Algebraic and topological perspectives on the Khovanov homology

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April 26, 2013

Abstract

We investigate the Khovanov homology, introduced in [4], of an embedded link. A detailed computation for the trefoil is provided, along with two diffirent proofs of invariance under Reidemeister moves - using a Frobenius algebra and a 2 -dimensional topological quantum field theory.

Contents

0.0 Introduction

This is an expository paper on the properties of Khovanov's homology of a knot, with extensions to tangles. The approach mirrors [2] and [3], but with a more complete analysis. The method of this paper may be described as understanding and proving the commutativity of the following diagram:

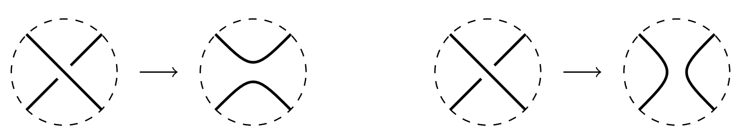
$$
\begin{tikzcd}
\text{(a link } L\text{)} \arrow[d, "\text{embedding in the plane}"] & & & J(L) \\
\text{(a diagram } D\text{)} \arrow[r, "\text{resolutions}"] \arrow[urrr, bend left, "\text{Jones polynomial}"] \arrow[d, "\text{Reidemeister moves}"] & \left(\begin{array}{c}\text{a cubical}\\ \text{simplicial}\\ \text{complex}\\ \text{in } 2Cob\end{array}\right) \arrow[r, "\text{field theory}"] & \left(\begin{array}{c}\text{a cubical}\\ \text{simplicial}\\ \text{complex}\\ \text{in Vect}\end{array}\right) \arrow[r, "\text{cohomology}"] & \left(\begin{array}{c}\text{Khovanov's}\\ \text{homology}\\ \text{of } L\end{array}\right) \arrow[u, "\text{Euler charateristic}"] \arrow[d, "\text{Poincare polynomial}"] \\
\text{(a diagram } D'\text{)} \arrow[r, "\text{resolutions}"'] & \left(\begin{array}{c}\text{a cubical}\\ \text{simplicial}\\ \text{complex}\\ \text{in } 2Cob\end{array}\right) \arrow[r, "\text{field theory}"'] & \left(\begin{array}{c}\text{a cubical}\\ \text{simplicial}\\ \text{complex}\\ \text{in Vect}\end{array}\right) \arrow[ur, bend right, "\text{cohomology}"'] & Kh(L)
\end{tikzcd}
$$

The terms introduced here are explained further below, with the exception of Vect, the category of vector spaces. It is used throughout, but further no reference will be made to it. Also, the reduction of Khovanov's homology to the Jones polynomial will only be discussed briefly at the very end.

Credit is given to David M.R. Jackson and Aaron Smith for endless support and advice on the material. The proofs (the last two at the end of the paper) that are left incomplete are as such solely due to time and mental constraints of the author.

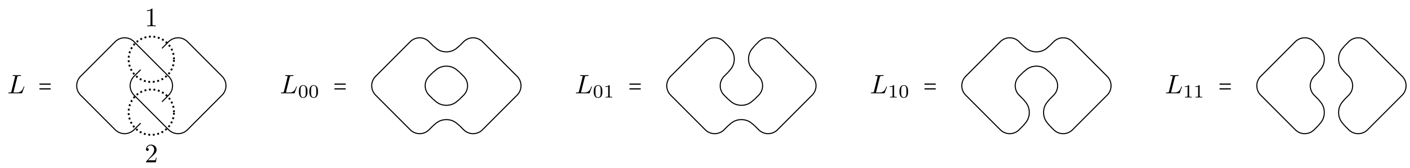
1 Motivating remarks

Definition 1.1. (*resolution*) Given a link L embedded in the plane as a diagram D, and a crossing c of the D, a resolution at c is a small neighborhood of $c$areplaced by one of two neighborhoods. The neighborhoods are called a 0-resolution and a 1-resolution.

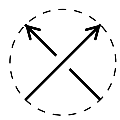


0-resolution 1-resolution

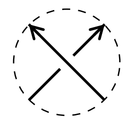
A diagram D with resolutions at all crossings is termed a complete resolution of D, or simply a resolution of D. If we order all the k crossings of a diagram D, every complete resolution of D may be assigned a string in $\{0,1\}^k$. For example



Definition 1.2. $(x(L), y(L))$ Given a link L and an orientation on each component of L, an embedding in the plane of L yields at mosttwo types of crossings. Let $x(L)$ denote the number of crossings of the first type, and $y(L)$ the number of crossings of the second type.



contributes 1 to $x(L)$



contributes 1 to $y(L)$

Although $x(L)$ and $y(L)$ are dependent upon the embedding D of L, we will use the described notation, hopefully without confusion.

1.1 Associating a link to a chain complex

Given a link L and a complete resolution of its embedded diagram D, we will have a number of simple closed curves that do not intersect in the plane. To each such curve we may associate a 2-dimensional vector space, and express the whole diagram as a tensor product of the assigned vector spaces. Note that in the case of tangles, we may not make such an assumption in general.

If D has n crossings, then we may form a chain group by taking the direct sum for $m = 0, \dots, n$ of all the spaces with m 0-resolutions and $(n-m)$ 1-resolutions. We may then form a cochain complex indexed by m, with an appropriate boundary operator, described in more detail in the next section.

It is natural to take the cohomology of this cochain complex, and in this case, the resulting cohomology theory is termed the Khovanov homology of L. Instead of using the whole theory as an invariant, we may represent the main information in a polynomial.

Definition 1.3. $(Kh(L))$ Given a link L and a chain complex $C(L)$ associated to an embedding of L, define the Khovanov homology of L to be

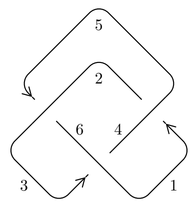
$$Kh(L) = \sum_r t^r \operatorname{qdim}(H^r(C(L)))$$

This is a polynomial (termed the Khovanov polynomial) in t and q. The q-dimension comes into play by, given a graded (over k) vector space W, letting $\operatorname{qdim}(W) = \sum_k q^k \dim(W_k)$.

