Compact course notes

Pure Math 467/667, Winter 2013

Algebraic Topology

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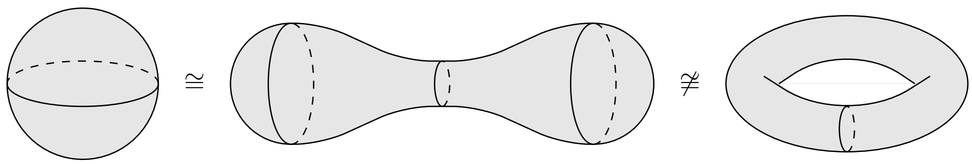
0.0 Motivating remarks

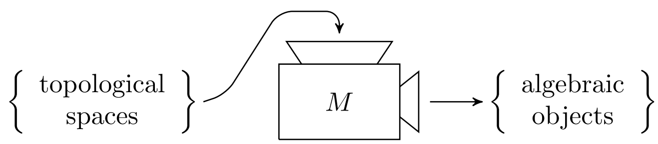
Topology is the study of continuity. It may be equivalently defined as:

1. The study of topological spaces and their properties which are invarint under topological equivalences
A *topological equivalence* may be a homeomorphism, homotopy equivalence, cobordism, etc.

2. Rubber sheet geometry
We may then pose the heuristic question: what properties of a surface are invariant under transformations that involve only stretching and bending, and not cutting, tearing or gluing? For example, we find that



The study of topology in this class will be concerned with the study of machines:

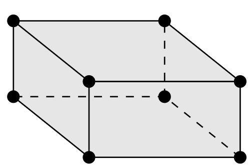


Then it should follow that $X \cong Y \implies M(X) \cong M(Y)$.

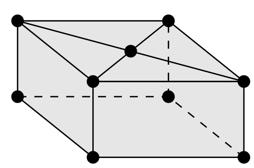
Example 0.0.1. The Euler characteristic is an old example of an algebro-topological invariant. For X a convex polyhedron, it follows that

$$(\# \text{ of faces of } X) - (\# \text{ of edges of } X) + (\# \text{ of vertices of } X) = 2$$

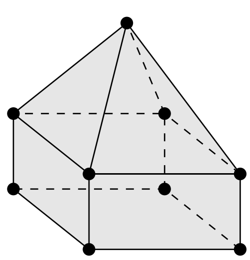
Consider the sphere, with different representations as a polyhedron.



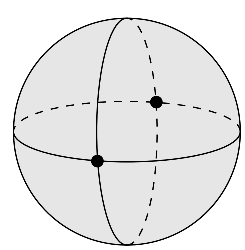
$6 - 12 + 8 = 2$



$9 - 16 + 9 = 2$



$9 - 16 + 9 = 2$



$4 - 4 + 2 = 2$

This course will focus on two specific machines, with a third if time permits:

$$\begin{aligned} \pi_1 &: \{\text{topological spaces}\} \to \{\text{groups}\} \\ H &: \{\text{topological spaces}\} \to \{\text{graded abelian groups}\} \\ QFT_2 &: \{\text{category of cobordisms of 1-dim manifolds}\} \to \{\text{algebras}\} \end{aligned}$$

The first is termed the fundamental group and the second denotes homology. We will study cellular and singular homology.

Example 0.0.2. There are several questions to be asked of the nature of invariants:

· *Does $M(X) \ncong M(Y)$ imply that $X \ncong Y$?*

An old problem asked if $\mathbb{R}^n \cong \mathbb{R}^m$, for $n \neq m$. With homology (and compactification), it is easily shown that $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$.

· *How good, or complete, is a given invariant M?*

Another old problem (with recent resurgent interest) asks if X is an n-dimensional compact manifold without boundary, and there exists a homotopy equivalence $f : X \to \mathbb{S}^n$, then is X homeomorphic to $\mathbb{S}^n$? This is termed the generalized Poincare conjecture, and is true. It has been proved in parts by:

$$\begin{array}{rcl} n = 3 & : & \text{Perelman, Hamilton, Ricci} \\ n = 4 & : & \text{Freedman} \\ n \geqslant 5 & : & \text{Smale} \end{array}$$

1 Point-set topology

1.1 Foundations

Definition 1.1.1. A <u>topological space</u> is a pair (X, τ), where X is a set and τ is a collection of subsets of X such that:

1. $\emptyset, X \in \tau$
2. arbitrary unions of elements of τ are in τ
3. finite intersection of elements of τ are in τ

The elements of τ are termed <u>open sets</u>, and for $U \in \tau$, $U^c = X \setminus U$ is termed a <u>closed set</u>. Note that not closed does not imply open, and not open does not imply closed.

A topology is often presented as generated by some collection of subsets of X, say B, for which every $x \in X$ must be in an element of B, and $\emptyset, X \in B$. Then the topology generated by B is given by closing B under arbitrary unions and finite intersections of elements of B.

Example 1.1.2. These are some examples of topologies.

· <u>The standard topology on $\mathbb{R}$</u>: The topology generated by $\{(a, b) \ : \ a < b \in \mathbb{R}\}$

· <u>The metric topology</u>: For $(M, d(\cdot, \cdot))$ a metric space, let $B_{c,r} = \{m \in M \ : \ d(c, m) < r\}$ be the open ball of radius r centered at c. Then the metric topology is the topology generated by $\{B_{c,r} \ : \ c \in M, r \in \mathbb{R}_{>0}\}$

· <u>The Zariski topology</u>: For A a commutative ring, let $\text{Spec}(A)$ be the set of prime ideals of A, and I an ideal of A. Define $V(I) = \{P \in \text{Spec}(A) \ : \ P \supseteq I\}$, and then the Zariski topology on A is generated by $\{V(I) \ : \ I \text{ is an ideal of } A\}$, which contains all the closed sets in the topology.

Consider further the Zariski topology, and let $A = \mathbb{C}[x]$, the single-variable polynomials over $\mathbb{C}$. As a prime ideal in $\mathbb{C}[x]$ is an ideal generated by irreducible polynomials, we have that

$$\text{Spec}(A) = \{\langle x - \alpha \rangle \ : \ \alpha \in \mathbb{C}\}$$

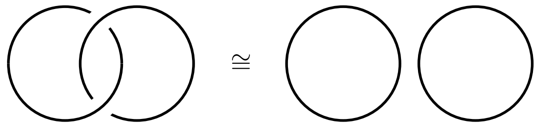
Note that for any $\alpha \in \mathbb{C}$, $V(\langle x - \alpha \rangle) = \{\langle x - \alpha \rangle\}$, and so every point in $\text{Spec}(A)$ is closed. Hence all open sets in the Zariski topology of A are of the form $\mathbb{C} \setminus \{\text{finite number of points}\}$, and therefore no open set separates two points in this topology, and so it is not Hausdorff.

1.2 Morphisms

Definition 1.2.1. Let X, Y be topological spaces, and f a map from X to Y. Then f is termed <u>continuous</u> iff $f^{-1}(U) \subset X$ is open, for $U \subset Y$ open.

A bijective map $f : X \to Y$ with f, f^{-1} continuous is termed a <u>homeomorphism</u>, and we write $X \cong Y$.

Example 1.2.2. Consider a link in $\mathbb{R}^3$, and an unlink in $\mathbb{R}^3$.



The homeomorphism is given by simply mapping one ring of the link to one ring of the unlink, and the other to the other. However the homeomorphism cannot be associated with a deformation in $\mathbb{R}^3$.

There are several common constructions of topological spaces.

· The product topology: let X, Y be topological spaces. Give $X \times Y$ the topology generated by the set $\{U \times V \ : \ U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$

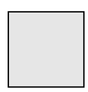
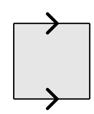
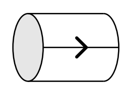
· The quotient topology: Let X be a topological space and Y a set, with $f : X \twoheadrightarrow Y$ surjective. Define a topology on Y by $U \subseteq Y$ open iff $f^{-1}(U) \subseteq X$ is open.

Example 1.2.3. The Hopf fibration gives a way of describing the homeomorphic spaces $S^3/S^1 \cong S^2$.

1.3 The quotient topology

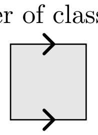
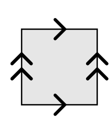
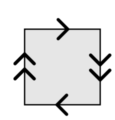
Using the definition for an open set in the quotient topology as given above, we choose an equivalence relation $\sim$ on the elements in X. Then the quotient topology is denoted $Y = X/\sim$.

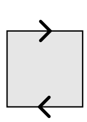
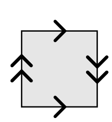
Example 1.3.1. Let X be the unit square on $\mathbb{R}^2$. Define a relation $\sim$ on X be identifying the top and bottom edges of the square in the same direction. Then Y is a cylinder of length 1 and circumfrence 1, with $f : X \to Y$ the transforming map.

X =  $Y = X/\sim$ =  = 

Open sets in X do not always correspond to open sets in Y. For instance, the open ball B_0 of radius .3 in X around the point $(.5, 1)$ is an open set in X, but its image in Y is not. This is because $B((.5, 1), \epsilon) \nsubseteq f(B_0)$ in Y for all $\epsilon > 0$. However, the union of the open balls around $(.5, 1)$ and $(.5, 0)$ in Y, or simply the open ball around $(.5, 1)$ in Y, as they are the same point in Y, is an open set in Y, and so is its preimage in X.

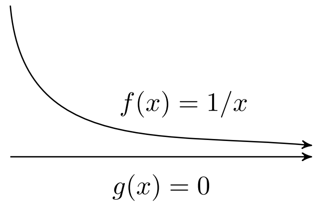
Example 1.3.2. Using the idea presented above, of identifying edges of the square in $\mathbb{R}^2$, we may construct a number of classical surfaces.

 = cylinder  = torus  = projective plane

 = Mobius strip  = Klein bottle

Definition 1.3.3. Let X be a topological space. Then X is termed connected if and only if X may not be expressed as $X = V \mathbin{\dot\cup} W$, for $V, W \subseteq X$ open nonepmty sets. The relation $\dot\cup$ indicates disjoint union by construction, i.e. $V \cap W = \emptyset$.

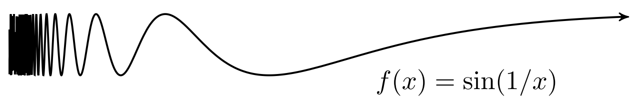
Example 1.3.4. The union of the graph of these two functions, $X = f(\mathbb{R}_{>0}) \cup g(\mathbb{R}_{>0})$, is not connected.



Definition 1.3.5. Let X be a topological space. Then X is termed path-connected if and only if for any two points $x, y \in X$, there exists a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Note that if X is path-connected, then X is connected, but the converse does not necessarily hold.

Example 1.3.6. The space below $X = f(\mathbb{R}_{>0}) \cup \{(0,y) \ : \ y \in [-1,1]\}$ is connected as it is the closure af a continuous function, but not path connected, as for $p(0) = x$ on the vertical strip and $p(1) = y$ somewhere on the graph of f, any path $p(t)$ between the two does not converge as $t \to 0^+$.



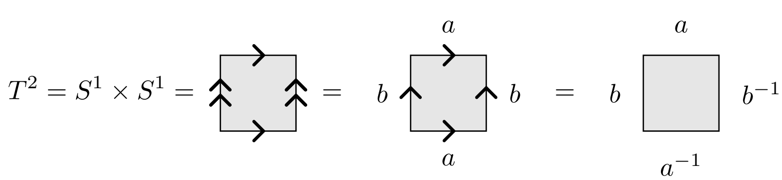
Definition 1.3.7. A space X is termed compact if for any open covering $\{U_\alpha\}_{\alpha \in I}$ of X, there exists a finite subcover.

A space X is termed sequentially compact if any sequence of points in X has a convergent subsequence, using the metric topology of X.

These are in fact equivalent definitions of compactness.

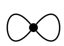
1.4 CW-complexes

Example 1.4.1. We have already seen a cell complex, the torus T^2, given by

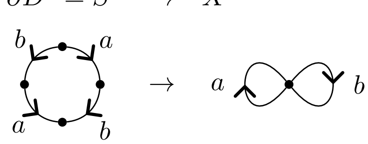


The corners are 0-cells, the sides are 1-cells, and the shaded area is a 2-cell. The construction is done in several steps:

$X^0 = \{*\}$ is a point, the 0-skeleton of the cell complex. Then take two 1-cells and glue them to X^0 by their boundaries, to get X^1.

$X^1 =$  is two loops, the 1-skeleton of the complex. Then take a 2-cell  and glue it to X^1 via

$$\varphi : \partial D^2 = S^1 \quad \to \quad X^1$$



$X^2 = T^2$, the 2-skeleton of $X = D^2 \sqcup X^1 / \big(s \in \partial D^2 \sim \varphi(s) \in X^1\big) = D^2 \sqcup_\varphi X^1$.

Definition 1.4.2. In general, a CW-complex X is built inductively from X^0 upward, following:

· Start with the 0-skeleton X^0, a discrete topology

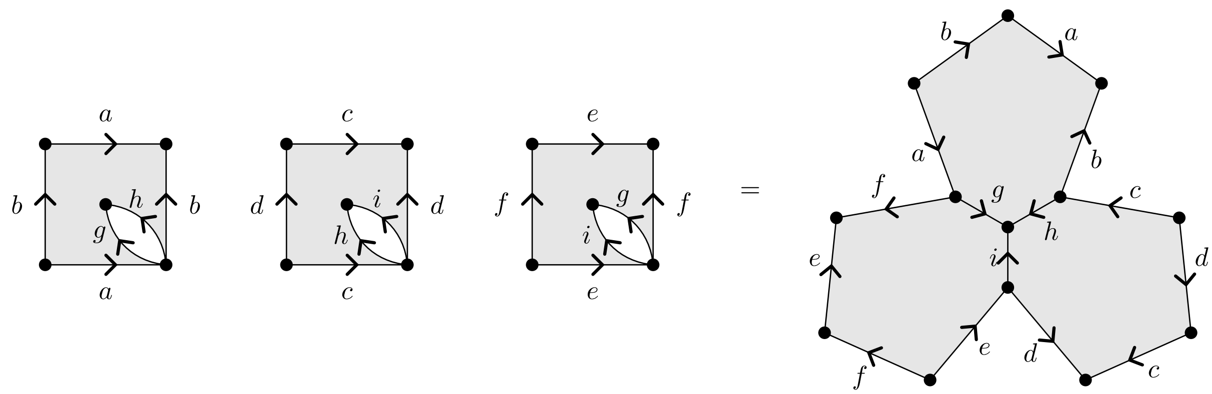
· For every $n > 1$, specify some number of n-cells e^n_α for $\alpha \in I_n$, the indexing set

  · Specify gluing maps $\varphi_\alpha : \partial D^n_\alpha = S^{n-1} \to X^{n-1}$

  · Construct $X^n = \displaystyle\bigsqcup_{\alpha \in I_n} \left(D^n_\alpha \sqcup X^{n-1}\right) \Big/ s \in \partial D^n_\alpha \sim \varphi_\alpha(s) \in X^{n-1}$

Stopping at the desired n, it will be that $X^n = X$, a finite-dimensional complex.

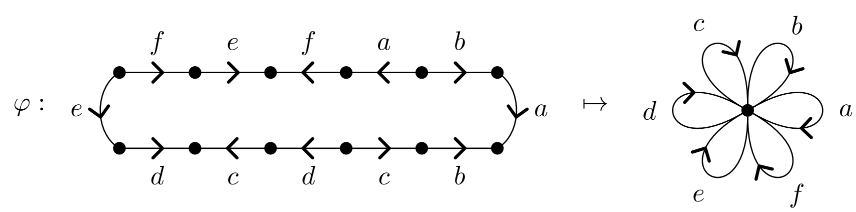
Example 1.4.3. The triple torus may be represented as a CW-complex, by noting that it is essentially three tori glued together. Begin with three separate tori, each with a hole, and identify the commonly labeled edges of the holes.



The border, which will be the 1-skeleton of the complex, simplifies to the wedge product of six circles, as all outside 0-cells are identified, and so:

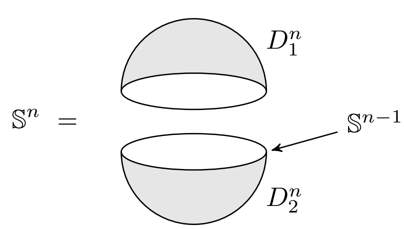
$X^0 = \{*\}$, then take six 1 cells and glue them by their boundaries to X^0

$X^1 =$ the wedge product of six circles. Take a 2-cell and glue it to X^1 via $\varphi : \partial D^2 \to X^1$ given by



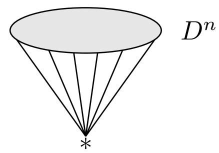
This ordering on ∂D^2 is taken from the 12-sided figure above.

Example 1.4.4. The n-sphere $\mathbb{S}^n$ may be constructed in two different ways. One is to do it inductively, by $X^0 = \{*, *\}$, and for each $n > 1$, $X^n = D^n_1 \sqcup D^n_2 \sqcup \mathbb{S}^{n-1} / \left(\partial D^n_1 \sim \mathbb{S}^{n-1}, \partial D^n_2 \sim \mathbb{S}^{n-1}\right)$.



