n\left(F, \frac{\pi}{n}\right) - F\left(\frac{\pi}{n}\right)\right] \quad \approx \quad 0.08949$$

It is related to the amount by which $s_n(F)$ over- and under- estimates F around 0. Here we have $s_6(F)$, with maxima and minima occuring at $\pm\frac{\pi}{6}$.



Here we have $s_{20}(F)$, with maxima and minima at $\pm\frac{\pi}{20}$.



The Gibbs constant describes this phenomenon of sharp error near the discontinuities. This is generalized for any $f \in L(\mathbb{T})$ with such discontinuities.

Theorem 5.3.4. [Gibbs]
Let $f \in L(\mathbb{T})$ be bounded and piecewise differentiable, i.e. $f'(t)$ exists except at finitely many points, and $|f'(t)| \leqslant M$ where it exists. Let $r_1, \ldots, r_m$ be the points where differentiability fails. Then the limits $f(r_j^{\pm}) = \lim_{h\to 0^+} [f(s_j \pm h)]$ exist, and for $\gamma_j = \gamma_f(r_j) = f(r_j^+) - f(r_j^-)$ we have

$$\lim_{n\to\infty}\left[s_n\left(f, r_j \pm \frac{\pi}{n}\right) - f\left(r_j \pm \frac{\pi}{n}\right)\right] = \pm\gamma_f(r_j) G_s$$

Theorem 5.3.5. Let $f \in L(\mathbb{T})$ satisfy:

· f is piecewise differentiable (i.e. differentiable except at finitely many points)

· $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'| = \|f'\|_1 < \infty$

· f is bounded, i.e. $\|f\|_\infty < \infty$

Then $\sup\limits_{k\in\mathbb{Z}} \{|kc_k(f)|\} < \infty$.

Remark 5.3.6. The above can be used to apply Hardy's Tauberian theorem, as its conclusion is one of Hardy's conditions.