2 An extended example

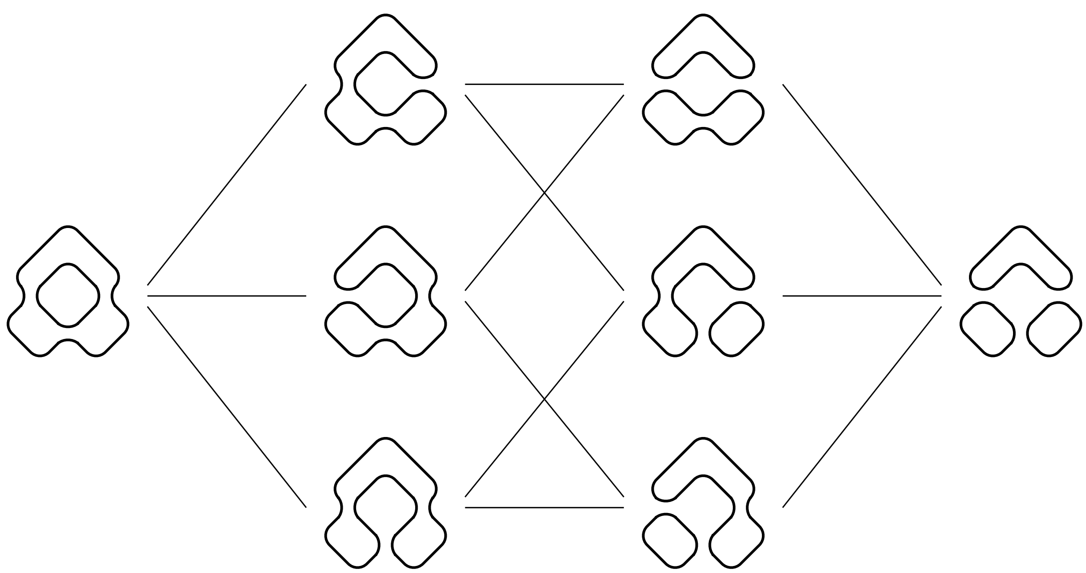
In this section we will go through a meticulous calculation of the Khovanov homology of a knot. Let L be the trefoil knot. Give L an embedding and an orientation, and label the edges between intersections in sequence, choosing an arbitrary starting point. We choose to fix the data as below.

L =



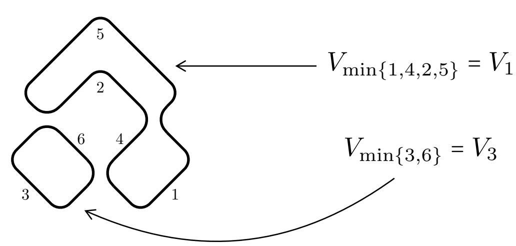
(2.1)

This embedding of L has three intersections, and each intersection has one of two possible resolutions, so there are $2^3 = 8$ diagrams, associated with L, of simple closed curves that do not intersect each other. Arrange these 8 diagrams on the vertices of a 3-square so that the strings in $\{0,1\}^3$ associated with ends of each edge differ in exactly one coordinate. The presentation in (2.2) does this, also placing diagrams with the same number of 1-resolutions in the same column, beginning with no 1-resolutions in the left-most column.



(2.2)

To each diagram associate a tensor product of as many vector spaces V as connected components (equivalently, closed curves). This space is $V = \mathrm{span}\{v_+, v_-\}$, where $\deg_V(v_+) = 1$ and $\deg_V(v_-) = -1$. The assigned vector spaces are labeled V_i, where the index i is the smallest edge label of edges contained in the component. For example,

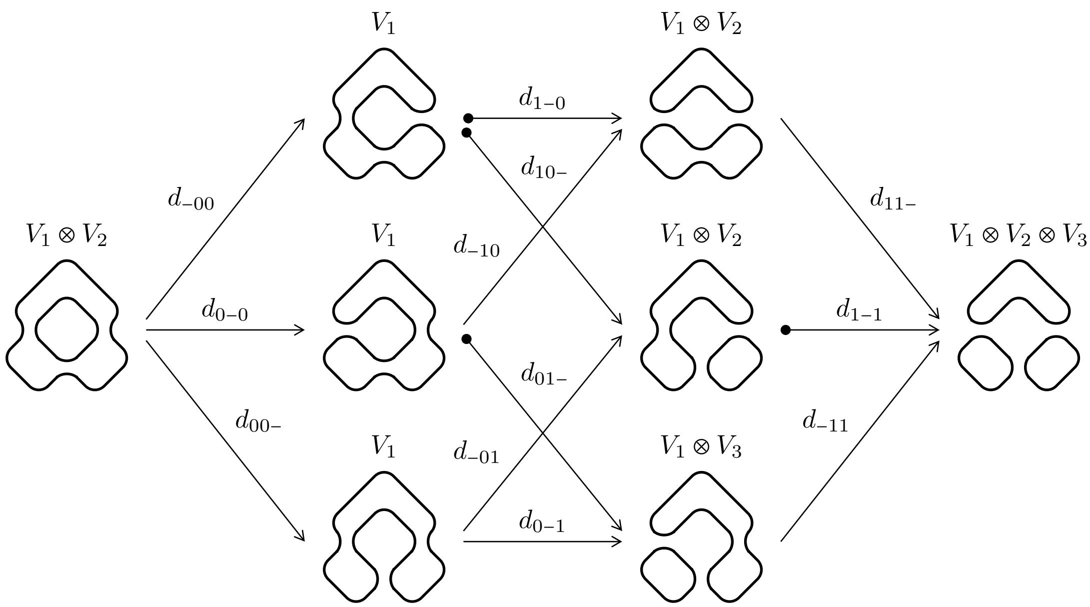


The vector spaces are arranged in increasing numerical order, so the diagram above is labelled $V_1 \otimes V_3$.

Next, to each edge of the 3-cube we associate a boundary map, moving left to right (left to right duo to the specific presentation in (2.2)). These maps are between spaces with a different number of tensored spaces V_i, so we need a standard way of increasing and decreasing the number of tensored spaces. This is done by using a multipliciation and a co-multiplication, defined as follows:

$$
\begin{array}{rccl}
m: & V \otimes V & \to & V \\
& v_- \otimes v_- & \mapsto & 0 \\
& v_- \otimes v_+ & \mapsto & v_- \\
& v_+ \otimes v_- & \mapsto & v_- \\
& v_+ \otimes v_+ & \mapsto & v_+
\end{array}
\qquad
\begin{array}{rccl}
\Delta: & V & \to & V \otimes V \\
& v_- & \mapsto & v_- \otimes v_- \\
& v_+ & \mapsto & v_+ \otimes v_- + v_- \otimes v_+
\end{array}
\tag{2.3}
$$

We present the cube of diagrams of (2.2) again with all maps and spaces labeled in (2.4). The boundary maps d_α for a string $\alpha \in \{0,1,-\}^3$ are labeled to indicate which intersections do and do not change resolution types.