A simpler way to construt $\mathbb{S}^n$ is to let $X^0 = \{*\}$, and $X^i = \emptyset$ for $1 < i < n$. Then $X = X^n = D^n \sqcup \{*\}/s \in$

$\partial D^n \sim *$. So the boundary of D^n is drawn to the single point $*$ to create $\mathbb{S}^n$.



Definition 1.4.5. Let X be a CW-complex. Then there exists a topological invariant $\chi(X)$, termed the Euler characteristic of X, given by

$$\chi(X) = \sum_{n \text{ even}} (\text{number of } n\text{-dim. cells of } X) - \sum_{n \text{ odd}} (\text{number of } n\text{-dim. cells of } X)$$

Example 1.4.6. Here is the Euler characteristic for some complexes that we have seen.

$$\chi(S^n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \qquad\qquad \chi(T^2) = 0$$

Definition 1.4.7. The real projective space of dimension n, denoted $\mathbb{R}P^n$, may be thought of as the set of lines in $\mathbb{R}^{n+1}$ through the origin, or as $\mathbb{S}^n/x \sim -x$. The same way that the sphere $\mathbb{S}^n$ was defined recursively, so may $\mathbb{R}P^n$ be defined recursively:

$$\mathbb{R}P^n = D^n \sqcup \mathbb{R}P^{n-1}/\varphi \qquad \begin{array}{rrcl} \varphi: & \partial D^n = \mathbb{S}^{n-1} & \to & \mathbb{R}P^{n-1} \\ & x & \mapsto & [x] \end{array}$$

where $[x] = \{x, -x\}$, i.e. all points are identified with their antipodes. Further, we find that

$$\chi(\mathbb{R}P^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Definition 1.4.8. The complex projective space of dimension n, denoted $\mathbb{C}P^n$, may be thought of as the set of complex lines in $\mathbb{C}^{n+1}$ through the origin, or as $\mathbb{S}^{2n+1}/\mathbb{S}^1$. The same way that $\mathbb{S}^n$ and $\mathbb{R}P^n$ were defined recursively, so may $\mathbb{C}P^n$ be defined:

$$\mathbb{C}P^n = D^{2n} \sqcup \mathbb{C}P^{n-1}/\varphi \qquad \begin{array}{rrcl} \varphi: & \partial D^{2n} = \mathbb{S}^{2n-1} & \to & \mathbb{C}P^{n-1} \\ & \lambda x & \mapsto & x \quad \text{for all } \lambda \in U(1) \end{array}$$

The sphere may be seen as embedded in the complex projective space, by

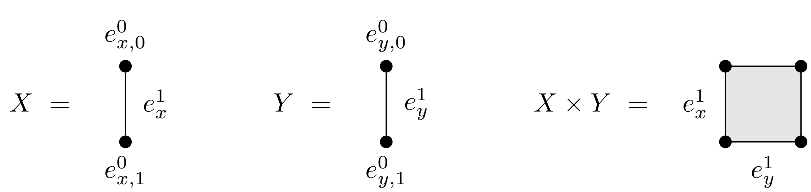
$$\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1} = \{(v_1, \dots, v_{n+1}) \in \mathbb{C}^{n+1} \ : \ \sum_i |v_i|^2 = 1\}$$

Note that $(v_1, \dots, v_{n+1}) = (v, w)$ for $v \in \mathbb{C}^n$ and $w \in \mathbb{C}$, in which case on the sphere $\mathbb{S}^{2n+1}$ we have $|v| = \sqrt{1 - |w|^2}$. Next we may use $U(1)$ to fix the phase of $w \neq 0$, i.e. assume that w is real. Then the set $\{(v, w) \ : \ w \in \mathbb{R}, |v| = \sqrt{1 - |w|^2}\} = D^{2n}$ with boundary where $w = 0$, and hence $\partial D^{2n} = \mathbb{S}^{2n-1}$.

Therefore by gluing ∂D^{2n} to $\mathbb{C}p^{n-1}$, we get $\mathbb{C}P^{n-1} = \mathbb{S}^{2n-1}/U(1)$. This also gives that $\chi(\mathbb{C}P^n) = n + 1$.

1.5 Standard operations on cell complexes

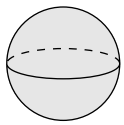
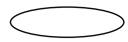
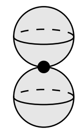
Definition 1.5.1. Product: For X, Y CW-complexes, the product $X \times Y$ is a CW-complex with cells that are the cartesian products of the cells of X and Y. For example,



The cells of $X \times Y$ are given by:

$$\begin{aligned} \text{0-cells}: &\quad e^0_{x,0} \times e^0_{y,0},\ e^0_{x,0} \times e^0_{y,1},\ e^0_{x,1} \times e^0_{y,0},\ e^0_{x,1} \times e^0_{y,1} \\ \text{1-cells}: &\quad e^1_{x,0} \times e^1_{y},\ e^1_{x,1} \times e^1_{y},\ e^1_{y,0} \times e^1_{x},\ e^1_{y,1} \times e^1_{x} \\ \text{2-cells}: &\quad e^1_x \times e^1_y \end{aligned}$$

Definition 1.5.2. Quotient: For X a CW-complex and A a subcomplex of X, the quotient X/A consists of the cells of $X \setminus A$ along with an extra 0-cell. For example, we may quotient by the equator of a sphere.

$X =$  $A =$  $X/A =$ 

Let X, Y be topological spaces with $x_0 \in X$ and $y_0 \in Y$.

Definition 1.5.3. Wedge sum:

$$(X, x_0) \vee (Y, y_0) = X \sqcup Y / x_0 \sim y_0$$

Definition 1.5.4. Suspension:

$$SX = X \times [0,1] \Big/ \substack{\{x\}\times\{0\}\sim\{x'\}\times\{0\} \\ \{x\}\times\{1\}\sim\{x'\}\times\{1\}}$$

For example, note that $S\mathbb{S}^n = \mathbb{S}^{n+1}$.

Definition 1.5.5. Smash product: Let X be based at x_0 and Y at y_0.

$$X \wedge Y = X \times Y / X \vee Y = X \times Y / \{X \times \{y_0\} \cup \{x_0\} \times Y\}$$

For example, note that $\mathbb{S}^n \wedge \mathbb{S}^m = \mathbb{S}^{n+m}$.

2 Homotopy theory

2.1 Retracts and deformations

Definition 2.1.1. Let X, Y be topological spaces with $f, g : X \to Y$ continuous maps. A continuous map $H : [0,1] \times X \to Y$ such that

$$\begin{aligned} H(\cdot, 0) : X \to Y = f(\cdot) \\ H(\cdot, 1) : X \to Y = g(\cdot) \end{aligned}$$

is termed a homotopy between f and g. This relationship is denoted $f \sim g$ or $f \overset{H}{\sim} g$. Note that homotopy is an equivalence relation.

Definition 2.1.2. Let X, Y be topological spaces. A map $f : X \to Y$ is termed a homotopy equivalence iff there exists a map $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Note that homotopy equivalence is an equivalence relation.

We will be interested in classifying spaces up to homotopy equivalence.

Definition 2.1.3. Let X be a topological space. A continuous map $f : X \to X$, which for some $A \subseteq X$ has $f|_A = \mathrm{id}_A$, is termed a retraction onto A.

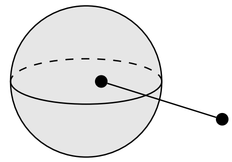
For X, A, f as above, with the addition that f is homotopy equivalent to id_X, the map f is termed a deformation retraction. Further, a deformation retraction onto A constitutes a homotopy equivalence between A and X:

Let $\iota : A \hookrightarrow X$ be the inclusion map. Then

$$\begin{aligned} f \circ \iota &= \mathrm{id}_A \\ \iota \circ f &= \mathrm{id}_X \end{aligned}$$

Example 2.1.4. Let $A = \mathbb{S}^2$ and the space X be defined as

$$X \;=\; (\mathbb{S}^2, *) \vee (I, \{0\}) \;=$$



Then the map $f : X \to X$ given by

$$\begin{array}{rcl} s \in \mathbb{S}^2 & \mapsto & s \in \mathbb{S}^2 \\ t \in I & \mapsto & \{*\} \end{array}$$

is a deform retraction onto A. Further, the homotopy map between f and id_X is given by $H : X \times [0,1] \to X$, for which

$$H_t : \begin{array}{rcl} s \in \mathbb{S}^2 & \mapsto & s \in \mathbb{S}^2 \\ r \in I & \mapsto & tr \end{array} \qquad \begin{array}{rcl} f = H_0 : X \times \{0\} & \to & X \\ \mathrm{id}_X = H_1 : X \times \{1\} & \to & X \end{array}$$

Hence the space X and A are homotopy equivalent.

Example 2.1.5. Not all retractions are deformation retractions. For example, the space $X = \bullet \;\; \bullet \;\; \cdots \;\; \bullet$ that consists of n distinct points may be retracted, but not deformation retracted, completely onto one of those points $\bullet$.

Example 2.1.6. Considering the shapes of the latin alphabet (without serifs) as one-dimensional, we may classify them by homeomorphism and homotopy type.

$$\begin{aligned} [A] &= \{A, R\} \\ [B] &= \{B\} \\ [C] &= \{C, G, L, M, N, S, U, V, W, Z\} \\ [D] &= \{D, O\} \\ [E] &= \{E, F, T, Y\} \\ [H] &= \{H, K\} \\ [I] &= \{I, J\} \\ [X] &= \{X\} \\ [P] &= \{P\} \\ [Q] &= \{Q\} \end{aligned}$$

$$\begin{aligned} [A] &= \{A, D, O, P, Q, R\} \\ [B] &= \{B\} \\ [C] &= \{C, E, F, G, H, I, J, K, L, M, N, S, T, U \\ &\qquad V, W, X, Y, Z\} \end{aligned}$$

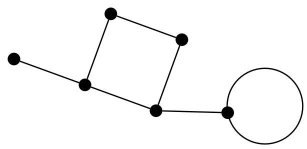
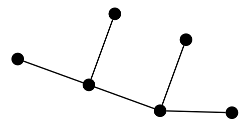
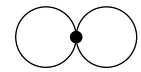
Example 2.1.7. For $GL_+(n, \mathbb{R})$ the set of $n \times n$ positive invertible matrices over $\mathbb{R}$, by scaling the determinant to 1,

$$GL_+(n, \mathbb{R}) \xrightarrow{\text{deformation retraction}} SL(n, \mathbb{R})$$

Definition 2.1.8. Let X, Y be topological space. Then X is termed <u>contractible</u> iff $X \approx \{*\}$. A map $f : X \to Y$ is termed <u>null homotopic</u> iff $f \approx$ constant map.

Proposition 2.1.9. Let (X, A) be a CW-pair with A contractible. Then $q : X \to X/A$ is a homotopy equivalence.

Example 2.1.10. Any connected graph is homotopy equivalent to a wedge sum of circles. For example,

$X =$  $A =$  $X/A =$  $= \mathbb{S}^1 \vee \mathbb{S}^1$

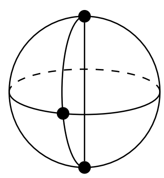
As here, a maximal spanning tree A works to show the previously made claim. Moreover, this observation demonstrates that any CW-complex is homotopic to a CW-complex with one 0-cell.

Proposition 2.1.11. Let (X, A) be a CW-pair and Y another CW-complex. Given $f : A \to Y$, we may glue X and Y along A via f. That is,

$$X \sqcup_f Y = X \sqcup Y / a \in A \sim f(a) \in Y$$

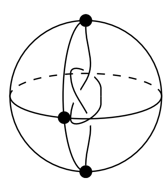
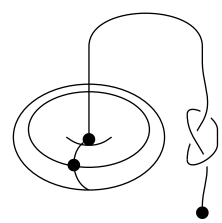
Further, if $g : A \to Y$ is another map with $f \approx g$, then $X \sqcup_f Y \approx X \sqcup_g Y$.

Example 2.1.12. Consider the folllowing application of the previous proposition. Let X be $\mathbb{S}^2$ with a diameter A dropped down. So X has three 0-cells, four 1-cells, and two 2-cells. Let Y be a knotted line.

$X =$  $Y =$ 

$f : A \to Y$ by id
$g : A \to Y$ by $x \mapsto *$ of Y

Then gluing X to Y by way of f and g results in

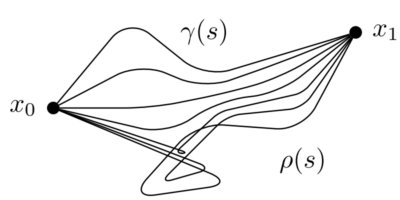
$X \sqcup_f Y =$  $X \sqcup_g Y =$ 

Note that the sphere with a dropped diameter becomes a pinched torus in $X \sqcup_g Y$. The previous proposition tells us that these two CW-complexes are homotopic to each other.

2.2 Path homotopy

Definition 2.2.1. Given a topological space X, a path γ in X is a continuous map $\gamma : I \to X$. The endpoints of γ are $\gamma(0)$ and $\gamma(1)$.

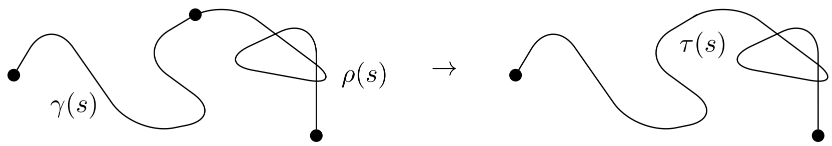
Given paths γ, ρ in X with $\gamma(0) = \rho(0) = x_0 \in X$ and $\gamma(1) = \rho(1) = x_1 \in X$, a fixed endpoint homotopy of paths between γ and ρ in X is a map $H : I \times I \to X$ such that $H(s, 0) = \gamma(s)$ and $H(s, 1) = \rho(s)$.



Note that $H(t, 0) = x_0$ and $H(t, 1) = x_1$ for all $t \in I$. Here we write $\gamma \approx \rho$ via H, which is an equivalence relation on paths with given enpoints x_0, x_1. This fact may be checked in a straighforward manner:

· *Refelexivity*: $\gamma(s) \approx \gamma(s)$ via $H = \text{id}$
· *Syymetry*: $\gamma(s) \approx \rho(s)$ implies $\rho(s) \approx \gamma(s)$ via $H(s, 1-t)$
· *Transitivity*: $\gamma(s) \approx \rho(s)$ and $\rho(s) \approx \tau(s)$ implies $\gamma(s) \approx \tau(s)$ by $H'(s,t) = \left\{\begin{smallmatrix} H_1(s,2t) & t\in[0,1/2] \\ H_2(s,2t-1) & t\in[1/2,1] \end{smallmatrix}\right.$

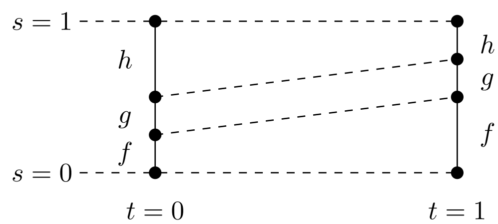
On such paths we have a kind of multiplication, if the paths are composable, that is if $\gamma(1) = \rho(0)$. If this holds, then we may define $\gamma(s)\cdot\rho(s) = \tau(s) = \left\{\begin{smallmatrix} \gamma(2s) & s\in[0,1/2] \\ \rho(2s-1) & s\in[1/2,1] \end{smallmatrix}\right.$ For example,



This multiplication is not associative, but is rather homotopy associative. Consider a triple af paths f, g, h, wtih $f(1) = g(0)$ and $g(1) = h(0)$. Then $(f\cdot g)\cdot h \neq f\cdot(g\cdot h)$, but $(f\cdot g)\cdot h \approx f\cdot(g\cdot h)$ via H.

$$(f\cdot g)\cdot h = \begin{cases} f(4s) & s\in[0,1/4] \\ g(4s-1) & s\in[1/4,1/2] \\ h(2s-1) & s\in[1/2,1] \end{cases} \qquad f\cdot(g\cdot h) = \begin{cases} f(2s) & s\in[0,1/2] \\ g(4s-2) & s\in[1/2,3/4] \\ h(4s-3) & s\in[3/4,1] \end{cases}$$

$H : I\times I \to X$



Definition 2.2.2. Given a topological space $X \ni x_0$, let $\pi_1(X, x_0)$ be the set of homotopy equivalence classes of paths which start and end at x_0 (i.e. loops based at x_0).

Theorem 2.2.3. The set $\pi_1(X, x_0)$ is a group, with the product $[f][g] = [fg]$.

Proof: The binary operation on this group, composition, is clear. Above we showed that the multipliciation is associative, i.e. $([f][g])[h] \approx [f]([g][h])$.

Now suppose that $[f] = [f']$. Let us check that $[fg] = [f'g]$. Now $f \approx f'$, so there exists a homotopy $H : I\times I \to X$ such that $H_0 = f$ and $H_1 = f'$. Multiplying both these maps by g, we have

$$fg = \begin{cases} f(2s) & s\in[0,1/2] \\ g(2s-1) & s\in[1/2,1] \end{cases} \qquad f'g = \begin{cases} f'(2s) & s\in[0,1/2] \\ g(2s-1) & s\in[1/2,1] \end{cases}$$

Define a homotopy $G : I\times I \to X$ given by

$$G(s,t) = \begin{cases} H(2s,t) & s\in[0,1/2] \\ \text{id}(2s-1,t) & s\in[1/2,1] \end{cases}$$

Then G shows that $fg \approx f'g$, meaning that $[fg] = [f'g]$.