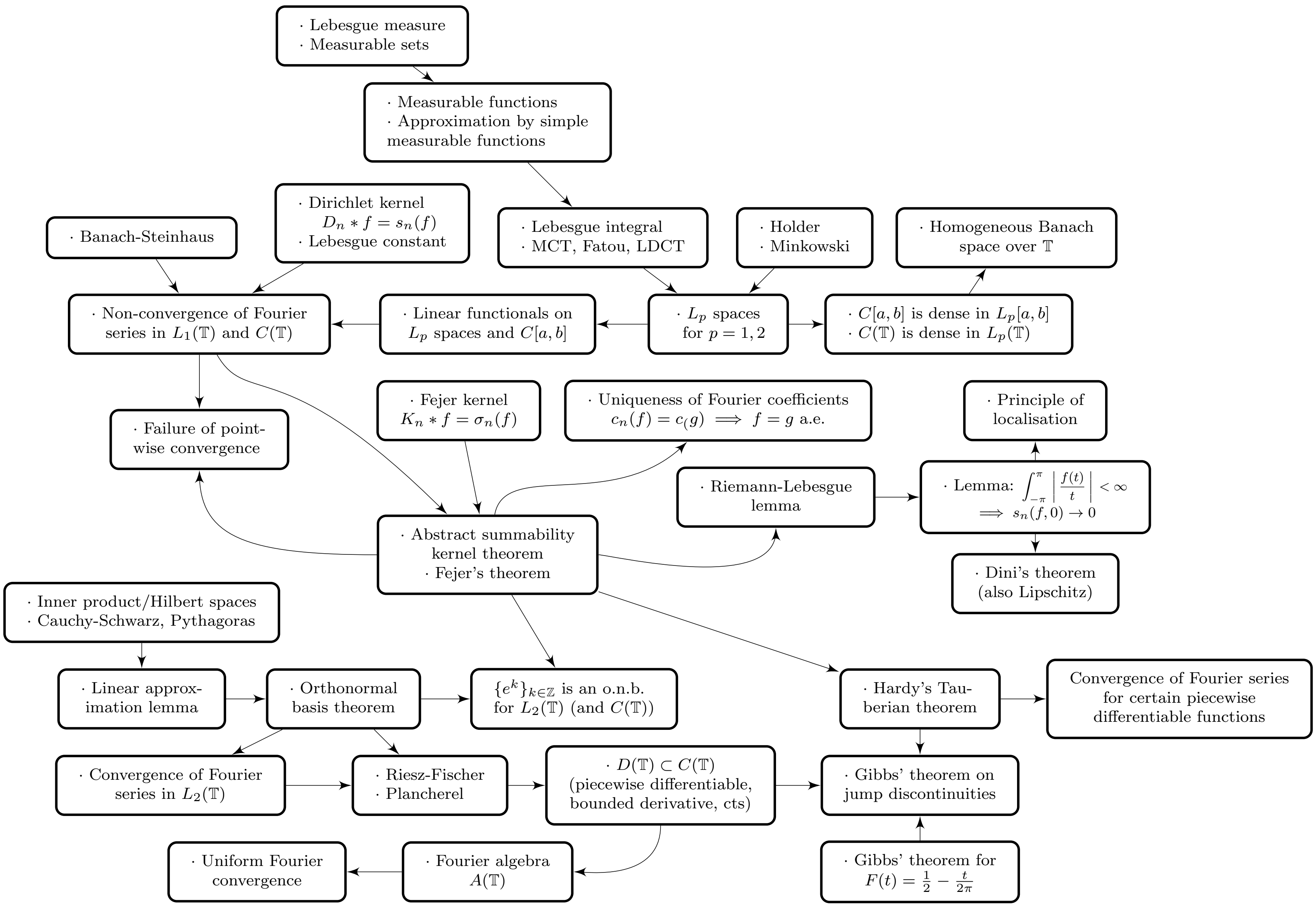
6 Review

These are some important function spaces and relations:

$$\begin{array}{ccccccccc} A(\mathbb{T}) & \subsetneq & C(\mathbb{T}) & \subsetneq & L_2(\mathbb{T}) & \subsetneq & L_1(\mathbb{T}) & \subset & C^*(\mathbb{T}) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \ell_1(\mathbb{Z}) & \subset & C^*(\mathbb{Z}) & \subset & \ell_2(\mathbb{Z}) & \subset & A(\mathbb{Z}) & \subsetneq & \mathcal{C}_0(\mathbb{Z}) \end{array}$$

The C^* algebras are included for completeness; we have not discussed them in this course.

On the next page we have an overview of the main topics covered in this course, and their relations with each other.



· Lebesgue measure
· Measurable sets
· Measurable functions
· Approximation by simple measurable functions
· Banach-Steinhaus
· Dirichlet kernel $D_n * f = s_n(f)$
· Lebesgue constant
· Lebesgue integral
· MCT, Fatou, LDCT
· Holder
· Minkowski
· Homogeneous Banach space over $\mathbb{T}$
· Non-convergence of Fourier series in $L_1(\mathbb{T})$ and $C(\mathbb{T})$
· Linear functionals on L_p spaces and $C[a,b]$
· L_p spaces for $p = 1, 2$
· $C[a,b]$ is dense in $L_p[a,b]$
· $C(\mathbb{T})$ is dense in $L_p(\mathbb{T})$
· Failure of point-wise convergence
· Fejer kernel $K_n * f = \sigma_n(f)$
· Uniqueness of Fourier coefficients $c_n(f) = c_(g) \implies f = g$ a.e.
· Principle of localisation
· Riemann-Lebesgue lemma
· Lemma: $\int_{-\pi}^{\pi} \left| \frac{f(t)}{t} \right| < \infty \implies s_n(f, 0) \to 0$
· Abstract summability kernel theorem
· Fejer's theorem
· Dini's theorem (also Lipschitz)
· Inner product/Hilbert spaces
· Cauchy-Schwarz, Pythagoras
· Linear approx-imation lemma
· Orthonormal basis theorem
$\{e^k\}_{k \in \mathbb{Z}}$ is an o.n.b. for $L_2(\mathbb{T})$ (and $C(\mathbb{T})$)
· Hardy's Tau-berian theorem
Convergence of Fourier series for certain piecewise differentiable functions
· Convergence of Fourier series in $L_2(\mathbb{T})$
· Riesz-Fischer
· Plancherel
· $D(\mathbb{T}) \subset C(\mathbb{T})$ (piecewise differentiable, bounded derivative, cts)
· Gibbs' theorem on jump discontinuities
· Uniform Fourier convergence
· Fourier algebra $A(\mathbb{T})$
· Gibbs' theorem for $F(t) = \frac{1}{2} - \frac{t}{2\pi}$