(2.4)

The edges maps d_α explicitly are:

$$
\begin{array}{ll}
d_{-00}: & v_1 \otimes v_2 \mapsto m(v_1 \otimes v_2) \\
d_{0-0}: & v_1 \otimes v_2 \mapsto m(v_1 \otimes v_2) \\
d_{0-0}: & v_1 \otimes v_2 \mapsto m(v_1 \otimes v_2)
\end{array}
\qquad
\begin{array}{ll}
d_{1-0}: & v_1 \mapsto \Delta(v_1) \\
d_{10-}: & v_1 \mapsto \Delta(v_1) \\
d_{-10}: & v_1 \mapsto \Delta(v_1) \\
d_{01-}: & v_1 \mapsto \Delta(v_1) \\
d_{-01}: & v_1 \mapsto \Delta(v_1) \\
d_{0-1}: & v_1 \mapsto \Delta(v_1)
\end{array}
\qquad
\begin{array}{ll}
d_{11-}: & v_1 \otimes v_2 \mapsto \Delta(v_1)_{[1]} \otimes v_2 \otimes \Delta(v_1)_{[2]} \\
d_{1-1}: & v_1 \otimes v_2 \mapsto v_1 \otimes \Delta(v_2) \\
d_{-11}: & v_1 \otimes v_3 \mapsto \Delta(v_1) \otimes v_3
\end{array}
$$

The symbol $\Delta(v_i)_{[j]}$ represents the jth of the two tensored elements in $\Delta(v_i)$. The expression $\Delta(v_1)$ in the case of the map d_{11-} had to be split, as the component labeled V_1 becomes $V_1 \otimes V_3$, but due to ordering, V_2 is tensored between them in the image.

The next step is to combine the spaces into chain groups, and "collapse" the whole cube into a chain complex. Each column of diagrams will become a space represented by the direct sum of the associated vector spaces:

$$C^0(L) = V^{\otimes 2} \qquad C^1(L) = V^{\oplus 3} \qquad C^2(L) = (V^{\otimes 2})^{\oplus 3} \qquad C^3(L) = V^{\otimes 3}$$

The superscript m in $C^m(L)$ indicates how many 1-resolutions the the associated diagrams have. Further, each set of maps between a pair of columns becomes a boundary map of the cochain complex, by an adjusted sum of the maps d_α. Defining the new boundary maps d^i, we introduce $|\alpha|$, which represents the sum of the elements in the string α, with "−" having value 0, and $\bar{\alpha}$, a coefficient that ensures anti-commutativity among the subcomplexes. Anti-commutativity in the cube of diagrams in (2.4) is necessary for the cochain boundary map te have the property $d^r \circ d^{r-1} = 0$.

$$d^r : C^r \to C^{r+1} = \sum_{|\alpha|=r} (-1)^{\bar{\alpha}} d_\alpha \qquad\qquad \bar{\alpha} = \sum_{i \,<\, \text{index of } -} |\alpha_i|$$

Note that in (2.4) the maps d_α with $(-1)^{\bar{\alpha}} = -1$ are indicated by a dot $\bullet$ at one end. Hence we have:

$$\begin{aligned} d^0 &= d_{-00} + d_{0-0} + d_{00-} \\ d^1 &= -d_{1-0} - d_{10-} + d_{-10} - d_{01-} + d_{-01} + d_{0-1} \\ d^2 &= d_{11-} - d_{1-1} + d_{-11} \end{aligned}$$

It remains to check that these maps are indeed boundary maps, i.e. that they satisfy the chain complex boundary map condition.

Proposition 2.1. The map d^r is a boundary map. Equivalently, $d^{r+1} \circ d^r(x) = 0$ for all $x \in C^r(L)$.

Proof: Proving this statement is equivalent to making each "face" of the whole cube in (2.4) anti-commute. Since each face has an odd number of boundary maps with negative multipliers, the faces do indeed anti-commute. ■

These maps and chain groups produce a cochain complex, termed $[\![L]\!]$:

$$V^{\otimes 2}\{0\} \xrightarrow{d^0} V^{\oplus 3}\{1\} \xrightarrow{d^1} \left(V^{\otimes 2}\right)^{\oplus 3}\{2\} \xrightarrow{d^2} \left(V^{\otimes 3}\right)\{3\}$$

The elements in the brackets $\{\cdot\}$ denote the degree shift of each complex, which encodes the number of 1-resolutions of diagrams associated to the each cochain space. We construct another, adjusted cochain complex, termed $C(L)$, where each cochain space $W\{i\}$ from $[\![L]\!]$ becomes $W\{i + x(L) - 2y(L)\}$, and each boundary operator d^r from $[\![L]\!]$ becomes $d^{r-y(L)}$. In our case, $x(L) = 3$ and $y(L) = 0$, so $C(L)$ is the complex

$$V^{\otimes 2}\{3\} \xrightarrow{d^0} V^{\oplus 3}\{4\} \xrightarrow{d^1} \left(V^{\otimes 2}\right)^{\oplus 3}\{5\} \xrightarrow{d^2} \left(V^{\otimes 3}\right)\{6\}$$

Definition 1.3 now follows naturally, as $Kh(L) = \sum_r t^r \mathrm{qdim}(H^r(C(L)))$ is the Poincare polynomial of the graded cochain complex $C(L)$.