The identity of this group exists as $[c_{x_0}]$, the equivalence class of all paths homotopic to the constant path at x_0. It is clear that $[fc_{x_0}] = [c_{x_0}f]$.

For inverses, if $f(s)$ is a loop at x_0, then $\bar{f}(s) = f(1-s)$ is its inverse. It is easily checked that $[f][\bar{f}] = [c_{x_0}]$, through the homotopy H.

$$H(s,t) = \begin{cases} f(2st) & s\in[0,1/2] \\ \bar{f}((2s-1)t) & s\in[1/2,1] \end{cases}$$

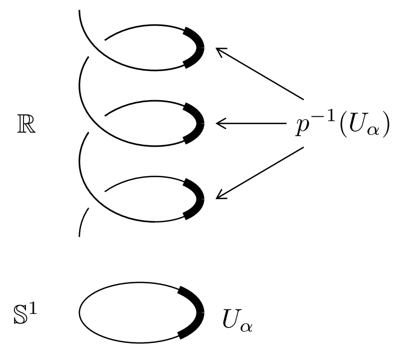
This shows that $\pi_1(X, x_0)$ is indeed a group. ■

Remark 2.2.4. If X is path connected, then $\pi_1(X, x_0) = \pi_1(X, x_1)$ for any two basepoints $x_0, x_1 \in X$. This is made clear by defining a path γ on X with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, for which $\gamma g \bar{\gamma}$ is a loop at x_1, if g is a loop at x_0.

2.3 The fundamental group of $\mathbb{S}^1$

Proposition 2.3.1. $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

Proof: Consider the map $p : \mathbb{R} \to \mathbb{S}^1$ given by $r \mapsto (\sin(2\pi r), \cos(2\pi r))$. Observe that it is possible to cover $\mathbb{S}^1$ by open sets $\{U_\alpha\}$ such that $p^{-1}(U_\alpha)$ is o disjoint union of open sets in $\mathbb{R}$, each of which maps homeomorphically onto U_α.



Let Φ be our potential isomorphism, given by

$$\begin{array}{rccc} \Phi : & \mathbb{Z} & \to & \pi_1(\mathbb{S}^1, (0,1)) \\ & r & \mapsto & [p \circ \tilde{f}] \end{array}$$

where $\tilde{f}$ is any path in $\mathbb{R}$ from 0 to r. Now we check that this defines a homomorphism. Clearly $0 \mapsto [c_{(0,1)}]$. Next observe that

$$\Phi(r+s) = [p \circ \tilde{f}_{r+s}]$$
$$\Phi(r)\Phi(s) = [p \circ \tilde{f}_r][p \circ \tilde{f}_s] = [p \circ (\tilde{f}_r \cdot \tilde{f}_s)] = [p \circ \tilde{f}_{r+s}]$$

Hence Φ is the desired isomorphism, and $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. ■

Lemma 2.3.2. Let Y be a topological space, and $F : Y \times I \to \mathbb{S}^1$ a continuous map. A lifting $\tilde{F} : Y \times \{0\} \to \mathbb{R}$ of F may be extended uniquely to a lift $\tilde{F} : Y \times I \to \mathbb{R}$.

Proof: Let $N \subseteq Y$ be an open neighborhood of $y_0 \in Y$. Suppose that $\{U_\alpha\}_{\alpha \in J}$ is a cover of $\mathbb{S}^1$. Then each $(y_0, t) \in Y \times I$ has an assigned product neighborhood $N_t \times (a_t, b_t)$, such that $F(N \times (a_t, b_t)) \subseteq U_\alpha$ for some $\alpha \in J$.

By the compactness of $y_0 \times I$, a finite number of $N_t \times (a_t, b_t)$ cover $y_0 \times I$. Label those indeces $0 < t_0 < t_1 < \cdots < t_m = 1$, such that $\{N_{t_i} \times (a_{t_i}, b_{t_i}) \ : \ i = 1, \ldots m\}$ covers $y_0 \times I$. Now redefine $N = \bigcap_{i=0}^m N_i$, so for each i, $F(N \times [t_i, t_{i+1}]) \subseteq U_\alpha$, for some $\alpha \in J$. For each $i = 0, \ldots, m-1$, let U_i be the U_α for which $F(N \times [t_i, t_{i+1}]) \subseteq U_\alpha$.

Assume inductively that $\tilde{F}$ has been constructed up to $N \times [0, t_i]$. Then as $F(N \times [t_i, t_{i+1}]) \subseteq U_i$, there exists $\tilde{U}_i \subseteq \mathbb{R}$ such that $p : \tilde{U}_i \to U_i$ is a homomorphism, and $\tilde{F}(\{y_0\} \times \{t_i\}) \in \tilde{U}_i$. Now reduce the size of N so that $\tilde{F}(N \times \{t_{i+1}\})$ is completely in $\tilde{U}_i$. Then on $N \times [t_i, t_{i+1}]$ define $\tilde{F} = p^{-1} \circ F$.

After a finite number of steps, we get $N \ni y_0$ and a lift $\tilde{F}$ such that the diagram below commutes.

$$\begin{tikzcd} & \mathbb{R} \arrow[dr, "p"] & \\ N \times I \arrow[ur, "\tilde{F}"] \arrow[rr, "F"'] & & \mathbb{S}^1 \end{tikzcd}$$

■

Proposition 2.3.3. The map $\Phi : \mathbb{Z} \to \pi_1(\mathbb{S}^1)$ that maps n to the path from 0 to n in $\mathbb{R}$ is an isomorphism.

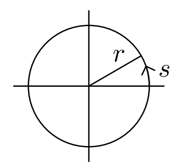
Proof: Left as an exercise. ■

The fact that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ has several important consequences, with the fundamental theorem of algebra and the Poincare-Hopf Index theorem.

Corollary 2.3.4. [Fundamental theorem of algebra]
If $f(z) \in \mathbb{C}[z]$, then $f(z)$ has a root in $\mathbb{C}$.

Proof: Suppose that $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ has no roots in $\mathbb{C}$, so normalize it by setting $f(z) \leftarrow \frac{f(z)}{|f(z)|}$, and consider the following function along concentric circles on $\mathbb{C}$.

$$f_s(r) = \frac{f(re^{2\pi i s})}{|f(re^{2\pi i s})|}$$



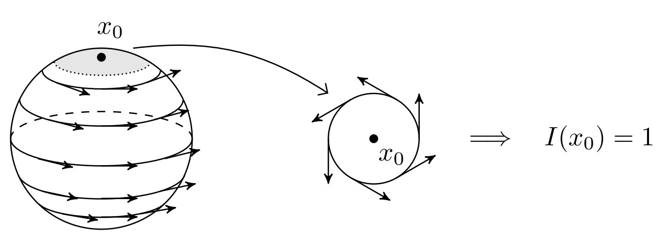
Note that $f_r : \mathbb{S}^1 \to \mathbb{S}^1$, and that $f_0(s)$ is the constant loop on $\mathbb{S}^1$. Varying r gives a homotopy of loops in $\mathbb{S}^1$, so define $R > |a_1| + \cdots + |a_n|$ and $R > 1$. Then on the circle where $|z| = R$, we have

$$|z^n| = R^n = R \cdot R^{n-1} > (|a_1| + \cdots + |a_n|)|z^{n-1}| \geqslant |a_1 z^{n-1} + \cdots + a_n|$$

Define a new function $g_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$, so by our assumption, $g_t(z)$ has no roots on $|z| = R$, for $0 \leqslant t \leqslant 1$. Going from $t = 1$ to $t = 0$ gives a homotopy between $f_r(s)$ and $\tilde{f}_r(s) = p_t(re^{2\pi i s})/|p_t(re^{2\pi i s})|$. This function, $\tilde{f}_r(s) = re^{2\pi i s n}/|re^{2\pi i s n}|$, when $t = 0$, is associated to the polynomial z^n. But when $t = 0$, $\tilde{f}_r(s) > n \in \pi_1(\mathbb{S}^1)$, and hence $[0] = [n]$. This contradicts the above. ■

Definition 2.3.5. Let X be a surface, i.e. a 2-dimensional differentiable manifold with no boundary. Let V be a vector field on X which vanishes only at discrete points. For x_0 a zero of the vector field and $N \subset X$ an open neighborhood containing x_0 and no other zeros, we may view the vector field on N as a vector field on $\mathbb{R}^2$. Then define the winding number $I(x_0)$ of x_0 to be the number of counter-clockwise cycles of the vector field along a loop in N around x_0.

Example 2.3.6. Consider a vector field on the sphere $\mathbb{S}^2$, and x_0 as the north pole.



Theorem 2.3.7. [Poincare-Hopf index theorem]
Let X be a surface, V a vector field on X, and $S \subset X$ the set of zeros of X. Then

$$\sum_{x \in S} I(x) = \chi(X)$$

2.4 The functor π_1

Definition 2.4.1. Let $f : (X, x_0) \to (Y, y_0)$ be a based map of topological spaces (i.e. $f(x_0) = y_0$). Then there exists an induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, with $[\gamma] \mapsto [f \circ \gamma]$. Moreover, this is a group homomorphism, and may be viewed functorially as coming from the application of π_1.

$$\begin{tikzcd}
(X, x_0) \arrow[r, "f"] \arrow[d, squiggly, "\pi_1"] & (Y, y_0) \\
\pi_1(X, x_0) \arrow[r, "f_*"] & \pi_1(Y, y_0)
\end{tikzcd}$$

it remains to check that f_* is well-defined. Suppose that $H : I \times I \to X$ is a based homotopy between loops γ, γ' in X. Then

$$[f \circ \gamma] = [(f \circ H) \circ \gamma] = [f \circ (h \circ \gamma)] = [f \circ \gamma']$$

This follows as $f \circ H$ is a homotopy between $f \circ \gamma$ and $f \circ \gamma'$. Now check that f_* is indeed a group homomorphism. First note that $f_*[c_{x_0}] = [f \circ c_{x_0}] = [c_{y_0}]$, and next

$$f_*([\gamma][\gamma']) = f_*([\gamma\gamma']) = [f_* \circ \gamma][f_* \circ \gamma'] = f_*[\gamma]f_*[\gamma']$$

This is the desired result. Composition works as follows:

$$\gamma\gamma' = \begin{cases} \gamma(2s) & s \in [0, 1/2] \\ \gamma(2s-1) & s \in [1/2, 1] \end{cases}$$

Definition 2.4.2. Let (X, x_0) be a based space, with $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$. Then π_1 is termed a <u>functor</u> if and only if the following conditions hold:

1. $\mathrm{id}_* : \pi_(X, x_0) \to \pi_1(X, x_0)$ is the same as $\mathrm{id} : \pi_1(X, x_0) \to \pi_1(X, x_0)$
2. $g_* \circ f_* = (g \circ f)_*$

Lemma 2.4.3. There are no retractions $r : D^2 \to \partial D^2$.

<u>Proof:</u> Suppose that such a retraction exists. Let $\iota : \partial D^2 \to D^2$ be the inclusion map. Pick $x_0 \in \partial D^2$ as a basepoint. Consider the folowing sequence of maps:

$$\begin{tikzcd}
\partial D^2 \arrow[r, hook, "\iota"] \arrow[rr, bend right, "r \circ \iota"'] & D^2 \arrow[r, "r"] & \partial D^2
\end{tikzcd}$$

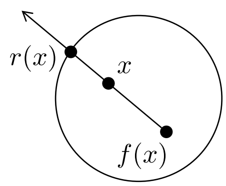
Then by assumption, $\iota \circ r = \mathrm{id}|_{\partial D^2}$. Now apply the functor π_1 to the above diagram to get the following diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, equal] & 0 \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\
\pi_1(\partial D^2) \arrow[r, hook, "\iota_*"] \arrow[rr, bend right, "r_* \circ \iota_*"'] & \pi_1(D^2) \arrow[r, "r_*"] & \pi_1(\partial D^2)
\end{tikzcd}$$

From the definition of π_1, we know that $r_* \circ \iota_* = (r \circ \iota)_*$. However, the only maps $\mathbb{Z} \to 0 \to \mathbb{Z}$ are the zero maps, so this composition is the zero map. From previously, $r \circ \iota = \mathrm{id}|_{\partial D^2}$, so $(r \circ \mathrm{id})_* = \mathrm{id} : \mathbb{Z} \to \mathbb{Z}$. This contradicts the fact the composition was the zero map. ■

Theorem 2.4.4. [Fixed point theorem, Brouwer]
Let $f : D^2 \to D^2$ be a continuous map. The there exists $x \in D^2$ with $f(x) = x$.

<u>*Proof:*</u> Suppose that there exists $f : D^2 \to D^2$ with $f(x) \neq x$ for all $x \in D^2$, i.e. no fixed points. Then we may define a retraction $r : D^2 \to \partial D^2$ by letting $r(x)$ be the intersection point of the ray from $f(x)$ through x with ∂D^2. This is a well-defined (as no $f(x) = x$) and continuous map.



Note that for $x \in \partial D^2$, $r(x) = x$, so the map is surjective. Note if we would have chose the arc from x to $f(x)$, then surjectivity need not necessarily hold. This yields a retraction $r : D^2 \to \partial D^2$, contradicting the previous lemma. ∎

Proposition 2.4.5. Let X be a topological space with a basepoint $x_0 \in A \subseteq X$. Let $\iota : A \to X$ be the standard inclusion map. If $r : X \to A$ is a retraction, then $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective. Furthermore, if r is a deformation retraction, then ι_* is an isomorphism.

<u>*Proof:*</u> Suppose that $r : X \to A$ is a retraction. Consider $(\iota \circ r)_* = \iota_* \circ r_*$, with

$$\pi_1(A, x_0) \xrightarrow{\iota_*} \pi_1(X, x_0) \xrightarrow{r_*} \pi_1(A, x_0)$$

We know that $(\iota \circ r) = \mathrm{id}_A$, meaning that $(\iota \circ r)_* = \mathrm{id}_{\pi_1(A,x_0)}$. Since $(\iota \circ r)_*$ is an isomorphism, ι_* is injective.

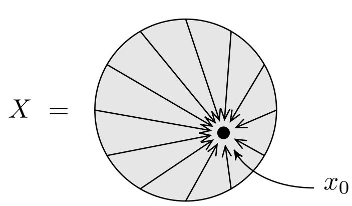
Suppose further that r is a deformation retraction, i.e. there exists $H : I \times X \to X$ sugh that $H_0 : X \to X = r$ and $H_1 : X \to X = \mathrm{id}_X$. Consider any loop $\gamma(s)$ at $x_0 \in X$, and take the family of loops $H_t \circ \gamma(s)$ as a homotopy of loops:

At $t = 0$, $H_0 \circ \gamma(s)$ is a loop in A based at x_0
At $t = 1$, $H_1 \circ \gamma(s)$ is a loop in X based at x_0, more specifically, is $\gamma(s)$

So we have that $[\gamma] = [H_0 \circ \gamma]$ in $\pi_1(X, x_0)$, but $H_0 \circ \gamma$ being a loop in A is the image of ι_*. Therefore the map ι_* is surjective, and hence is an isomorphism. ∎

Example 2.4.6. Let X be contractible, i.e. $X \approx \{x_0 \in X\}$, with $r : X \to \{x_0\}$ a deformation retraction. Then $\pi_1(X, x_0) = 0$.

Example 2.4.7. Let $X = D^2 \ni x_0$. Then there exists a deformation retraction from X onto x_0.



Proposition 2.4.8. $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$

<u>*Proof:*</u> Any map $f : \mathbb{Z} \to X \times Y$ is continuous if and only if its component maps $f = (f_1, f_2)$ are individually continuous. So, given a loop at $(x_0, y_0) \in X \times Y$, we have that $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, and γ_1, γ_2 are loops at x_0, y_0, respectively. Likewise, any homotopy $H : I \times I \to X \times Y$ of loops is a pair of homotopies between the coordinate loops. So $[\gamma] \to [\gamma_1] \times [\gamma_2]$ is an isomorphism. ∎

Example 2.4.9. The implications of the above theorem are immediate:

$$\pi_1(T^2 = \mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$$
$$\pi_1(D^2 \times \mathbb{S}^1) = 0 \times \mathbb{Z} = \mathbb{Z}$$

2.5 The van Kampen theorem

Definition 2.5.1. Let $\{G_\alpha\}_{\alpha \in J}$ be a collection of groups. Then $\mathop{\ast}\limits_{\alpha \in J} G_\alpha = G_\alpha * G_\beta * G_\gamma * \cdots$ is termed the free product of groups G_α, and is itself a group. It conforms to the following rules:

1. $g \in \mathop{\ast}\limits_{\alpha \in J} G_\alpha$ is of the form $g = g_1 g_2 \cdots g_k$ for $g_i \in G_{\alpha_i}$, with $g_i \neq e$ and g_i, g_{i+1} not in the same group

2. the empty word is in $\mathop{\ast}\limits_{\alpha \in J} G_\alpha$ and acts as the identity

3. multiplication is concatenation of words

Note that the free product of groups is the coproduct on the category of groups.

Example 2.5.2. $\mathbb{Z} * \mathbb{Z} = F_2$, and in general $\mathbb{Z}^{*k} = F_k$, the free group on k elements.

Remark 2.5.3. Note that any G_α has a canonical inclusion ι_α into $\mathop{\ast}\limits_{\alpha \in J} G_\alpha$, given by $g \in G_\alpha \mapsto g \in \mathop{\ast}\limits_{\alpha \in J} G_\alpha$. Moreover, given any family of homomorphisms $\{\varphi_\alpha : G_\alpha \to H\}$, there exists a unique map $\varphi : \mathop{\ast}\limits_{\alpha \in J} G_\alpha \to H$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathop{\ast}\limits_{\alpha \in J} G_\alpha \arrow[rr, "\varphi"] & & H \\
& G_\alpha \arrow[ul, hook, "\iota_\alpha"'] \arrow[ur, "\varphi_\alpha"'] &
\end{tikzcd}$$

It follows that $\varphi(g_1 g_2 \cdots g_k) = \varphi_{\alpha_1}(g_1)\varphi_{\alpha_2}(g_2) \cdots \varphi_{\alpha_k}(g_k)$ for $g_i \in G_{\alpha_i}$.

Theorem 2.5.4. [van Kampen]
Let a topological space X be defined as $X = \bigcup_{\alpha \in J} A_\alpha$, for $\{A_\alpha\}_{\alpha \in J}$ a collection of open, path-connected sets in X. Let $x_0 \in A_\alpha$ for all $\alpha \in J$, with the added conditions that:

1. For all $\alpha, \beta \in J$, $A_\alpha \cap A_\beta$ is path-connected

2. For all $\alpha, \beta, \gamma \in J$, $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected

If **1.** holds, then $\pi_1(X, x_0) = \mathop{\ast}\limits_{\alpha \in J} \pi_1(A_\alpha, x_0) / \ker(\varphi)$, for $\varphi : \mathop{\ast}\limits_{\alpha \in J} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ the induced map. If condition **2.** also holds, then $\ker(\varphi)$ is generated by elements of the form $\iota_{\alpha\beta *}(w)\iota^{-1}_{\beta\alpha *}(w)$ for all $\alpha, \beta \in J$.

Note that we have the inclusion maps

$$\begin{array}{rrcl}
\iota_\alpha : & A_\alpha & \hookrightarrow & X \\
\iota_{\alpha\beta} : & A_\alpha \cap A_\beta & \hookrightarrow & A_\alpha
\end{array}$$

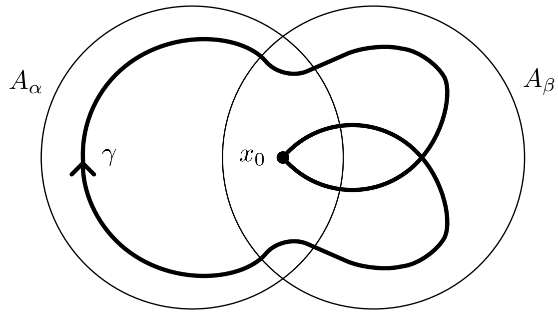
Apply the functor π_1 to find that $\iota_{\alpha *} : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$, meaning that $\{\iota_{\alpha *} : \alpha \in J\}$ induces

$$\varphi : \mathop{\ast}\limits_{\alpha \in J} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$$

Proof: We want to show that φ is surjective, namely that for any $\gamma(s) \in \pi_1(X, x_0)$, there exist paths

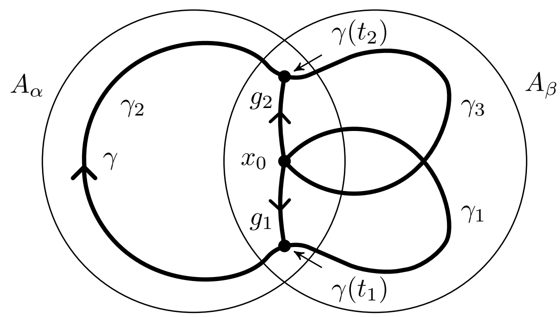
$\gamma_1, \gamma_2, \ldots, \gamma_k$ with $\gamma_i \in A_{\alpha_i}$ such that $[\gamma] = [\gamma_1][\gamma_2]\cdots[\gamma_k]$. We prove this part by pictures.

$X \;=\; A_\alpha \cup A_\beta \;=$



Suppose that $\gamma(s)$ running over I is a loop in both A_α and A_β as above. It is possible to partition I into $0 = t_0 < t_1 < t_2 < t_3 = 1$ such that $\gamma([t_i, t_{i+1}])$ lies completely in either A_α or A_β. Connect each partition point $\gamma(t_i)$ to x_0 by a path g_i. Label the components of the complete path γ.

$X \;=$



Then $\gamma = \gamma_1\gamma_2\gamma_3 = \underbrace{\gamma_1\overline{g_1}}_{\in A_\beta}\underbrace{g_1\gamma_2\overline{g_2}}_{\in A_\alpha}\underbrace{g_2\gamma_3}_{\in A_\beta}$, and so $\gamma \in \mathrm{Im}(\varphi)$. Hence φ is surjective, and the first claim is satisfied.

Now assume that $A_\alpha \cap A_\beta \cap A_\delta$ is path-connected for any $\alpha, \beta, \delta \in J$. By generalizing the above, any element $\gamma \in \pi_1(X, x_0)$ is in $\mathrm{Im}(\varphi)$, so there exist $\gamma_1, \gamma_2, \ldots, \gamma_k$ with $[\gamma] = [\gamma_1][\gamma_2]\cdots[\gamma_k]$ such that each γ_i is a loop completely in some A_{α_i}. There are two points to note:

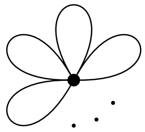
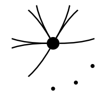
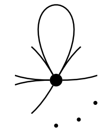
· Suppose that γ_1, γ_{i+1} are in the same A_α. Then reducing the factorization $[\gamma_i][\gamma_{i+1}] \to [\gamma_i\gamma_{i+1}]$ does not change the element of $\mathop{\ast}\limits_{\alpha \in J} \pi_1(A_\alpha, x_0)$ represented by $\gamma_1\gamma_2\cdots\gamma_k$.

· Suppose that $\gamma_i \subseteq A_\alpha \cap A_\beta$. Then γ_i represents a loop in either A_α or A_β. Then changing,for example, $[\gamma_i] \in \pi_1(A_\alpha, x_0)$ to $[\gamma_i] \in \pi_1(A_\beta, x_0)$ does change the element of $\mathop{\ast}\limits_{\alpha \in J} \pi_1(A_\alpha, x_0)$ represented by $\gamma_1\gamma_2\cdots\gamma_k$. As a result, we want to say that for $\iota_{\alpha\beta*} : \pi_1(A_\alpha \cap A_\beta, x_0) \hookrightarrow \pi_1(A_\alpha, x_0)$ has $\iota_{\alpha\beta*}(\gamma_i) = \iota_{\beta\alpha*}(\gamma_i)$, implying that $\iota_{\alpha\beta_*}(\gamma_i)\iota_{\beta\alpha*}^{-1}(\gamma_i) = 0$, which was the second claim. ■

The van Kampen theorem has several consequences.

Proposition 2.5.5. Let $X = \bigvee_{i=1}^{n} \mathbb{S}^1$. Then $\pi_1(X) = \mathbb{Z}^{*n} = F_n$,the free group on n generators.

Proof: Describe X as a bouquet of n ordered circles, all connected at the point x_0. Let U be a small open neighborhood around x_0, and let $A_i = U \cup \mathbb{S}^1$, where $\mathbb{S}^1$ is the ith circle in the bouquet.

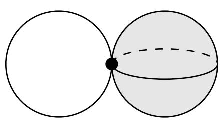
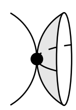
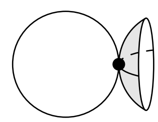
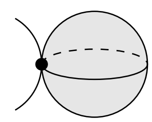
$X =$  $U =$  $A_i =$ 

As U deformation retracts to x_0, A_i deformation retracts to the ith circle $\mathbb{S}^1$, for which $\pi_1(A_i, x_0) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$. The inclusion maps $\iota_{ij} : A_i \cap A_j \hookrightarrow A_i$ induce the maps ι_{ij*}, and these are the 0-homomorphism. This follows as $A_i \cap A_j = U$ and U is conractible to x_0, hence $\pi_1(A_i \cap A_j) = 0$. Therefore $\iota_{ij*}(w)\iota_{ji*}(w)$ is always zero, and by van Kampen,

$$\pi_1(X, x_0) = \mathop{\ast}_{i=1}^{n} \pi_1(A_i, x_0) \,/ \langle \iota_{ij*}(w)\iota_{ji*}^{-1}(w) \rangle = \mathop{\ast}_{i=1}^{n} \mathbb{Z}/\{0\} = \mathbb{Z}^{*n}$$

■

Example 2.5.6. More generally, the same argument works for X a wedge product of different types of spheres. For example, if $X = \mathbb{S}^1 \vee \mathbb{S}^2$, then $\pi_1(X, x_0) = \pi_1(\mathbb{S}^1, x_0) * \pi_1(\mathbb{S}^2, x_0)$, where we choose U, A_1 and A_2 as below.

$X =$  $U =$  $A_1 =$  $A_2 =$ 

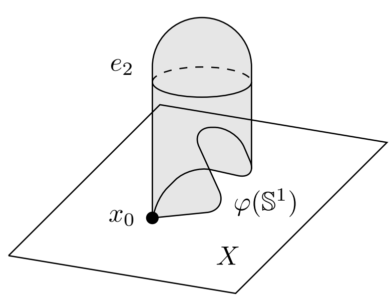
Remark 2.5.7. Even more generally than in the example, if $X = A \vee B$ with A and B locally contractible (for example, topological manifolds) and path-connected spaces, it will follow that $\pi_1(X, x_0) = \pi_1(A, x_{)} * \pi_1(B, x_0)$ by the same argument as above.

Example 2.5.8. Let $X = \mathbb{R}^3 \backslash$(the Hopf link), for which $X \approx \mathbb{S}^2 \vee \mathbb{T}^2$. Then by van Kampen,

$$\pi_1(X, x_0) = \pi_1(\mathbb{S}^2, x_0) * \pi_1(\mathbb{T}^2, x_0) = 0 * \mathbb{Z} \times \mathbb{Z} = \mathbb{Z} \times \mathbb{Z}$$

This is the free abelian product on two groups. Note that $\langle a, b \rangle = \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z} \times \mathbb{Z} = \langle a, b \ : \ ab = ba \rangle$, where the group operation in $\mathbb{Z} \times \mathbb{Z}$ is addition, and $a = (0, 1)$, $b = (1, 0)$.

Remark 2.5.9. Consider the effect on π_1 of gluing 2-cells to a CW-complex. Let X be the CW-complex, with basepoint $x_0 \in X$. Form Y by attaching a 2-cell along a map $\varphi : \mathbb{S}^1 \to X$, so $Y = X \sqcup_\varphi e^2$. Suppose without loss of (much) generality that φ is a based map, so $\varphi : (\mathbb{S}^1, s_0) \to (X, x_0)$. Then note that $[\varphi] \in \pi_1(X, x_0)$.

$Y =$ 

We use van Kampen to compute $\pi_1(Y, x_0)$, in terms of $\pi_1(X, x_0)$ and $[\varphi]$. Let $y \in e^2$ be fixed, and set $A_1 = X \cup (e^2 \setminus \{y\})$ and $A_2 = e^2$, which are both open. Note that A_1 deformation retracts onto X, so

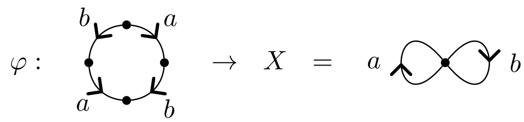
$\pi_1(A_1, x_0) = \pi_1(X, x_0)$. Also note that A_2 does not contain x_0, but any other choice that is not y in e^2 is a good basepoint for use in van Kampen.

Van Kampen tells us that $\pi_1(Y, x_0) = \pi_1(A_1, x_0) * \pi_1(A_2, x_0)/N$. Note that $\pi_1(A_2, x_0) = 0$ and $\pi_1(A_1 \cap A_2) = \pi_1(e^2 \setminus \{y\}) = \mathbb{Z}$. Since the triple intersection hypothesis is trivially satisfied, $N = \langle \iota_{ij*}(w)\iota_{ji*}^{-1}(w)\rangle$. Next note that

$$\begin{array}{rrcl} \iota_{12}: & A_1 \cap A_2 & \hookrightarrow & A_2 \\ \iota_{12*}: & \pi_1(A_1 \cap A_2) & \to & \pi_1(A_2) = \pi_1(X) \end{array} \qquad \begin{array}{rrcl} \iota_{21}: & A_1 \cap A_2 & \hookrightarrow & A_1 \\ \iota_{21*}: & \pi_1(A_1 \cap A_2) & \to & \pi_1(A_1) = 0 \end{array}$$

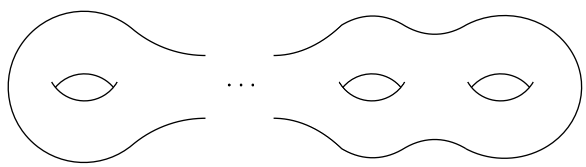
Since $\pi_1(A_1 \cap A_2) = \mathbb{Z}$ and ι_{21*} is trivial, we are left with $\iota_{12*} : [1] \mapsto [\varphi]$, so $\pi_1(Y, x_0) = \pi_1(X, x_0)/\langle[\varphi]\rangle$.

Example 2.5.10. Consider $X = \mathbb{S}^1 \vee \mathbb{S}^1$, and let $Y = T^2$. Then $\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$, and let $\varphi : \mathbb{S}^1 \to X$ be given by:

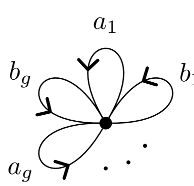


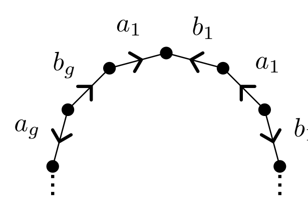
Then $\pi_1(Y, x_0) = \langle a, b \ : \ aba^{-1}b^{-1} = e\rangle = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \times \mathbb{Z}$, as $[\varphi] = aba^{-1}b^{-1}$ in $\pi_1(X, x_0)$.

Example 2.5.11. Let M_g be the orientable surface of genus g, i.e.

$M_g =$ 

For this it is fairly simple to make the cell decomposition, with a map $\varphi : \partial e^2 \to X^1$.

$X^0 = \bullet$ $\qquad X^1 =$ 

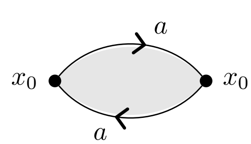
$\qquad X^2 = X^1 \bigsqcup_{\varphi} e^2 \qquad \varphi :$ 

$\to X^1$

This construction allows a clear construction of the fundamental group of M_g. Note that for all $g > 1$, this group is not abelian.

$$\pi_1(M_g) = \pi_1(\underbrace{\mathbb{S}^1 \vee \mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{2g \text{ times}}) \Big/ \langle[\varphi]\rangle = \langle a_1, b_1, \ldots, a_g, b_g \ : \ a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1} = e\rangle$$

Example 2.5.12. Consider the projective plane, $\mathbb{R}P^2$, constructed as:



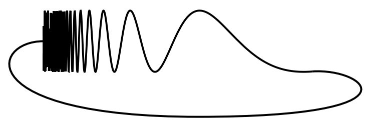
By this construction, $\pi_1(\mathbb{R}P^2) = \pi_1(\mathbb{S}^1)/\langle a^2\rangle = \langle a \ : \ a^2 = e\rangle = \mathbb{Z}/2\mathbb{Z}$. Note that this group has no torsion, i.e. there exists a loop γ in $\mathbb{R}P^2$ for which $[\gamma] \neq 0$, but $[\gamma][\gamma] = 0$.

3 Covering spaces

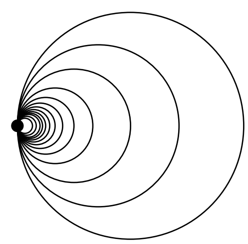
In this section, the folowing assumptions will be made about the topological space X:

1. X is path-connected
2. X is locally path-connected, i.e. for all $x \in X$ and open sets $U \subseteq X$ containing x, there exists $V \subseteq U$ such that V is path-connected
3. X is semi-locally simply connected, i.e. for all $x \in X$ there exists an open set $U \subseteq X$ containing x such that the map $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by $\iota : U \hookrightarrow X$ is trivial

Example 3.0.1. Consider the following example of a path-connected but not locally path-connected space.



This next example, a union of circles of radii $1/n$ centered at $(1/n, 0)$ in $\mathbb{R}^2$, is path-connected, locally path-connected, but not semi-locally simply connected, as every open neighborhood of the point $(0, 0)$ has non-trivial fundamental group.