To calculate the rth cohomolgy of $C(L)$, it is necessary to calculate $\mathrm{im}(d^r)$ and $\ker(d^r)$ for all $r = 0, 1, 2$ and then take the appropriate quotients. This we will now do in explicit detail. The tensor symbol $\otimes$ is omitted where convenient.

$$\boxed{d^0 : V^{\otimes 2}\{3\} \to V^{\oplus 3}\{4\}}$$

$$\begin{bmatrix} d_{-00} \\ d_{0-0} \\ d_{00-} \end{bmatrix} \begin{bmatrix} v_- v_- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} d_{-00} \\ d_{0-0} \\ d_{00-} \end{bmatrix} \begin{bmatrix} v_- v_+ \end{bmatrix} = \begin{bmatrix} v_- \\ v_- \\ v_- \end{bmatrix} \quad \begin{bmatrix} d_{-00} \\ d_{0-0} \\ d_{00-} \end{bmatrix} \begin{bmatrix} v_+ v_- \end{bmatrix} = \begin{bmatrix} v_- \\ v_- \\ v_- \end{bmatrix} \quad \begin{bmatrix} d_{-00} \\ d_{0-0} \\ d_{00-} \end{bmatrix} \begin{bmatrix} v_+ v_+ \end{bmatrix} = \begin{bmatrix} v_+ \\ v_+ \\ v_+ \end{bmatrix}$$

$$\begin{aligned} \mathrm{im}(d^0) &= \langle v_-^{\oplus 3}, v_+^{\oplus 3} \rangle & \qquad \mathrm{qdim}(\mathrm{im}(d^0)) &= q^3 + q^5 \\ \ker(d^0) &= \langle v_- v_-, v_- v_+ - v_+ v_- \rangle & \qquad \mathrm{qdim}(\ker(d^0)) &= q^1 + q^3 \end{aligned}$$

$$\boxed{d^1 : V^{\oplus 3}\{4\} \to \left(V^{\otimes 2}\right)^{\oplus 3}\{5\}}$$

$$\begin{bmatrix} -d_{1-0} & d_{10-} & 0 \\ -d_{10-} & 0 & -d_{-01} \\ 0 & d_{01-} & d_{0-1} \end{bmatrix}\begin{bmatrix} v_- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -v_-v_- \\ -v_-v_- \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -d_{1-0} & d_{10-} & 0 \\ -d_{10-} & 0 & -d_{-01} \\ 0 & d_{01-} & d_{0-1} \end{bmatrix}\begin{bmatrix} v_+ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -v_+v_- - v_-v_+ \\ -v_+v_- - v_-v_+ \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -d_{1-0} & d_{10-} & 0 \\ -d_{10-} & 0 & -d_{-01} \\ 0 & d_{01-} & d_{0-1} \end{bmatrix}\begin{bmatrix} 0 \\ v_- \\ 0 \end{bmatrix} = \begin{bmatrix} v_-v_- \\ 0 \\ -v_-v_- \end{bmatrix} \qquad \begin{bmatrix} -d_{1-0} & d_{10-} & 0 \\ -d_{10-} & 0 & -d_{-01} \\ 0 & d_{01-} & d_{0-1} \end{bmatrix}\begin{bmatrix} 0 \\ v_+ \\ 0 \end{bmatrix} = \begin{bmatrix} v_+v_+ - v_-v_+ \\ 0 \\ -v_+v_- - v_-v_+ \end{bmatrix}$$

$$\begin{bmatrix} -d_{1-0} & d_{10-} & 0 \\ -d_{10-} & 0 & -d_{-01} \\ 0 & d_{01-} & d_{0-1} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_- \end{bmatrix} = \begin{bmatrix} 0 \\ v_-v_- \\ v_-v_- \end{bmatrix} \qquad \begin{bmatrix} -d_{1-0} & d_{10-} & 0 \\ -d_{10-} & 0 & -d_{-01} \\ 0 & d_{01-} & d_{0-1} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_+ \end{bmatrix} = \begin{bmatrix} 0 \\ v_+v_- + v_-v_+ \\ v_+v_- + v_-v_+ \end{bmatrix}$$

$$\mathrm{im}(d^1) = \left\langle \begin{bmatrix} -v_-v_- \\ -v_-v_- \\ 0 \end{bmatrix}, \begin{bmatrix} v_-v_- \\ 0 \\ -v_-v_- \end{bmatrix}, \begin{bmatrix} -v_+v_- - v_-v_+ \\ -v_+v_- - v_-v_+ \\ 0 \end{bmatrix}, \begin{bmatrix} v_+v_+ - v_-v_+ \\ 0 \\ -v_+v_- - v_-v_+ \end{bmatrix} \right\rangle \qquad \mathrm{qdim}(\mathrm{im}(d^1)) = 2q^3 + 2q^5$$

$$\ker(d^1) = \left\langle \begin{bmatrix} v_- \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_- \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_- \end{bmatrix}, \begin{bmatrix} v_+ \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_+ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_+ \end{bmatrix} \right\rangle \qquad \mathrm{qdim}(\ker(d^1)) = q^3 + q^5$$

$$\boxed{d^2 : \left(V^{\otimes 2}\right)^{\oplus 3}\{5\} \to V^{\otimes 3}\{6\}}$$

$$\begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} v_-v_- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_-v_-v_- \end{bmatrix} \qquad \begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} v_-v_+ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_-v_+v_- \end{bmatrix}$$

$$\begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} v_+v_- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_+v_-v_- + v_-v_-v_+ \end{bmatrix} \qquad \begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} v_+v_+ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} v_+v_+v_- + v_-v_+v_+ \end{bmatrix}$$

$$\begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ v_-v_- \\ 0 \end{bmatrix} = \begin{bmatrix} -v_-v_-v_- \end{bmatrix} \qquad \begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ v_-v_+ \\ 0 \end{bmatrix} = \begin{bmatrix} -v_-v_+v_- - v_-v_-v_+ \end{bmatrix}$$

$$\begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ v_+v_- \\ 0 \end{bmatrix} = \begin{bmatrix} -v_+v_-v_- \end{bmatrix} \qquad \begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ v_+v_+ \\ 0 \end{bmatrix} = \begin{bmatrix} -v_+v_+v_- - v_+v_-v_+ \end{bmatrix}$$

$$\begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_-v_- \end{bmatrix} = \begin{bmatrix} v_-v_-v_- \end{bmatrix} \qquad \begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_-v_+ \end{bmatrix} = \begin{bmatrix} v_-v_-v_+ \end{bmatrix}$$

$$\begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_+v_- \end{bmatrix} = \begin{bmatrix} v_+v_-v_- + v_-v_+v_- \end{bmatrix} \qquad \begin{bmatrix} d_{11-} & -d_{1-1} & d_{-11} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ v_+v_+ \end{bmatrix} = \begin{bmatrix} v_+v_-v_+ + v_-v_+v_+ \end{bmatrix}$$

$$\mathrm{im}(d^2) = \langle [v_-v_+v_-], [v_-v_+v_-], [-v_+v_-v_-], [v_-v_-v_+], [v_+v_-v_- + v_-v_-v_+], [-v_+v_+v_- - v_+v_-v_+], [v_+v_-v_+ + v_-v_+v_+] \rangle$$