3.1 The universal cover

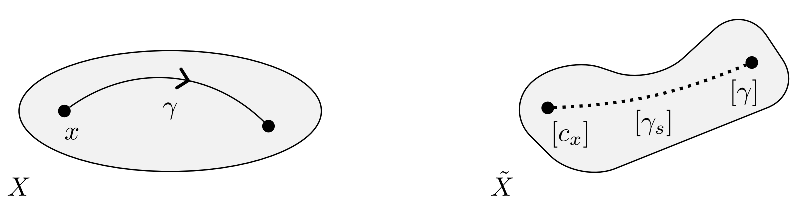
Definition 3.1.1. Let X, Y, Z be topological spaces with continuous maps $f : X \to Y$ and $g : Z \to Y$. A lifting of the map f is a map $h : X \to Z$ such that $f = g \circ h$, or equivalently, such that the following diagram commutes.

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[ur, "h"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Definition 3.1.2. Let X be a topological space. a covering space of X is a space $\tilde{X}$ along with a map $p : \tilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}_{\alpha \in J}$ of X for which the space $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U_α, for all $\alpha \in J$.

Proposition 3.1.3. Given a topological space X, its universal covering $\tilde{X}$ is path-connected.

Proof: Let γ be a path in X starting at x. Consider the family of paths $\gamma_s(t) = \gamma(st)$ for $s, t \in [0, 1]$.



By moving from $s = 1$ to $s = 0$ we may move from the class $[y]$ to the class of the constant path $[c_x]$ at x. Hence any path $[\gamma] \in \tilde{X}$ may be connected to the constant path $[c_x] \in \tilde{X}$, so $\tilde{X}$ is path-connected. ■

Proposition 3.1.4. Given a topological space X, its universal covering $\tilde{X}$ is simply-connected.

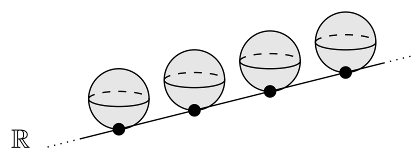
Proof: Recall that if $p_*(\pi_1(\tilde{X}, [c_x])) \subseteq \pi_1(X, x)$, then $\pi_1(\tilde{X}, [c_x]) = 0$ iff $p_*(\pi_1(\tilde{X}, [c_x])) = 0$. So let γ be a path in X with $[\gamma] \in \mathrm{Im}(p_*)$. First note that elements in $\mathrm{Im}(p_*)$ are loops in X that lift to loops in $\tilde{X}$.

Suppose that there exists a path γ' with $[\gamma'] = [\gamma]$ in X, such that there exists a path $\tilde{\gamma}'$ in $\tilde{X}$ with $p \circ \tilde{\gamma}' = \gamma'$. Observe that we can lift γ by the construction in the path-connectedness proof. Then apply the homotopy lifting property between γ and γ' as the starting data using this observation. Then γ and hence γ' are null-homotopic in X, implying that $\tilde{\gamma}'$ is null-homotopic in $\tilde{X}$, meaning that $\tilde{X}$ is simply connected. ■

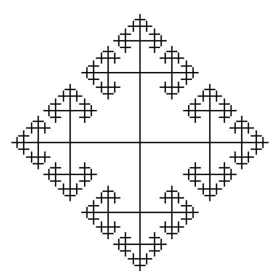
Example 3.1.5. Consider the following spaces and their universal coverings.

$$\begin{array}{ll} X = \mathbb{S}^1 & p : \mathbb{R} \to \mathbb{S}^1 \\ \tilde{X} = \mathbb{R} & \quad t \mapsto e^{it} \end{array} \qquad\qquad \begin{array}{ll} X = \mathbb{S}^1 \times \mathbb{S}^1 & p : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1 \\ \tilde{X} = \mathbb{R}^2 & \quad (t, s) \mapsto (e^{it}, e^{is}) \end{array}$$

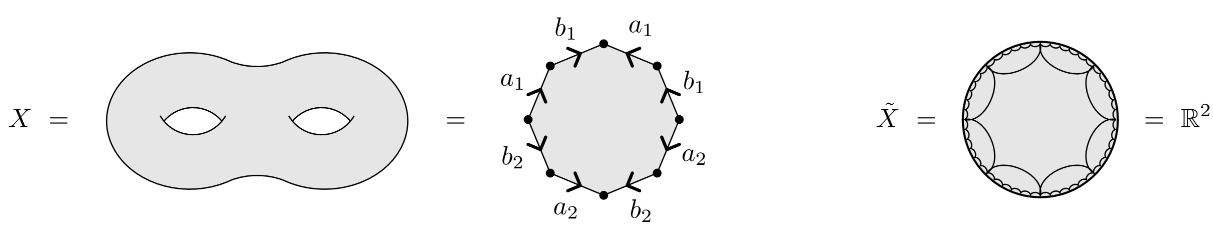
For $X = \mathbb{S}^2$, $\tilde{X} = \mathbb{S}^2$, giving for $X = \mathbb{S}^1 \vee \mathbb{S}^2$ a universal covering of spheres attached to the real line $\mathbb{R}$ at every integer:



For $X = \mathbb{S}^1 \vee \mathbb{S}^1$, the covering space $\tilde{X}$ is the Cayley graph:



For $X = M_2$, the surface of genus two, the covering space $\tilde{X}$ is $\mathbb{R}^2$, by tiling $\mathbb{R}^2$ with octagons. This is possible if the geometry of $\mathbb{R}^2$ is made hyperbolic.



Remark 3.1.6. Given a space (B, b) and its universal cover $(\tilde{B}, \tilde{b})$, we assign the fundamental group $\pi_1(B, b)$ to (B, b), and the trivial group $\{0\}$ to $(\tilde{B}, \tilde{b})$. Then for every subgroup $H \leqslant \pi_1(B, b)$ there exists a covering

$(\tilde{B}_H, \tilde{b}_H)$ of (B, b). This relation may be presented in diagram form:

$$\begin{tikzcd}
(\tilde{B}, \tilde{b}) \arrow[r, mapsto] \arrow[dd, bend right=60, "p"'] & \{0\} \arrow[d, phantom, "\rotatebox{90}{$\leqslant$}"] \\
(\tilde{B}_H, \tilde{b}_H) \arrow[r, mapsto] \arrow[d, "p_H"] & H \arrow[d, phantom, "\rotatebox{90}{$\leqslant$}"] \\
(B, b) \arrow[r, mapsto] & \pi_1(B, b)
\end{tikzcd}$$

To construct $\tilde{B}_H$, we put an equivalence relation on $\tilde{B}$. We say that $[\gamma] \sim [\gamma']$ if and only if $\gamma(1) = \gamma'(1)$ and $[\gamma\bar{\gamma}']$. This defines an equivalence relation, letting us state that $\tilde{B}_H = \tilde{B}/\sim$ and $\tilde{b}_H = [\tilde{b}]$.

To see that $\tilde{B}_H \xrightarrow{p_H} B$ is a cover, note that if $[\gamma\eta] \in U_{\alpha[\gamma]} \sim [\gamma'\eta] \in U_{\alpha[\gamma']}$, then $U_{\alpha[\gamma]} \sim U_{\alpha[\gamma']}$.

3.2 Equivalence of covers

Definition 3.2.1. Suppose that (E, e_0) and (E', e_0') both cover (B, b_0), by p and p', respectively. A morphism between these two covers is the data of a continuous map $\varphi : E \to E'$ such that the following diagram commutes.

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& B &
\end{tikzcd}$$

If the map φ is a homeomorphism, then φ is termed an isomorphism of covers.

The following two lemmae will be used in the proof of the succeeding proposition.

Lemma 3.2.2. [Lifting criterion]
Let $(E, e_0) \xrightarrow{p} (B, b_0)$ be a covering and $f : (Y, y_0) \to (B, b_0)$ a continuous based map. There exists a lift $\tilde{f} : (Y, y_0) \to (E, e_0)$ of f if and only if $\mathrm{Im}(f_*) \leqslant \mathrm{Im}(p_*)$.

Lemma 3.2.3. [Uniqueness statement]
With the conditions as in the previous lemma, let $\tilde{f}$ and $\tilde{f}'$ be two such lifts of f, such that there exists a point $y \in Y$ with $\tilde{f}(y) = \tilde{f}'(y)$. Then $\tilde{f} = \tilde{f}'$.

Proposition 3.2.4. Two covers $(E, e_0) \xrightarrow{p} (B, b_0)$ and $(E', e_0') \xrightarrow{p'} (B, b_0)$ are isomorphic if and only if $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(\pi_1(E', e_0'))$.

Proof: Suppose that $\varphi : (E, e_0) \to (E', e_0')$ is an isomorphism of covers. Apply the functor π_1 to the commutative diagram to get another commutative diagram.

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] & \xrightarrow{\pi_1} & \pi_1(E, e_0) \arrow[rr, "\varphi_*"] \arrow[dr, "p_*"'] & & \pi_1(E', e_0') \arrow[dl, "p'_*"] \\
& B & & & & \pi_1(B, b_0) &
\end{tikzcd}$$

The map φ_* is still an isomorphism, and since this diagram commutes, $p'_* \circ \varphi_* = p_*$, so $\mathrm{Im}(p_*) = \mathrm{Im}(p'_*)$.

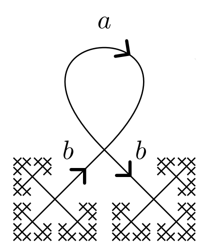
Now suppose that $p_*(\pi_1(E, e_0)) = p'_*(\pi_1(\pi_1(E', e_0'))$, and apply the lifting criterion to get a commutative diagram:

$$\begin{tikzcd}
 & & (E', e_0') \arrow[d, "p'"] \arrow[dd, dashed, bend left=60, "p'"] \\
 & (E, e_0) \arrow[ur, "\tilde{p}"] \arrow[r, "p"'] \arrow[d, "p"] & (B, b_0) \arrow[d, "\mathrm{id}"] \\
(E', e_0') \arrow[ur, "\tilde{p}'"] \arrow[r, "p'"'] \arrow[uurr, dashed, bend left=60, "\mathrm{id}"] \arrow[rr, dashed, bend right=40, "p"'] & (B, b_0) \arrow[r, "\mathrm{id}"'] & (B, b_0)
\end{tikzcd}$$

By the uniqueness of lifts, we have that $\tilde{p} \circ \tilde{p}' = \mathrm{id}$. By a symmetric diagram, with the spaces E and E' interchanged, we have that $\tilde{p}' \circ \tilde{p} = \mathrm{id}$. Therefore $\varphi = \tilde{p} = (\tilde{p}')^{-1}$ is the desired isomorphism of covers. ■

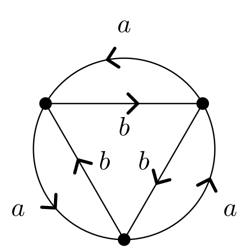
Example 3.2.5. Let $B = \mathbb{S}^1 \vee \mathbb{S}^1$, for which $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \langle a, b \rangle$. Let $H = \langle a \rangle \subset \langle a, b \rangle$. Then

B_H =



Note that this diagram is not maximally symmetric, as moving the basepoint does not preserve symmetry. Next let $B = \mathbb{S}^1 \vee \mathbb{S}^1$ again, but with $H = \langle a, b \ : \ a^3 = b^3 = ab = ba \rangle$. Then

B_H =



This is a 3-sheeted cover.

Theorem 3.2.6. [Nielsen-Schreier theorem]
Every subgroup of a freely-generated group is free.

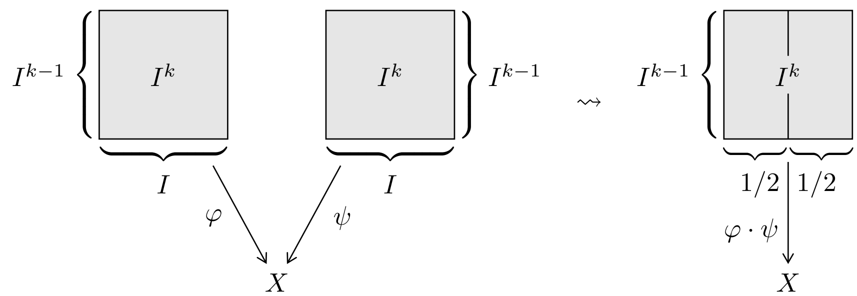
Remark 3.2.7. Suppose that F is a free group on n generators. Then $F = \pi_1(\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1)$, the wedge product of n 1-spheres. Moreover, any connected cover of $\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1$ is a connected graph.

It follows that given a maximal subtree of a connected graph, quotienting by this subtree yields a bouquet of circles. Hence any subgroup of F is $\pi_1(\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1)$, the wedge product of $k \leqslant n$ 1-spheres.

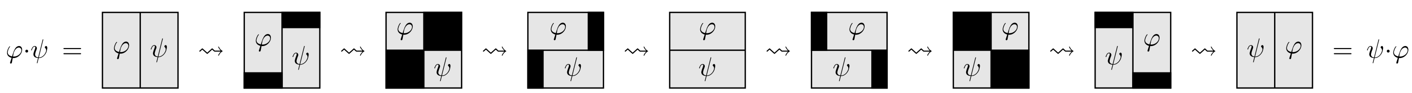
3.3 Higher homotopy groups

Definition 3.3.1. Let (X, x_0) be a based space. For $k \geqslant 1$, let $\pi_k(X, x_0)$ be the set of classes of maps $\varphi : I^k \to X$ such that $\varphi|_{\partial I^k} = x_0$, with respect to fixed boundary homotopy, denoted $[(I^k, \partial I^k) \to (X, x_0)]$.

Note that for all $\pi_k(X, x_0)$ is a group for all $k \geqslant 1$, where the group action is concatenation of cubes, $I^k = I \times \cdots \times I$ representing the k-dimensional cube.



Moreover, π_k is abelian for $k > 1$. This can be seen by noting that since $\varphi|_{\partial I^k} = x_0$, the whole boundary of the cube goes to x_0, so by thickening and thinning this boundary in certain parts, commutativity may be demonstrated.



For $k = 0$, the object π_0 has $I^0 = *$, and $\partial I^0 = \emptyset$ - it detects path-connected components of X, and unlike the other homotopy groups, is not a group.

Example 3.3.2. For the circle $\mathbb{S}^1$, we have the following associated groups:

k	0	1	2	3	$\cdots$
$\pi_k(\mathbb{S}^1)$	0	$\mathbb{Z}$	0	0	0

For the sphere $\mathbb{S}^2$, we have the following associated groups:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
$\pi_k(\mathbb{S}^2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\cdots$

As these examples show, both the calculation of the homotopy group and the group itself may be very complex.

Remark 3.3.3. Serre proved that, for all $q \geqslant 1$, there exist $k, n \in \mathbb{N}$ such that $\pi_k(\mathbb{S}^n)$ has q-torsion (i.e. a subgroup of order q).

Even though higher homotopy groups are very complex, the field of *stable homotopy theory* studies $\lim\limits_{n\to\infty} [\pi_{k+n}(\mathbb{S}^n)]$. However, we now move to homology theory for the sake of computability.

4 Homology

To a space X, we associate a countable family of abelian groups $H_k(X)$, for $k \in \mathbb{N}$.

4.1 Background

The groups $H_k(X)$ are both very similar and very distinct from the homotopy groups $\pi_k(X)$. For example,

$$\pi_1(X)_{ab} = H_1(X)$$

That is, the abelianization of $\pi_1(X)$ is equal to $H_1(X)$. In general, the Hurewicz theorem states that if j is the smallest index for which $\pi_j(X) \neq 0$, it follows that

$$\pi_{0<k\leqslant j}(X) = H_{0<k\leqslant j}(X)$$

Example 4.1.1. For the n-dimensional sphere $\mathbb{S}^n$, we have the following homology groups:

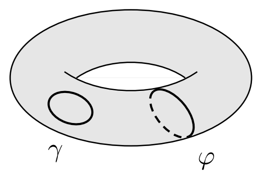
k	0	1	2	$\cdots$	$n-1$	n	$n+1$	$\cdots$
$H_k(\mathbb{S}^n)$	$\mathbb{Z}$	0	0	$\cdots$	0	$\mathbb{Z}$	0	$\cdots$

Example 4.1.2. Let $n \in \mathbb{N}$ and G be a group (abelian if $n > 1$). Then there exists a space $K(G,n)$, termed the Eilenberg-MacLane space, with $\pi_k(K(G,n)) = \left\{ \begin{smallmatrix} 0 & k=n \\ G & k\neq n \end{smallmatrix} \right.$. For $G = \mathbb{Z}$ and small n, we have the following:

$$\begin{aligned} K(\mathbb{Z},1) &= \mathbb{S}^1 \\ K(\mathbb{Z},2) &= \mathbb{C}P^\infty \end{aligned}$$

4.2 Singular homology

Consider a space X, with loops embedded in X. Not all have the same properties.

$X =$ 

- $\partial\varphi = 0,\ \partial\gamma = 0$
- $\varphi \neq \partial$(closed subsurface)
- γ is a boundary of a disk in X

We will study spaces that have no boundary and that are not the boundary of a subspace of the whole space.

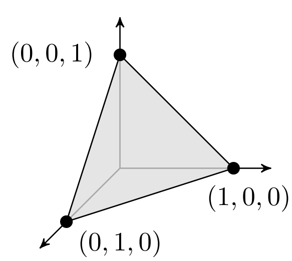
Definition 4.2.1. For $n \in \mathbb{N}$, an <u>n-simplex</u> is the smallest convex set containing $n+1$ generic points in $\mathbb{R}^{n+1}$ (i.e. not all on a hyperplane in $\mathbb{R}^n$). A simplex is a generalization of a triangle. For example,

0-simplex	-	point
1-simplex	-	line segment
2-simplex	-	triangle
3-simplex	-	tetrahedron
	$\vdots$	

The stanard n-simplex is described as a subset of $\mathbb{R}^{n+1}$, by

$$\Delta^n = \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} \ : \ t_i \geqslant 0,\ \sum_{i=0}^n t_i = 1\}$$

For example, the standard 2-simplex in $\mathbb{R}^3$ is embedded as:



Definition 4.2.2. Let X be a space. Then $C_n(X)$ is termed the <u>nth chain group</u> of X, with elements termed <u>chains</u>.

$$C_n(X) = \left(\text{free abelian group generated by } \{\Delta^n \xrightarrow{\sigma} X\} = \left\{\sum_i a_i\sigma_i\right\}\right)$$

Here the maps σ, $\sigma_i : \Delta^n \to X$ are continous, and $a_i \in \mathbb{Z}$ for all i. Further, we have boundary maps ∂, which may be extended to C_n linearly, by the following definition:

$$\begin{array}{rrcl} \partial: & C_n(X) & \to & C_{n-1}(X) \\ & \sigma & \mapsto & \displaystyle\sum_{i=0}^{n} (-1)^i \sigma|_{[x_0,\ldots,\widehat{x_i},\ldots,x_n]} \end{array}$$

The definition implies that $\partial \circ \partial = 0$. With these maps, the singular chain complex of X is given by:

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

Definition 4.2.3. With respect to the previous definitions, the kth homology group of X is given by

$$H_k(X) = \underbrace{\ker(\partial : C_k \to C_{k-1})}_{\text{contains } k\text{-cycles}} \Big/ \underbrace{\mathrm{Im}(\partial : C_{k+1} \to C_k)}_{\text{contains } k\text{-boundaries}}$$

Remark 4.2.4. If the chain complex of X is exact (i.e. $\ker(\partial) = \mathrm{Im}(\partial)$), then $H_k = 0$ for all k. Hence H_k measures the failure to be exact.

Proposition 4.2.5. Let $X = \{\mathrm{pt}\}$. Then $H_0 = \mathbb{Z}$ and $H_{k\neq 0} = 0$.

Proof: Since $X = \{\mathrm{pt}\}$, for each $n \in \mathbb{N}$, there exists a unique singular n-simplex $\sigma_n : \Delta^n \to \{\mathrm{pt}\}$, so $C_n = \mathbb{Z}$. The boundary maps ∂ are defined by

$$\begin{array}{rrcl} \partial: & C_n & \to & C_{n-1} \\ & a\sigma_n & \mapsto & a\partial(\sigma_n) = a\displaystyle\sum_{i=0}^{n} (-1)^i \underbrace{\sigma_n|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}}_{\sigma_{n-1}} = \begin{cases} 0 & n \text{ odd} \\ a\sigma_{n-1} & n \text{ even} \end{cases} \end{array}$$

Therefore the chain complex of X is given by

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, equal] & C_n \arrow[r, "\partial"] \arrow[d, equal] & C_{n-1} \arrow[r, "\partial"] \arrow[d, equal] & \cdots \arrow[r, "\partial"] & C_1 \arrow[r, "\partial"] \arrow[d, equal] & C_0 \arrow[r, "\partial"] \arrow[d, equal] & 0 \\ \cdots \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & \mathbb{Z} \arrow[r, "0"] \arrow[d, dashed] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, dashed] & \cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, dashed] & \mathbb{Z} \arrow[r, "0"] \arrow[d, dashed] & 0 \\ & H_{n+1} = 0 & H_n = 0 & H_{n-1} = 0 & & H_1 = 0 & H_0 = \mathbb{Z} & \end{tikzcd}$$

■

Proposition 4.2.6. The group H_0 detects the path-connected components of X. That is, if X has α path-connected components, then $H_0(X) = \bigoplus_\alpha \mathbb{Z}$.

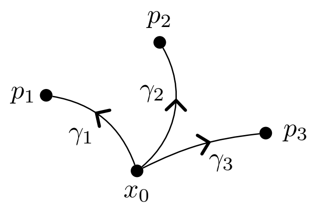
Proof: Assume that X is path-connected and non-empty, so

$$H_0(X) = \ker(\partial : C_0 \to 0) \Big/ \mathrm{Im}(\partial : C_1 \to C_0) = C_0 \Big/ \mathrm{Im}(\partial : C_1 \to C_0)$$

Define a new map $\varepsilon : C_0 \to \mathbb{Z}$ by $\sum a_i p_i \mapsto \sum a_i$, where the sums are finite, $a_i \in Z$, and p_i is a point in X. This allows us to claim that $\ker(\varepsilon) = \mathrm{Im}(\partial : C_1 \to C_0)$. To check this claim, we prove double inclusion.

Im(∂) $\subseteq$ ker(ε): Suppose that σ is a singular 1-simplex in X, so $\partial\sigma = \sigma|_{v_1} - \sigma|_{v_0}$, for $\sigma : [v_0, v_1] \to X$. Then $\varepsilon(\partial\sigma) = 1 - 1 = 0$, proving the first part.

ker(ε) $\subseteq$ Im(∂): Suppose that $\Sigma a_i p_i$ is such that $\varepsilon(\sigma a_i p_i) = 0$. This implies that $\sum a_i = 0$. Next pick a basepoint $x_0 \in X$, and for each p_i a path from x_0 to p_i called γ_i.



Consider the element $\Sigma a_i\gamma_i$ in C_1, for which

$$\begin{aligned}
\partial\left(\sum a_i\gamma_i\right) &= \sum a_i\left(\gamma_1(1) - \gamma_i(0)\right)\\
&= \sum a_i\left(p_i - x_0\right)\\
&= \sum a_i p_i - \sum a_i x_0\\
&= \sum a_i p_i
\end{aligned}$$

This proves the second part, and the claim. Now we have an exact sequence:

$$0 \longrightarrow \ker(\varepsilon) \hookrightarrow C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Since $\ker(\varepsilon) = \mathrm{Im}(\partial)$, it follows that $C_0/\mathrm{Im}(\partial) = H_0(X) = \mathbb{Z}$. A special case of the folowing proposition proves the second statement. ■

Proposition 4.2.7. Let $X = \bigsqcup_\alpha X_\alpha$ be the decomposition of X into its path-connected components. Then for any k, $H_k(X) = \bigoplus_\alpha H_k(X_\alpha)$.

Proof: If a singular complex $\sigma : \Delta^n \to X$ has image in some X_α, then so does $\partial(\sigma)$. ■

4.3 The functoriality of $H_\bullet(\cdot)$

Definition 4.3.1. The statement that $H_\bullet$ is functorial means that given $f : X \to Y$, there exists an induced map $f_* : H_\bullet(X) \to H_\bullet(Y)$ such that

1. For the map $\mathrm{id}_X : X \to X$, we have $(\mathrm{id}_X)_* = \mathrm{id}_{H_\bullet(X)} : H_\bullet(X) \to H_\bullet(X)$
2. For maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have $g_* \circ f_* = (g \circ f)_*$

The map f_* is defined by first writing the induced maps on chain complexes, given by

$$f_\# : C_k(X) \to C_k(Y)$$

For $\sigma : \Delta^k \to X$ a singular k-simplex in X, the map $f \circ \sigma$ becomes a singular k-simplex in Y. This allows us to say $f_\#\left(\sum a_i\sigma_i\right) = \sum a_i(f \circ \sigma_i)$. It is important to note that $f_\#$ is a chain map, or a chain morphism, equivalently, satisfies the property below, which is also equivalent to to the diagram below commuting.

$$f_\# \circ \partial = \partial \circ f_\# \qquad \iff \qquad \begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Remark 4.3.2. The fact that $f_\#$ is a chain map is crucial to defining f_*. As $f_\#$ satisfies the property above, we may check that it indeed induces a map $f_* : H_\bullet(X) \to H_\bullet(Y)$. Suppose that a class $[\sigma] \in H_k(X)$ is represented by a singular k-simplex σ. It follows that $f_*[\sigma] = [f_\# \sigma]$.

Note that given $\partial\sigma = 0$, we have $\partial(f_\#\sigma) = 0$ and $f_\#\partial\sigma = 0$.

Further, if σ is the sum of a k-cycle and the boundary of a $(k+1)$-chain, then

$$\begin{aligned} f_*[\sigma] &= f_*[\sigma' + \partial\alpha] \\ &= [f_\#(\sigma' + \partial\alpha)] \\ &= [f_\#\sigma' + f_\#\partial\alpha] \\ &= [f_\#\sigma'] + [f_\#\partial\alpha] \\ &= [f_\#\sigma'] \\ &= f_*[\sigma'] \end{aligned}$$

The properties given above with f_* as defined are similarly checked.

Proposition 4.3.3. Suppose that $f, g : X \to Y$ are continous and homotopic via $H : X \times I \to Y$ (so $H_0 = f$ and $H_1 = g$). Then $f_* = g_*$.

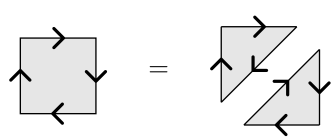
Proof: Recall that we have the following maps:

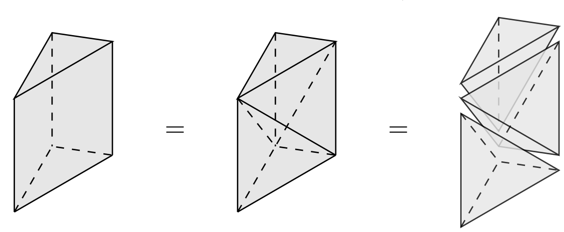
$$\begin{array}{rccc} f_\# : & C_\bullet(X) & \to & C_\bullet(Y) \\ g_\# : & C_\bullet(X) & \to & C_\bullet(Y) \end{array} \qquad \text{with} \qquad \begin{array}{rcl} \partial \circ f_\# & = & f_\# \circ \partial \\ \partial \circ g_\# & = & g_\# \circ \partial \end{array}$$

Now we construct a chain homotopy between $f_\#$ and $g_\#$, given by the operator $P : C_\bullet(X) \to C_{\bullet+1}(Y)$ such that $\partial P + P\partial = g_\# - f_\#$, or equivalently, so that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & C_{\bullet+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\# \, g_\#"'] \arrow[dl, "P"'] & C_\bullet(X) \arrow[r, "\partial"] \arrow[d, "f_\# \, g_\#"'] \arrow[dl, "P"'] & C_{\bullet-1}(X) \arrow[r, "\partial"] \arrow[d, "f_\# \, g_\#"'] \arrow[dl, "P"'] & \cdots \arrow[dl, "P"'] \\ \cdots \arrow[r, "\partial"] & C_{\bullet+1}(Y) \arrow[r, "\partial"] & C_\bullet(Y) \arrow[r, "\partial"] & C_{\bullet-1}(Y) \arrow[r, "\partial"] & \cdots \end{tikzcd}$$

To construct P, we need to express $\Delta^n \times I$ as a linear combination of simpleces. The general case will not be done here, but the smallest examples will be presented to give an idea of the procedure.

$\Delta^1 \times I =$ 

$\Delta^2 \times I =$ 

Given such a representation of $\Delta^n \times I$ as a sum of n-simplices, we may define P. Choose a generator $\sigma \in C_n(X)$ to get maps

$$\begin{array}{rccc} \sigma \times \mathrm{id} : & \Delta^n \times I & \to & X \times I \\ H \circ (\sigma \times \mathrm{id}) : & \Delta^n \times I & \to & Y \end{array}$$

This induces a map $P : C_n(X) \to C_{n+1}(Y)$ that satisfies $\partial P + P\partial = g_\# - f_\#$. Further, since $f_\#[\sigma] = g_\#[\sigma]$, it follows that

$$(g_\# - f_\#)[\sigma] = [g_\#\sigma - f_\#\sigma] = [\partial P\sigma - P\partial\sigma] = [0]$$

Here, $P\sigma$ is a boundary, so $\partial P\sigma = 0$. Further, as $\partial\sigma$ is a boundary, it is no longer a generator, so $P\partial\sigma = 0$, and the function has been defined as desired. ■

Corollary 4.3.4. If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n.

Definition 4.3.5. Let $A_\bullet$, $B_\bullet$, and $C_\bullet$ be chain complexes. A short exact sequence of chain complexes is a pair of chain maps $i : A_\bullet \to B_\bullet$ and $j : B_\bullet \to C_\bullet$, with $i\partial_A = \partial_B i$ and $j\partial_B = \partial_C j$, such that for each n, each row in the following diagram is a short exact sequence.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, hook, "i"] \arrow[d, "\partial_A"'] & B_{n+1} \arrow[r, two heads, "j"] \arrow[d, "\partial_B"'] & C_{n+1} \arrow[r] \arrow[d, "\partial_C"'] & 0 \\
0 \arrow[r] & A_n \arrow[r, hook, "i"] \arrow[d, "\partial_A"'] & B_n \arrow[r, two heads, "j"] \arrow[d, "\partial_B"'] & C_n \arrow[r] \arrow[d, "\partial_C"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, hook, "i"] \arrow[d] & B_{n-1} \arrow[r, two heads, "j"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{1}$$

Proposition 4.3.6. Let $A_\bullet$, $B_\bullet$, and $C_\bullet$ be chain complexes, and $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ a short exact sequence of chain complexes. Then this short exact sequence induces a long exact sequence in homology:

$$\cdots \xrightarrow{j_*} H_{n+1}(C_\bullet) \xrightarrow{\delta} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \xrightarrow{i_*} \cdots \xrightarrow{j_*} H_0(C_\bullet) \to 0$$

Proof: First we need to define the map δ. We will use the diagram presented above in (1). As each row there is exact, each component map i (and j) is injective (and surjective).

Consider $[c] \in H_{n+1}(C_\bullet)$, so c is a cycle in C_{n+1}, that is, $\partial c = 0$. Since j is surjective, there exists $b \in B_{n+1}$ with $j(b) = c$. And as $\partial j = j\partial$,

$$j(\partial b) = \partial j(b) = \partial c = 0$$

Therefore $\partial b \in B_n$ and $\partial b \in \ker(j)$. By the exactness of i and j, and as $\partial b \in \ker(j)$, there exists $a \in A_n$ such that $i(a) = \partial b$. Then we define $\delta[c] = [a]$.

Now check that this definition makes sense. As $i(a) = \partial b$, we have that $i(\partial a) = \partial i(a) = \partial\partial b = 0$, and as i is injective, this means that $\partial a = 0$. Therefore $[a] \in H_n(A_\bullet)$, which is what we wanted.

Next check this is well defined. Let $\tilde{c} = c + \partial c$, so $c' = \partial b'$ for some b'. If $i(\tilde{a}) = b + \partial b'$ and $i(a) = b$, then $i(\tilde{a} - a) = b + \partial b - b = \partial b$, and so $i_*[\tilde{a} - a] = [\partial b] = 0$

This shows that δ is a well-defined homomorphism. To complete the proof, it remains to check that the sequence produced is indeed long exact. This is a diagram-chasing calculation that involves six calculations, or three double inclusions, which are omitted here. ■

4.4 Relative and reduced homology

Definition 4.4.1. Let (A, X) be a pair of spaces, that is, $A \subseteq X$ a subspace. Then the relative chain complex of (X, A) is defined as

$$C_\bullet(X, A) = C_\bullet(X)/C_\bullet(A)$$

So $\partial_{X,A}$ descends from $C_\bullet(X)$ to the quotient, as $\alpha \in C_\bullet(A)$ implies $\partial_X \alpha = \partial_A \alpha$. So if $c = c' + \alpha$, for $c, c' \in C_\bullet(X)$ and $\alpha \in C_\bullet(A)$, then $\partial c = \partial c' + \partial\alpha$, meaning that ∂c is in the same class as $\partial c' + \partial\alpha$ in $C_{\bullet-1}(X)/C_{\bullet-1}(A)$.

By construction, there is an induced short exact sequence of complexes:

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i} C_\bullet(X) \xrightarrow{j} C_\bullet(X, A) \longrightarrow$$

Here i is the inclusion map and j is the quotient map. Further, we then say that $H_\bullet(X, A)$ is the homology of $C_\bullet(X, A)$. This produces a long exact sequence, by the previous proposition, in homolgy, termed a long exact sequence in relative homology:

$$\cdots \xrightarrow{j_*} H_{n+1}(X, A) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

Remark 4.4.2. With respect to the above definition, if A is non-empty, closed, and has an open neighborhood deformation retracting onto A, then (X, A) is termed a "good pair," and for $n \geqslant 1$, it follows from excision that $\tilde{H}_n(X/A) = H_n(X, A)$, where $\tilde{H}_n$ is the reduced homology of this pair. This implies that there is a long exact sequence in reduced homology associated to this pair:

$$\cdots \xrightarrow{j_*} \tilde{H}_{n+1}(X/A) \xrightarrow{\delta} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots$$

Note that for $n = 0$, we have $\tilde{H}_0(Y) = H_0(Y) \oplus \mathbb{Z}$ for any space Y.