$$\begin{aligned}\ker(d^2) = \Bigg\langle & \begin{bmatrix} v_-v_- \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_-v_- \\ 0 \end{bmatrix}, \begin{bmatrix} v_-v_- \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ v_-v_- \end{bmatrix}, \begin{bmatrix} v_-v_+ \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} v_+v_- \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_-v_+ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_+v_- \\ 0 \end{bmatrix}, \\ & -2\begin{bmatrix} v_-v_+ \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} v_+v_- \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ v_-v_+ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_+v_- \end{bmatrix}, \begin{bmatrix} v_-v_+ \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_-v_+ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_-v_+ \end{bmatrix} \Bigg\rangle\end{aligned}$$

$$\mathrm{qdim}(\mathrm{im}(d^2)) = 5q^5 + 2q^7 \qquad \mathrm{qdim}(\ker(d^2)) = 2q^3 + 3q^5$$

Now we calculate the cohomology groups.

$$t^0\mathrm{qdim}(H^0(C(L))) = \mathrm{qdim}(\ker(d^0)) = q^1 + q^3$$

More explicitly, q has the exponent 1 because in the kernel $\ker(d^0)$, the element v_-v_- has degree -2, and since it is located in $V^{\otimes 2}\{3\}$ which has a degree shift of 3, the qdim of that part is $q^{-2+3} = q^1$. The same analogously goes for the other exponent.

$$t^1\mathrm{qdim}(H^1(C(L))) = t^1\mathrm{qdim}(\ker(d^1)/\mathrm{im}(d^0)) = t^1\mathrm{qdim}(1) = 0$$

This follows as $\ker(d^1) = \mathrm{im}(d^0)$.

$$t^2\mathrm{qdim}(H^2(C(L))) = t^2\mathrm{qdim}(\ker(d^2)/\mathrm{im}(d^1)) = t^2\mathrm{qdim}(V\{5\}) = t^2q^5$$

This follows as the first three terms, as presented above, in both $\ker(d^2)$ and $\mathrm{im}(d^1)$ are the same, and the last term in $\mathrm{im}(d^1)$ is a linear combination of the last two terms in $\ker(d^2)$, which both have dimension 0, so the resulting space has only one of these terms, of dimension $0 + 5$.

$$t^3\mathrm{qdim}(H^3(C(L))) = t^3\mathrm{qdim}(\ker(d^3)/\mathrm{im}(d^2)) = t^3q^9$$

This follows as $\ker(d^3) = V^{\otimes 3}\{6\}$, the whole space to which d^2 maps, and the only basis element of this space that is not in the image of d^2 is $v_+v_+v_+$. This element has degree 3 in $V^{\otimes 3}\{6\}$, so its q-dimension is $q^{3+6} = q^9$. Therefore

$$Kh(L) = q^1 + q^5 + t^2q^5 + t^3q^9$$

This completes the example.

We now move to consider the tools applied here in a more abstract setting, describing the natural relations between the structures introduced here, and existing ones in other fields of study.

3 Parallels to other branches of mathematics

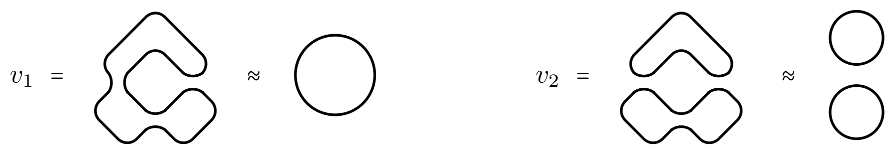
3.1 2-dimensional topological quantum field theories

In the interests of generality (while noting the lack of its recurrence in this paper), we present Atiyah's abstract definition, from [1], of a 2-dimensional topological quantum field theory over a ring Λ. It is:

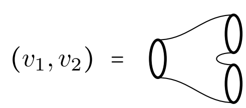
· A finitely-generated Λ-module $\Sigma(Z)$ associated to each oriented closed smooth 1-dimensional manifold Σ
· An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth 2-dimensional (with boundary) manifold M

Moreover, Z is functorial, involutory, and multiplicative. It may be viewed as a monoidal functor $Z : Cob_{1+1} \to Mod_R$ from the category of closed, oriented 1-manifolds.

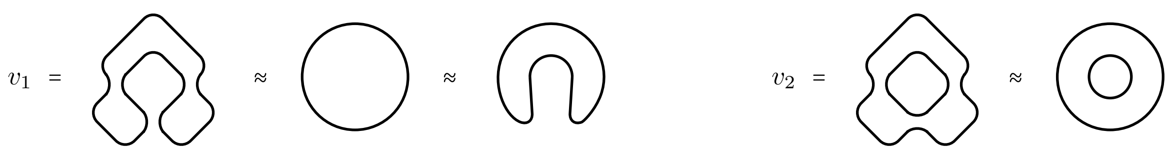
In our case, $\Sigma(Z) = V$, the 1-dimensional manifolds are closed simple curves, and hence the 2-dimensional manifolds are simply manifolds that connect these curves, cobordisms. Each vertex of a cube of resolutions, as in (2.2), is a collection of closed simple curves, or cycles, so the idea of a cobordism between the vertices may be applied to each edge. Take, for example, two adjacent vertices v_1 and v_2 from (2.2).



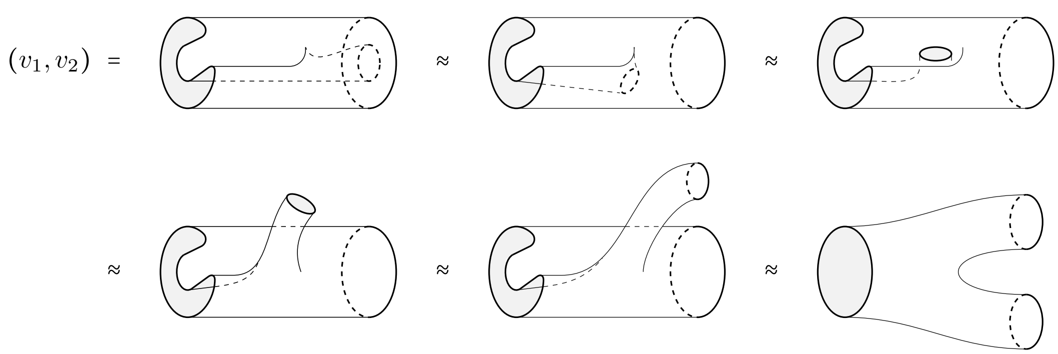
Being adjacent means they are cobordant, so the edge (v_1, v_2) may be represented as:



This cobordism is termed the pair of pants. Moreover, every edge of a cube of resolutions may be associated with a pair of pants (or an inverse pair of pants - given by the edge from v_2 to v_1 above) and a finite number of cylinders, where a cylinder is a cobordism between ◯ and ◯. This is almost immediately clear, with the observation that if one vertex has a nested pair of circles, then the cobordism relating it an adjacent vertex without this nested pair is still a pair of pants. Another example from (2.2) demonstrates this.



The smaller circle nested inside the larger circle is brought out for the cobordism to resemble the standard presentation of the pair of pants.



However, once we associate cycles in the TQFT to elements in the Frobenius algebra, the issue of an ambient space becomes irrelevant, as the algebra is associative. Although the above diagram helps visualize the scenario, we may simply move the nested cycle to the outside, while keeping both cycles on the same plane.

3.2 The Frobenius algebra

A Frobenius algebra A (see [6] for a very thorough investigation) is an associative bi-algebra over a field K with a bilinear form (termed the Frobenius form). Being a bi-algebra, it has a multiplication $A \otimes A \to A$ and a co-multiplication $A \to A \otimes A$. As previously we had pairs of vector spaces combining, and vector spaces splitting in two, the relation is natural.

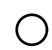
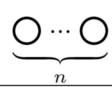
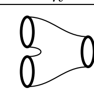
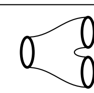
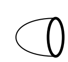
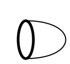
Indeed, the geometric intuition is identical to that of topological quantum field theories presented above, and as in [6], the identification is often made immediately. As defined in (2.3), the co-multiplication Δ corresponds to the pair of pants and the multiplication m corresponds to the inverse pair of pants.

3.3 The category $nCob$

There is also a relation to category theroy, where $nCob$ is the category containing n-dimensional cobordisms as morphisms, and $(n-1)$-dimensional manifolds as objects. The category $2Cob$ is the most relevant here, where the objects are all isomorphically circles, or cycles, and the morphisms are the 2-dimensional cobordisms already introduced.

3.4 A heuristic translation scheme

We now present a table comparing analogous ideas across the fields mentioned above. The relations are not meant as strict identifications, but rather as analogies of the same idea through different interpretations.

category theory	**algebra**	**cohomology theory**	**topology**
an object in $2Cob$	an algebra	a chain group of a single-component diagram	
n identical objects in $2Cob$	the tensor product of n identical algebras	a chain group of an n-component diagram	
a morphism from 2 objects to 1 object	multiplication	a boundary map $d : V \otimes V \to V$	
a morphism from 1 object to 2 objects	co-multiplication	a boundary map $d : V \to V \otimes V$	
the unit in a symmetric monoidal category	the unit		
the counit in a symmetric monoidal category	the co-unit		

Note that the last two cells in the cohomology theory column have been left empty. The interested reader is invited to check, ameliorate, and complete this table of concepts.

4 Invariance of the Khovanov homology

In this section we inspect the effect on the Khavonov homology under Reidemeister's three moves $R1$, $R2$, and $R3$. Extensive use will be made of the following proposition.

Proposition 4.1. Let C be a chain complex with a subcomplex C'. If C' has no homology, then $H^\bullet(C) = H^\bullet(C/C')$. Further, if C/C' has no homolgy, then $H^\bullet(C) = H^\bullet(C')$.

Proof: Consider the following short exact sequence of cochain complexes.

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C/C' \longrightarrow 0$$

To every cochain complex we may associate a cohomolgy group of every degree, giving a long exact sequence:

$$\cdots \longrightarrow H^{r-1}(C/C') \longrightarrow H^r(C') \longrightarrow H^r(C) \longrightarrow H^r(C/C') \longrightarrow H^{r+1}(C') \longrightarrow \cdots$$

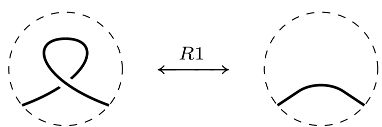
Suppose that C' is acyclic, so $H^r(C') = 0$ for all r. Since the long sequence is exact, the exact sequence $0 \to H^r(C) \to H^r(C/C') \to 0$ repeats for all r, and hence $H^\bullet(C) = H^\bullet(C/C')$.

Similarly if C/C' is acyclic, then $H^r(C/C') = 0$ for all r, and $0 \to H^r(C') \to H^r(C) \to 0$ is an exact sequence for all r. As above, this means that $H^\bullet(C) = H^\bullet(C')$. ∎

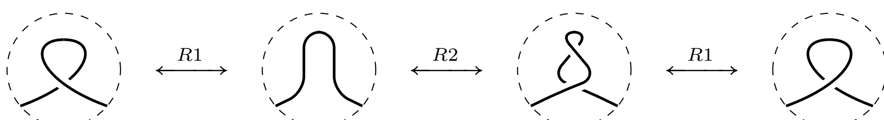
This will allow us to reduce large complexes to a more manageable size, while preserving all the relevant information. We now fix a link L and a diagram D of L.

4.1 Invariance under $R1$

Recall that $R1$ removes a "loop" from the knot. If we consider only a portion of a knot diagram, and leave everything outside the dashed circle constant, then



In this section we will consider only the right-twist loop, as under the Reidemeister moves, it is the same as the left-twist loop:

right-twist =


= left-twist

Let D be a link diagram embedded with an occurence of . Define

C = $[\![\ ]\!]$ = the cochain complex associated with D
· boundary map is $d_C^\bullet : C^\bullet \to C^{\bullet+1}$

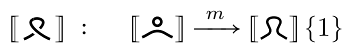
A = $[\![\ ]\!]$ = the cochain complex associated with D having a 0-resolution at the presented crossing
· boundary map is $d_A^\bullet : A^\bullet \to A^{\bullet+1}$

B = $[\![\ ]\!]$ = the cochain complex associated with D having a 1-resolution at the presented crossing
· boundary map is $d_B^\bullet : B^\bullet \to B^{\bullet+1}$