Example 4.4.3. Let $X = I$, $A = \partial I = \{0\} \sqcup \{1\}$, and so $X/A = \mathbb{S}^1$. Since $I \approx \text{pt}$, we have that $\tilde{H}_n(I) = 0$ for all n. As for the space A, we have

$$\tilde{H}_n(\{0\} \sqcup \{1\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

The long exact sequence of reduced homology of the pair (X, A) ends as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_2(\mathbb{S}^1) \arrow[r] & \tilde{H}_1(\{0\} \sqcup \{1\}) \arrow[r] \arrow[d, equal] & \tilde{H}_1(I) \arrow[r] \arrow[d, equal] & \tilde{H}_1(\mathbb{S}^1) \arrow[r] & \tilde{H}_0(\{0\} \sqcup \{1\}) \arrow[r] \arrow[d, equal] & \tilde{H}_0(I) \arrow[r] \arrow[d, equal] & \tilde{H}_0(\mathbb{S}^1) \arrow[r] & 0 \\
& & 0 & 0 & & \mathbb{Z} & 0 & &
\end{tikzcd}$$

This directly implies that

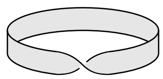
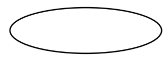
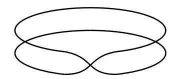
$$\tilde{H}_n(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

Example 4.4.4. Compute the homology of $Mb/\partial Mb$. This is an example of the general case that for any space X, $(X, \partial X)$ forms a good pair. Hence we apply the reduced long exact sequence, and have the following data initially:

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_2(Mb) \arrow[r, "j_*"] \arrow[d, equal] & \tilde{H}_2(Mb/\partial Mb) \arrow[r, "\delta"] & \tilde{H}_1(\partial Mb) \arrow[r, "i_*"] \arrow[d, equal] & \tilde{H}_1(Mb) \arrow[r, "j_*"] \arrow[d, equal] & \tilde{H}_1(Mb/\partial Mb) \arrow[r, "\delta"] & \tilde{H}_0(\partial Mb) \arrow[r, "i_*"] \arrow[d, equal] & \tilde{H}_0(Mb) \arrow[r, "j_*"] \arrow[d, equal] & \tilde{H}_0(Mb/\partial Mb) \arrow[r] & 0 \\
& 0 & & \mathbb{Z} & \mathbb{Z} & & 0 & 0 & &
\end{tikzcd}$$

To get the groups, we used the fact that $Mb \approx \mathbb{S}^1$ and $\partial Mb \approx \mathbb{S}^1$. Then as $\tilde{H}_n(\mathbb{S}^1) = 0$ for all $n \neq 1$, in which case it is $\mathbb{Z}$, we have the exact sequence $0 \to \tilde{H}_n(Mb/\partial Mb) \to 0$ for all $n \neq 1$, meaning that $\tilde{H}_n(Mb/\partial Mb) = 0$ for all $n \geqslant 3$.

Since the first j_* is the zero map, $\mathrm{Im}(j_*) = 0 = \ker(\delta)$, so the first δ is injective. Further, the map $i_* : \mathbb{Z} \to \mathbb{Z}$ takes a generator $\mathbb{S}^1$ from ∂Mb to Mb. The action is given by $i_* : a \mapsto 2a$, as the boundary of Mb maps twice to the circle $\mathbb{S}^1$ that Mb is homotopic to.

$Mb =$  $\approx$  $= \mathbb{S}^1$ $\qquad \partial Mb =$ 

Therefore this i_* is injective, so $\ker(i_*) = 0 = \mathrm{Im}(\delta)$. Then δ is injective with an empty image, so δ is the zero map, implying that $\tilde{H}_2(Mb/\partial Mb) = 0$.

Since the second δ is the zero map, $\ker(\delta) = \tilde{H}_1(Mb/\partial Mb) = \mathrm{Im}(j_*)$, so the second j_* is surjective. We also know that $\mathrm{Im}(i_*) = 2\mathbb{Z} = \ker(j_*)$. This gives a short exact sequence $\mathbb{Z} \to \mathbb{Z} \to \tilde{H}_1(Mb/\partial Mb)$, meaning that $\tilde{H}_1(Mb/\partial Mb) \cong \mathbb{Z}/\ker(j_*) = \mathbb{Z}/2\mathbb{Z}$.

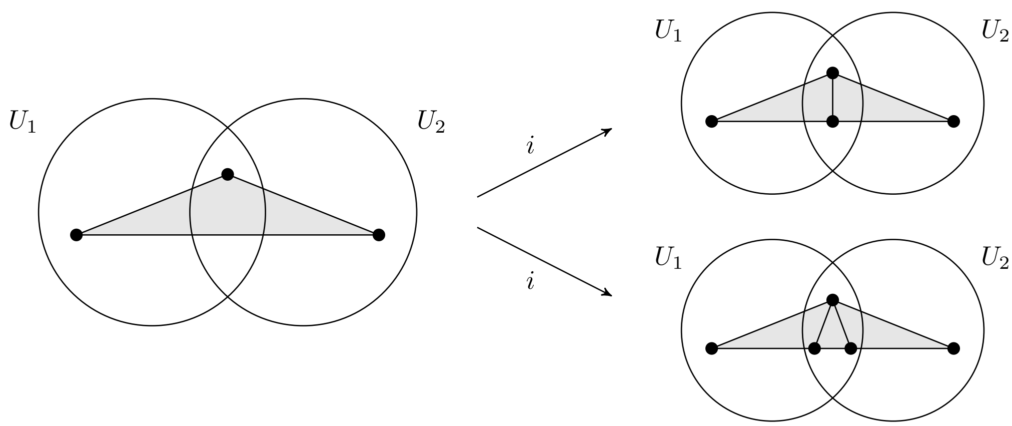
As for $\tilde{H}_0(Mb/\partial Mb)$, it is clearly 0 from above. This allows us to conclude that:

$$H_n(Mb/\partial Mb) = H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

Remark 4.4.5. Suppose U_1, U_2 are open sets covering $X = U_1 \cup U_2$. Consider the set

$$C_n(U_1 + U_2) = \{\alpha + \beta \ : \ \alpha \in C_n(U_1), \beta \in C_n(U_2)\} \subseteq C_n(X)$$

The boundary map $\partial : C_n(U_1 + U_2) \to C_{n-1}(U_1 + U_2)$ is the standard differential. Note that the map $i : C_n(U_1 + U_2) \to C_{n-1}(U_1 + U_2)$ induces an isomorphism in homology $i_* : H_n(U_1 + U_2) \to H_n(X)$. This isomorphism is not canonical, in the sense that there is no one natural way to induce it. To see this, consider the below situation, with 0-, 1-, and 2-cells as drawn.



Either way, this produces a short exact sequence of complexes:

$$\begin{array}{ccccccccc} 0 \longrightarrow C_n(U_1 \cap U_2) & \xrightarrow{\varphi} & C_n(U_1) \oplus C_n(U_2) & \xrightarrow{\psi} & C_n(U_1 + U_2) \longrightarrow 0 \\ x \longmapsto (x, -x) & & (x, y) \longmapsto x + y & & \end{array}$$

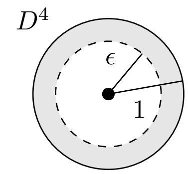
This in turn induces a long exact sequence in homology:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(U_1+U_2) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=-90] & H_n(U_1\cap U_2) \arrow[r, "\varphi_*"] & H_n(U_1)\oplus H_n(U_2) \arrow[r, "\psi_*"] & H_n(U_1+U_2) \arrow[r, "\delta"] \arrow[d, phantom, "\cong" rotate=-90] & H_{n-1}(U_1\cap U_2) \arrow[r] & \cdots \\
& H_{n+1}(X) & & & H_n(X) & &
\end{tikzcd}$$

Moreover, this approach may be generalized to the case that $X = U_1 \cup U_2 \cup \cdots \cup U_k$ canonically.

Example 4.4.6. Calculate the homology groups of $\mathbb{C}P^2$, using the Mayer-Vietoris sequence. First we note that this is a 4-manifold that is not the boundary of any 5-manifold. Second, we consider the following construction of this space:

$$\mathbb{C}P^2 \;=\; \begin{array}{c} D^4 \quad \partial D^4 = \mathbb{S}^3 \\ \Big\downarrow \varphi \\ \mathbb{C}P^1 \approx \mathbb{S}^2 \end{array} \qquad \begin{array}{l} U_1 = D^4 \\ U_2 = \mathbb{C}P^2 \setminus B_\epsilon \\ U_1 \cap U_2 = int(D^4) \setminus B_\epsilon \end{array}$$



Above, the gluing $\varphi : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map, and B_ϵ is the open ball centered at the center of D^4 with a radius of $\epsilon < 1$. Such a construction allows us to apply the Mayer-Vietoris sequence. First we state the homolgies that we already know:

$$H_n(U_1) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases} \qquad H_n(U_2) \cong H_n(\mathbb{C}P^1) = \begin{cases} \mathbb{Z} & n=0,2 \\ 0 & \text{else} \end{cases} \qquad H_n(U_1 \cap U_2) \cong H_n(\mathbb{S}^3) = \begin{cases} \mathbb{Z} & n=0,3 \\ 0 & \text{else} \end{cases}$$

So most of the groups in the long exact sequence are trivial, but not all. The non-trivial part of this sequence begins at $n = 4$, or $H_n(\mathbb{C}P^2) = 0$ for all $n \geqslant 4$. Sections will be considered separately.

$$\begin{tikzcd}
\cdots \arrow[r] & H_4(U_1)\oplus H_4(U_2) \arrow[r] \arrow[d, equal] & H_4(\mathbb{C}P^2) \arrow[r, "\delta"] & H_3(U_1\cap U_2) \arrow[r] \arrow[d, equal] & H_3(U_1)\oplus H_3(U_2) \arrow[r] \arrow[d, equal] & H_3(\mathbb{C}P^2) \arrow[r] & H_2(U_1\cap U_2) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 \oplus 0 & & \mathbb{Z} & 0 \oplus 0 & & 0 &
\end{tikzcd}$$

As the map before the map δ is the zero map, it has zero image, so δ has zero kernel, meaning that δ is injective. And as the map after δ is also the zero map, it has kernel $\mathbb{Z}$, so δ has image $\mathbb{Z}$, meaning that δ is surjective. Hence δ is an isomorphsm, and $H_4(\mathbb{C}P^2) = \mathbb{Z}$. Note that the generator (a cycle) of $H_4(\mathbb{C}P^2)$ is the whole space, i.e. the class $[\mathbb{C}P^2]$.

For the third homolgy group, both groups to either side of it are zero, so $H_3(\mathbb{C}P^2) = 0$ as well.

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(U_1\cap U_2) \arrow[r] \arrow[d, equal] & H_2(U_1)\oplus H_2(U_2) \arrow[r, "\psi_*"] \arrow[d, equal] & H_2(\mathbb{C}P^2) \arrow[r] & H_1(U_1\cap U_2) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 & 0 \oplus \mathbb{Z} & & 0 &
\end{tikzcd}$$

For the same reasons as above, ψ_* is an isomorphism, so $H_2(\mathbb{C}P^2) = \mathbb{Z}$. Here note that the generator (a cycle) of $H_2(\mathbb{C}P^2)$ is $\mathbb{C}P^1$, or $\mathbb{S}^2 = \mathbb{Z}[\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2]$.

$$\begin{tikzcd}
\cdots \arrow[r] & H_1(U_1\cap U_2) \arrow[r] \arrow[d, equal] & H_1(U_1)\oplus H_1(U_2) \arrow[r] \arrow[d, equal] & H_1(\mathbb{C}P^2) \arrow[r, "\delta"] & H_0(U_1\cap U_2) \arrow[r, "\varphi_*"] \arrow[d, equal] & H_0(U_1)\oplus H_0(U_2) \arrow[r] \arrow[d, equal] & H_0(\mathbb{C}P^2) \arrow[r] & 0 \\
& 0 & 0 \oplus 0 & & \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & &
\end{tikzcd}$$

Recall that $\varphi : \text{pt} \mapsto \text{pt} \oplus -\text{pt}$, so $\varphi : [\text{pt}] \mapsto [\text{pt}] \oplus -[\text{pt}]$. Hence $[1] \mapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so φ_* is injective. Therefore $\ker(\varphi_*) = 0 = \text{Im}(\delta)$, so δ is the zero map. This means that the map before δ is injective, directly implying that $H_1(\mathbb{C}P^2) = 0$.

For the last homology group, since $\mathbb{C}P^2$ is connected, $H_0(\mathbb{C}P^2) = \mathbb{Z}$. Therefore we have the following results, from whih we may generalize to the k-dimensional complex projective space:

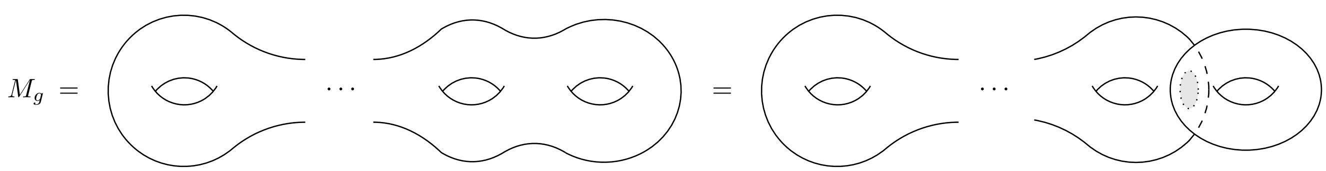
$$H_n(\mathbb{C}P^2) = \begin{cases} \mathbb{Z} & n = 0, 2, 4 \\ 0 & \text{else} \end{cases} \qquad\qquad H_n(\mathbb{C}P^k) = \begin{cases} \mathbb{Z} & n \in 2\mathbb{Z}, n \leqslant 2k \\ 0 & \text{else} \end{cases}$$

Definition 4.4.7. Let X, Y be connected surfaces. The surface $X\#Y$ is defined as follows:

1. Remove a contractible 2-disk D_1 from X and D_2 from Y.
2. Pick a homeomorphism $\mathbb{S}^1 \xrightarrow{\varphi} \mathbb{S}^1$
3. Glue $X \setminus int(D_1)$ to $Y \setminus int(D_2)$ via φ, so

$$X\#Y = X \setminus int(D_1) \bigsqcup_{\varphi} Y \setminus int(D_2)$$

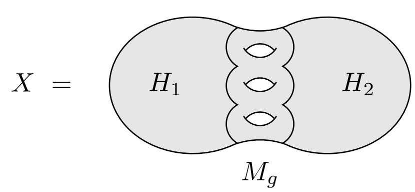
Remark 4.4.8. The connect sum $\#$ is well-defined with respect to the choice of φ. For example, $Kl = T^2\#\mathbb{R}P^2$, and $M_g = \underbrace{T^2\# \cdots \#T^2}_{g \text{ times}}$.



Now compute $H_n(M_g)$ From this construction. Recall that $\pi_1(M_g) = \mathbb{Z}^{*2g}$, so $H_1(M_g) = \left(\mathbb{Z}^{*2g}\right)_{ab} = \mathbb{Z}^{\oplus 2g}$.

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{\oplus 2g} & n = 1 \\ 0 & \text{else} \end{cases}$$

Definition 4.4.9. Let X be a connected 3-manifold. Then a Heegaard splitting is a way of expressing $X = H_1 \cup H_2$, where H_1 and H_2 are isomorphic, and $\partial(H_1) = \partial(H_2) = M_g$ for some $g \in \mathbb{N}$.



4.5 Relating $\pi_1(X)$ and $H_1(X)$

Given a topological space X with basepoint x_0, there is a natural map $h : \pi_1(X, x_0) \to H_1(X)$ that takes $[f]$ to $[f]$, where $f : I \to X$ is a loop based at x_0, so $\partial f = [x_0] - [x_0] = 0$.

Definition 4.5.1. Suppose that σ, σ' are two n-cycles in X, and there exists an $(n+1)$-cycle τ in X such that $\partial\tau = \sigma - \sigma'$. Then σ is termed homologous to σ', and the relation is expressed as $\sigma \sim \sigma'$. That is, $[\sigma] = [\sigma']$ i]n $H_n(X)$.

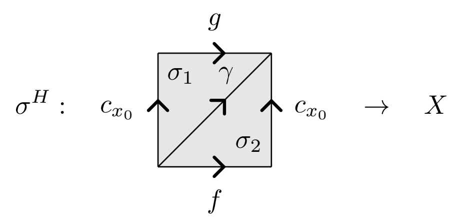
Theorem 4.5.2. The map $h : \pi_1(X, x_0) \to H_1(X)$ is a group homomorphism, and if X is path-connected, then h is surjective and $\ker(h) = [\pi_1(X, x_0), \pi_1(X, x_0)]$, and

$$H_1(X) = \pi_1(X, x_0) \Big/ [\pi_1(X, x_0), \pi_1(X, x_0)] = \pi_1(X, x_0)_{ab}$$

Proof: We make some basic observations first.

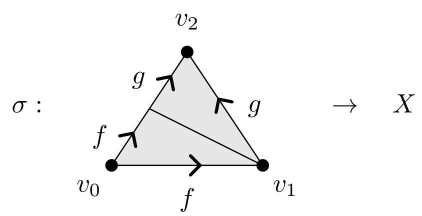
Obs.1: If $f = c_{x_0}$, then $f \sim 0$. That is, if $f = c_{x_0}$, then represents a cycle in $C_1(\{x_0\})$, and as we know $H_1(\{x_0\}) = 0$, it follows that $f \sim 0$.

Obs.2: $f \approx g$ implies $f \sim g$. That is, the existence of the homotopy$H : I \times I \to X$ with $H_0 = f$ and $H_1 = g$ is the data of a 2-chain $\sigma^H = \sigma_1 - \sigma_2$, given diagrammatically by:

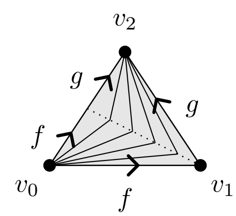


$$\partial\sigma^H = \partial\sigma_1 - \partial\sigma_2 = (c_{x_0} + g - \gamma) - (-\gamma + f + c_{x_0}) = g - f \iff g \sim f$$

Obs.3: $f \cdot g \sim f + g$. That is, the map h takes multiplication to addition, or $[h(fg)] = [h(f)] + [h(g)]$. This may be described by the following diagram:



where the action of σ is described by:

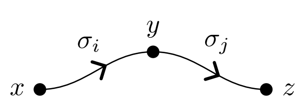


$$\partial\sigma = -f \cdot g + g + f \implies f \cdot g = g + f - \partial\sigma \implies f \cdot g \sim g + f$$

Obs.4: $\overline{f} \sim -f$. This follows from the previous observations.

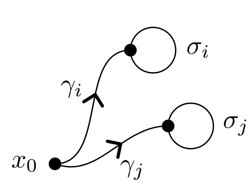
The above observations imply that h is a well-defined homomorphism from $\pi_1(X, x_0)$ to $H_1(X)$. Now suppose that X is path connected. We will now prove that h is surjective.

Let σ be a 1-cycle in $C_1(X)$, so $\sigma = \sum_i n_i\sigma_i$, for σ_i singular 1-simpleces and $n_i \in \mathbb{Z}$. By relabelling, we may assume that $n_i \in \{1, -1\}$. Further, since $\bar{f} \sim -f$, we may assume that $n_i = 1$ for all i, so $\sigma = \sum_i \sigma_i$. Lastly, we assume that $\partial\sigma = 0$. This means that for each i such that σ_i is not a loop, there is a j such that σ_j composes with σ_i.



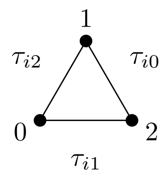
$$\begin{aligned} \partial\sigma_i &= [y] - [x] \\ \partial\sigma_j &= [z] - [y] \end{aligned}$$

Hence we may assume that $\sum_i \sigma_i$ is a sum of loops in X. Next, for each i, pick a path γ_i from x_0 to $\sigma_i(0)$.

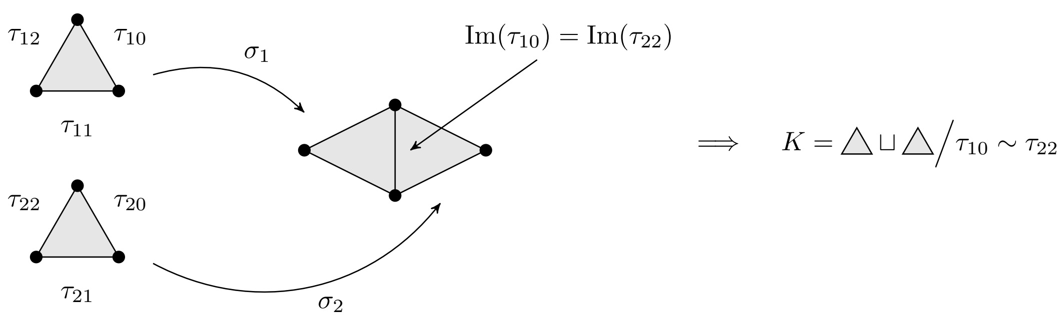


Then by the previous observations, $\gamma_i\sigma_i\overline{\gamma_i} \sim \sigma_i$, so $\sigma \sim \sum_i \gamma_i\sigma_i\overline{\gamma_i}$. Hence $[\sigma] \in \mathrm{Im}(h)$, and h is surjective.

Next we observe that $[\pi_1(X,x_0), \pi_1(X,x_0)] \subseteq \ker(h)$, as $H_1(X)$ is abelian. We claim that $\ker(h) \subseteq [\pi_1(X,x_0), \pi_1(X,x_0)]$. To see this, first suppose that $[f] \in \pi_1(X,x_0)$ and $h(f) = 0$, which implies that there exists a 2-chain $\sigma = \sum_i n_i\sigma_i$ such that $\partial\sigma = f$. As above, we assume that $n \in \{1,-1\}$, and additionally we assume that f appears as a single face of some σ_i. The σ_i are 2-simplices, and we may label their faces as follows:

$\sigma_i =$  $\to \quad X$

Recall that $\tau_{ij} = f$ for some j. Define a space $K = \sqcup_i \Delta^2/\varphi$, where φ identifies the faces of the simpleces that coincide in the image. For example,



Then we regard σ as going from K to X. Then it is possible to deform $\sigma : K \to X$ to a map $\sigma' : K \to X$ such that σ' maps all the vertices of K to x_0, and agrees with σ on f. This may be done by fixing a path from each vertex v_{ij} to x_0, which then defines a homotopy between $\sigma|_{f\cup K^o} \to X$ and $\sigma'|_{f\cup K^o} \to \mathrm{Im}(f) \cup \{x_0\}$.

Now we have $\sigma' : K \to X$ with all σ' loops at x_0, and as σ' is a class in $C_2(X)$, $\partial\sigma' = \sum_{i,j}(\pm 1)\tau_{ij}$, where the τ_{ij} are loops based at x_0, and these loops cancelling in pairs except for $f = \sum_{i,j}(\pm 1)\tau_{ij}$. But as $f = \sum_i \partial(\sigma_i)$, it follows that $[f] = 0$ in $\pi_1(X)_{ab}$, since $\partial(\sigma_1) = 0$ in $\pi_1(X)_{ab}$. This completes the proof. ■

Definition 4.5.3. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a map. The degree of f is the integer associated to f_* in $\mathrm{Hom}_{\mathbb{Z}}(H_n(\mathbb{S}^n), H_n(\mathbb{S}^n)) = \mathbb{Z}$. This map is induced canonically:

$$\begin{aligned} f : \mathbb{S}^n \to \mathbb{S}^n \rightsquigarrow f_* &: H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n) \\ &(\mathrm{id} : 1 \mapsto 1) \mapsto 1 \\ &(n : 1 \mapsto n) \mapsto n \end{aligned}$$

Example 4.5.4. Consider the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ given by $e^{i\theta} \mapsto e^{in\theta}$. Then $\deg(f) = n$.

Definition 4.5.5. Lat γ be a closed simple curve in $\mathbb{R}^2$, and p a point not on γ.

$$\begin{aligned} f_p : \quad \mathbb{S}^1 &\to \mathbb{S}^1 \\ \theta &\mapsto \frac{\overrightarrow{p\gamma(0)}}{\|\overrightarrow{p\gamma(0)}\|} \end{aligned}$$

Then $\deg(f_p)$ is termed the winding number of γ around p. The generalization of this map is termed the Gauss map.

More generally, the above implies that if X, Y are compact connected manifolds of dimension n, $H_n(X) = H_n(Y) = \mathbb{Z}$.

Definition 4.5.6. Let X be a space. Then the Euler characteristic of X is defined as the number

$$\chi(X) = \sum_i (-1)^i \text{rank}(H_i(X))$$

Here rank is the dimension of the free part of the argument. This agrees with the Euler characteristic for CW complexes. Moreover, the nth Betti number of X is $\text{rank}(H_n(X))$.

Definition 4.5.7. Let X be an n-dimensional topological space. An orientation of X is a generator for $\mathbb{Z}$, that is, either $[x]$ ar $[-x]$. The space X has no orientation of and only if it has no generator for $\mathbb{Z}$, or equivalently, if $H_n(X) = 0$.

5 An introduction to cohomology

5.1 The universal coefficient theorem

Definition 5.1.1. Let X be a topological space. From previously we know that $C_n(X)$, the singular chain groups of X, fit into the singular chain complex of X:

$$\cdots \xrightarrow{\delta} C_{n+1}(X) \xrightarrow{\delta} C_n(X) \xrightarrow{\delta} C_{n-1}(X) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C_1(X) \xrightarrow{\delta} C_0(X) \xrightarrow{\delta} 0$$

Let $C^n(X) = \text{Hom}_{\mathbb{Z}}(\mathbb{C}_n(X); \mathbb{Z})$, the let of linear functions on singular n-simpleces valued in $\mathbb{Z}$. This is an abelian group, or a $\mathbb{Z}$-module. It is also denoted $C_n^*(X)$, and is the dual of $C_n(X)$. These groups fit into what is termed a cochain complex, given by

$$\cdots \xleftarrow{\delta^*} C^{n+1}(X) \xleftarrow{\delta^*} C^n(X) \xleftarrow{\delta^*} C^{n-1}(X) \xleftarrow{\delta^*} \cdots \xleftarrow{\delta^{\;*}} C^1(X) \xleftarrow{\delta^*} C^0(X) \xleftarrow{\delta^*} 0$$

The boundary operator is described by $\delta^*(f)(\sigma) = (f \circ \delta)(\sigma)$, where $f \in C^n$ and σ is an $(n-1)$-simplex.

Definition 5.1.2. Using the definitions above, define the nth cohomology group of a space X,

$$H^n(X) = \underbrace{\ker(d : C^n \to C^{n+1})}_{\text{contains } n\text{-cocycles}} \Big/ \underbrace{\text{Im}(d : C_{n-1} \to C_n)}_{\text{contains } n\text{-coboundaries}}$$

Note in general $H^n(X) \ncong \text{Hom}_{\mathbb{Z}}(H_n(X); \mathbb{Z})$, although it sometimes might be the case. The actual relation is given by the following theorem.

Theorem 5.1.3. [Universal coefficient theorem]
Let G be a $\mathbb{Z}$-module. Then there exists a split short exact sequnce

$$0 \to \text{Ext}(H_{n-1}(X); G) \to H^n(X; G) \to \text{Hom}(H_n(X); G) \to 0$$

Equivalently, $H^n(X; G) = \text{Hom}(H_n(X); G) \oplus \text{Ext}(H_{n-1}(X); G)$.

Definition 5.1.4. Let G be a $\mathbb{Z}$-module. Define singular cohomology with coefficients in G to be the cohomology groups of the complex

$$\cdots \longleftarrow \text{Hom}(C_n; G) \xleftarrow{\delta^* = d} \text{Hom}(C_{n-1}; G) \xleftarrow{\delta^* = d} \cdots \longleftarrow \text{Hom}(C_0; G) \longleftarrow 0$$

The groups are denoted $H^n(X; G)$. If G is omitted, it is understood that $G = \mathbb{Z}$.

Definition 5.1.5. The group Ext has several important properties:

1. $\text{Ext}(H \oplus H'; G) = \text{Ext}(H; G) \oplus \text{Ext}(H'; G)$
2. $\text{Ext}(H; G) = 0$ if H is free
3. $\text{Ext}(\mathbb{Z}/n\mathbb{Z}; G) = G/nG$

Example 5.1.6. Consider the effect of the universal coefficient theorem with $G = \mathbb{Z}$ and $X = \mathbb{R}P^2$. Then $H^n(\mathbb{R}P^2) \cong \mathrm{Hom}(H_n(\mathbb{R}P^2);\mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(\mathbb{R}P^2);\mathbb{Z})$. From this and previously known facts we may calculate the cohomology groups:

$$\begin{aligned} H_0(\mathbb{R}P^2) = \mathbb{Z} &\implies H^0(\mathbb{R}P^2;\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z};\mathbb{Z}) \oplus \mathrm{Ext}(0;\mathbb{Z}) = \mathbb{Z} \\ H_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z} &\implies H^1(\mathbb{R}P^2;\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z};\mathbb{Z}) = 0 \\ H_2(\mathbb{R}P^2) = 0 &\implies H^1(\mathbb{R}P^2;\mathbb{Z}) = \mathrm{Hom}(0;\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

Here we see a key facet of cohomology groups. The torsion part of the nth homology group is always shifted to the $(n+1)$th cohomology group.

Definition 5.1.7. The object $H^\bullet = \bigoplus_n H^n(X;\mathbb{Z})$ is a graded ring, hence there exists a natural multiplication on it, termed the <u>cup product</u>, that respects the grading:

$$\begin{array}{rccl} \smile: & H^\bullet(X) \otimes H^\bullet(X) & \to & H^\bullet(X) \\ & \big(\alpha \in H^i(X)\big) \otimes \big(\beta \in H^j(X)\big) & \mapsto & \alpha \smile \beta \in H^{i+j}(X) \end{array}$$

If we consider the action on the chains,

$$\begin{array}{rcl} C_i(X) \otimes C_j(Y) & \to & C_{i+j}(X \times Y) \\ \sigma \otimes \tau & \mapsto & \sigma \times \tau \end{array} \qquad \text{yields maps} \qquad \begin{array}{rcl} H_i(X) \otimes H_j(X) & \to & H_{i+j}(X) \\ H^i(X) \otimes H^j(X) & \to & H^{i+j}(X) \end{array}$$

This comes from functoriality. Then the cup product is the composition of the induced map on cohomology (with $X = Y$) with another map Δ_*:

$$\smile: \ H^i(X) \otimes H^j(X) \to H^{i+j}(X \times X) \xrightarrow{\Delta_*} H^{i+j}(X)$$

The second map Δ_* is induced by the following inclusion map:

$$\begin{array}{rccc} \Delta: & X & \hookrightarrow & X \times X \\ & x & \mapsto & (x,x) \end{array}$$

More explicitly, the cup product is given by:

$$(\alpha \smile \beta)([v_0\ v_1\ \ldots\ v_{i+j}]) = \alpha([v_1\ \ldots\ v_i]) \cdot \beta([v_i\ \ldots\ v_{i+j}])$$

Multiplication is over $\mathbb{Z}$, and $[v_0\ v_1\ \ldots\ v_{i+j}]$ is a singular $(i+j)$-simplex in X. This action is well defined at the level of cohomology. That is, $[\alpha] \smile [\beta] = [\alpha \smile \beta]$. However, we require coefficients to be ins some ring for $\smile$ to be defined. There is a cup product on $H^\bullet(X;R)$ for any ring R.

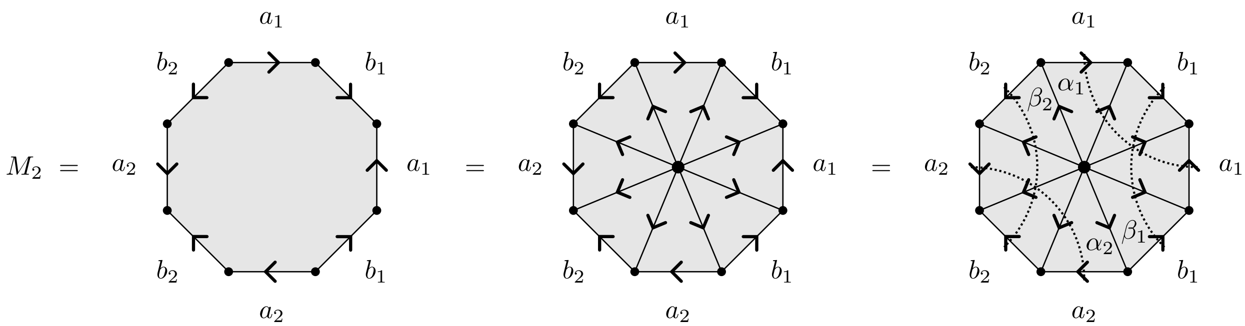
Example 5.1.8. We wish to compute $H^\bullet(M_2;\mathbb{Z})$ as a ring. First we compute the cohomology groups.

Via the universal coefficient theorem, $H^n(M_2;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(M_2);\mathbb{Z}) \cong H_n(M_2)$, as $H_n(M_2)$ is free for every n, i.e. has no torsion. Therefore the cohomology groups are

$$H^n(M_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0,2 \\ \mathbb{Z}^{\oplus 4} & n = 1 \\ 0 & \text{else} \end{cases}$$

Next we need to pick out generators for these cohomology classes. For $H^0(M_2;\mathbb{Z})$, the identity element is $e = 1 \in \mathbb{Z}$, which in $H^0(M_2;\mathbb{Z})$ corresponds to the function assigning the value 1 to any point in M_2.

For $H^1(M_2; \mathbb{Z}/2\mathbb{Z})$, we first fix a simpliciation of M_2 and define transverse curves α_i, β_i for $i = 1, 2$.



Here, α_1 represents the data of a 1-cocycle by the rule $\alpha_1(\tau) = \left\{\begin{smallmatrix} 1 & \text{if } \alpha_1 \text{ intersects } \tau \\ 0 & \text{else} \end{smallmatrix}\right.$, where τ is a 1-simplex. Now, using the explicit formula for the cup product, we may compute the following:

$$\alpha_i \smile \beta_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \alpha_i \smile \alpha_j = 0 \ \forall\ i, j \qquad \beta_i \smile \alpha_j = 0 \ \forall\ i, j$$

Note that $\alpha_i \smile \beta_j$ is generator of H^2, and is the dual of a fundamental cycle, which is a generator of H_2.

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