It is clear that each of A, B, C are cochain complexes with proper boundary maps, i.e. $d_A^{\bullet+1} d_A^\bullet = 0$ (equivalently for B and C). Recall that n is the number of crossings of D, so we may write down the cochain complex of C:

$$0 \longrightarrow [\![\ ]\!]^0 \xrightarrow{d_C^0} [\![\ ]\!]^1 \xrightarrow{d_C^1} \cdots \xrightarrow{d_C^{n-2}} [\![\ ]\!]^{n-1} \xrightarrow{d_C^{n-1}} [\![\ ]\!]^n \longrightarrow 0$$

The cochain complexes of A and B are similar, but with cochain spaces only up to index $n-1$. Next, we claim that C is a cochain complex with cochain spaces A and B, and boundary map m:



(4.1)

It is easier to see this by taking two copies of D, and giving resolutions to each crossing the same way in both copies, except for the one that has the right-twist loop in it. Then the copy with a 0-resolution at this crossing has one more cycle than the other copy, meaning that the multiplication map $m^\bullet : A^\bullet \to B^\bullet$ must be applied to move from the first copy to the other. This gives a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{\bullet+1}"] & A^\bullet \arrow[r, "d_A^\bullet"] \arrow[d, "m^\bullet"] & A^{\bullet+1} \arrow[r, "d_A^{\bullet-1}"] \arrow[d, "m^{\bullet+1}"] & \cdots \\
\cdots \arrow[r, "d_B^{\bullet+1}"'] & B^\bullet \arrow[r, "d_B^\bullet"'] & B^{\bullet+1} \arrow[r, "d_B^{\bullet-1}"'] & \cdots
\end{tikzcd}$$

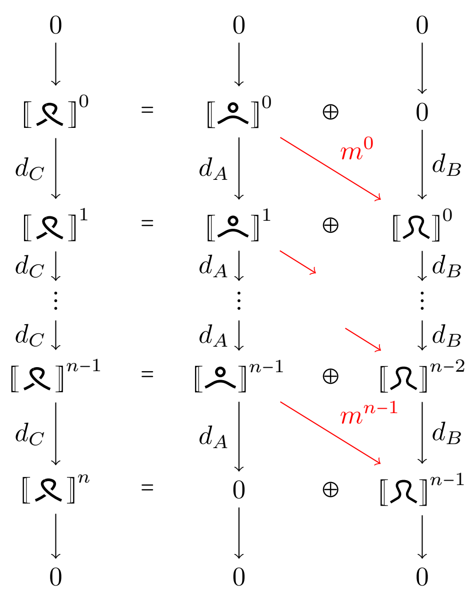
This implies that $m^{\bullet+1} d_A^\bullet = d_B^\bullet m^\bullet$. Using this, we have another expression for $d^\bullet C$, namely

$$d_C^\bullet = \begin{bmatrix} d_A^\bullet & 0 \\ m^\bullet & (-1)d_B^{\bullet-1} \end{bmatrix} \quad \text{with} \quad d_C^{\bullet+1} d_C^\bullet = \begin{bmatrix} d_A^{\bullet+1} d_A^\bullet & 0 \\ m^{\bullet+1} d_A^\bullet - d_B^\bullet m^\bullet & d_B^\bullet d_B^{\bullet-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.2}$$

Using the previous arguments, we may also express each cochain space of C as a direct sum of cochain spaces of A and B.

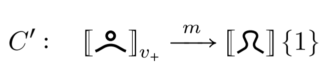
$$C^\bullet = A^\bullet \oplus B^\bullet\{1\} = A^\bullet \oplus B^{\bullet-1}$$

Writing down all of these cochain spaces in this decomposition gives the diagram below, where columns are, left to right, the cochain complexes C, A, and B.



(4.3)

As mentioned, there is one more cycle in $[\![\ \ ]\!]$ than in $[\![\ \ ]\!]$ (call it the "special cycle"). Let $[\![\ \ ]\!]_{v_+}$ denote the subspace of $[\![\ \ ]\!]$ with the special cycle assigned the vector space $V' = \mathrm{span}\{v_+\}$. This gives a subcomplex of C, denoted



To see that C' is acyclic, first observe that for any argument x, we have $m(x \otimes v_+) = m(v_+ \otimes x) = x$, so v_+ is a unit for m. Therefore the map m acts on $\llbracket \overset{\circ}{\frown} \rrbracket_{v_+}$ as an isomorphsim:

$$m : \llbracket \overset{\circ}{\frown} \rrbracket_{v_+} = v_+ \otimes \llbracket \Omega \rrbracket \{1\} \xrightarrow{m} \llbracket \Omega \rrbracket \{1\}$$

Since $\llbracket \Omega \rrbracket$ is a subcomplex of $\llbracket \overset{\circ}{\times} \rrbracket$, and $\llbracket \overset{\circ}{\frown} \rrbracket_{v_+} \cong \llbracket \Omega \rrbracket$, we have that $\llbracket \overset{\circ}{\frown} \rrbracket_{v_+}$ is a subcomplex of $\llbracket \overset{\circ}{\times} \rrbracket$. Now we have a sequence as below, which must be exact, as the two non-zero groups are isomorphic.

$$C' : \quad 0 \xrightarrow{f_1} \llbracket \overset{\circ}{\frown} \rrbracket_{v_+} \xrightarrow{m} \llbracket \Omega \rrbracket \{1\} \xrightarrow{f_2} 0$$

$$\begin{array}{rclcrclcl} \mathrm{im}(f_1) & = & 0 & \quad & \mathrm{im}(m) & = & \llbracket \Omega \rrbracket \{1\} & \quad\Longrightarrow\quad & H^0(C') = 0 \\ \ker(m) & = & 0 & & \ker(f_2) & = & \llbracket \Omega \rrbracket \{1\} & & H^1(C') = 0 \end{array}$$

All cohomology groups of C' are zero, so C' is acyclic. By Proposition 4.1, $H^\bullet(C) = H^\bullet(C/C')$. The complex C/C' is then given by

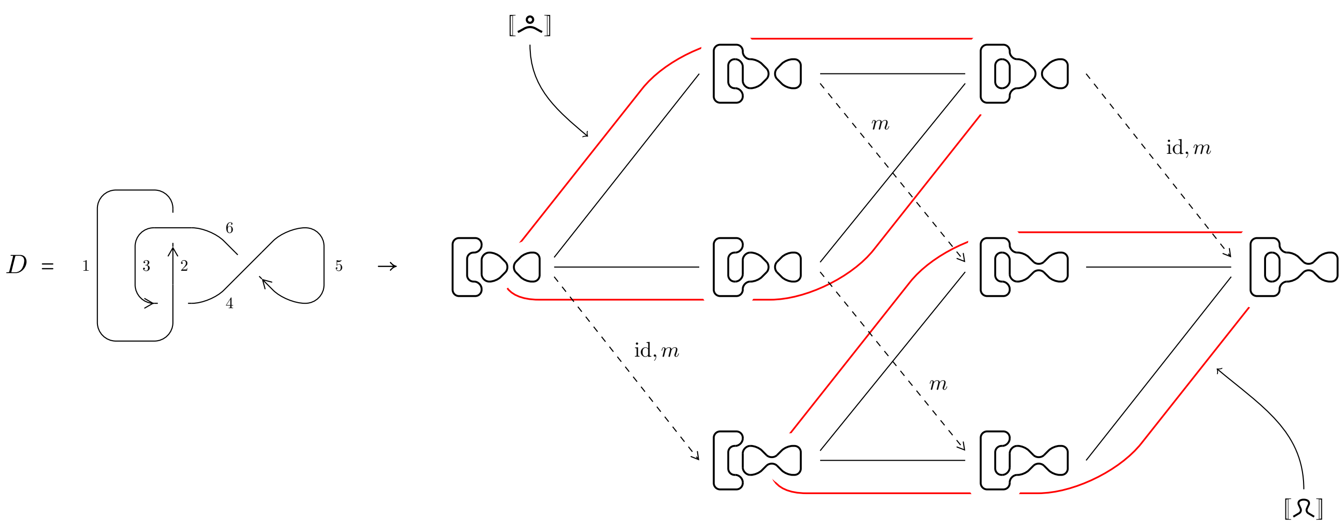
$$C/C' : \quad \llbracket \overset{\circ}{\frown} \rrbracket_{v_-} \longrightarrow 0$$

where the special cycle in $\llbracket \overset{\circ}{\frown} \rrbracket_{v_-}$ is assigned the vector space $V'' = \mathrm{span}\{v_-\}$. Therefore $\llbracket \overset{\circ}{\frown} \rrbracket_{v_-}$ is one-dimensional, and so is isomorphic to $\llbracket \frown \rrbracket$, the chain complex associated with D without a loop at the crossing under discussion.

The degree shift of $\llbracket \overset{\circ}{\frown} \rrbracket_{v_-}$ is the same as that of $\llbracket \frown \rrbracket$, as the crossing was a 0-smoothing. The height shift is also the same for the two, so indeed, they are isomorphic.

4.1.1 An example

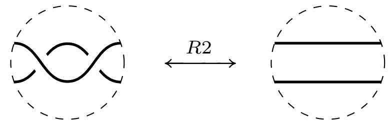
Consider the Hopf link L with $R1$ applied once, as in the following diagram.



The part of the diagram representing the subcomplex $\llbracket \overset{\circ}{\frown} \rrbracket$ of L is connected to the other part of the diagram, representing the subcomplex $\llbracket \Omega \rrbracket$, by either the map m or a combination of m and id on each edge, which take a tensor product of vector spaces to a single vector space.

4.2 Invariance under $R2$

Recall that $R2$ removes or introduces a pair of intersections from the knot. If we consider only a portion of a knot, and leave everything outside the dashed circle constant, then



Let D be a link diagram embedded with an occurence of $\times\!\!\times$. The cube of resolutions of D may be presented as a cube of resolutions with cochain complexes for vertices, denoted by C:

$$
\begin{tikzcd}
 & {[\![\supset\!\circ\!\subset]\!]\{1\}} \arrow[dr, "\Delta"] & \\
C: \; [\![\asymp\!\subset]\!] \arrow[ur, "m"] \arrow[dr] & & {[\![\supset\!\asymp]\!]\{2\}} \\
 & {[\![\asymp\!\asymp]\!]\{1\}} \arrow[ur] &
\end{tikzcd}
\tag{4.4}
$$

Only two maps are labeled, as the other ones could be either m or Δ, depending on what the rest of the diagram D looks like. Next we define a subcomplex C' of C and another complex C'', given by the diagrams:

$$
\begin{tikzcd}
 & {[\![\supset\!\circ\!\subset]\!]_{v_+}\{1\}} \arrow[dr, "\Delta"] & \\
C': \; 0 \arrow[ur] \arrow[dr] & & {[\![\supset\!\asymp]\!]\{2\}} \\
 & 0 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[dr] & \\
C'': \; 0 \arrow[ur] \arrow[dr] & & 0 \\
 & {[\![\asymp\!\asymp]\!]\{1\}} \arrow[ur] &
\end{tikzcd}
\tag{4.5}
$$

The complex C'' is essentially $[\![=]\!]$, but shifted, for which we wish to find an equal complex. The space indicated $[\![\supset\!\circ\!\subset]\!]_{v_+}$ in the subcomplex C' indicates, as previously, that the special cycle is assigned the subspace $V' = \operatorname{span}\{v_+\}$. Further, C' is acyclic for the same reasons as above, so $H^\bullet(C) = H^\bullet(C/C')$. The complex C/C' is computed also as above, and given below.

$$
\begin{tikzcd}
 & {[\![\supset\!\circ\!\subset]\!]_{v_-}\{1\}} \arrow[dr] & \\
C/C': \; [\![\asymp\!\subset]\!] \arrow[ur, "m"] \arrow[dr] & & 0 \\
 & {[\![\asymp\!\asymp]\!]\{1\}} \arrow[ur] &
\end{tikzcd}
$$

Given the boundary map $d : [\![\asymp\!\subset]\!] \to [\![\asymp\!\asymp]\!]$ and the bijective map m, define a new map $\tau : [\![\supset\!\circ\!\subset]\!] \to [\![\asymp\!\asymp]\!]$ given by $\tau = d \circ m^{-1}$. This induces a subcomplex C''' of C/C', given by

$$
\begin{tikzcd}[column sep=small]
 & & [\![\supset\!\circ\!\subset]\!]_{v_-}\{1\} \arrow[dd, dashed, "\tau"] \arrow[dr] & \\
C''' : & [\![\asymp\!\subset]\!] \arrow[ur, "m"] \arrow[dr] & & 0 \\
 & & \tau\left([\![\supset\!\circ\!\subset]\!]_{v_-}\{1\}\right) \arrow[ur] &
\end{tikzcd}
$$

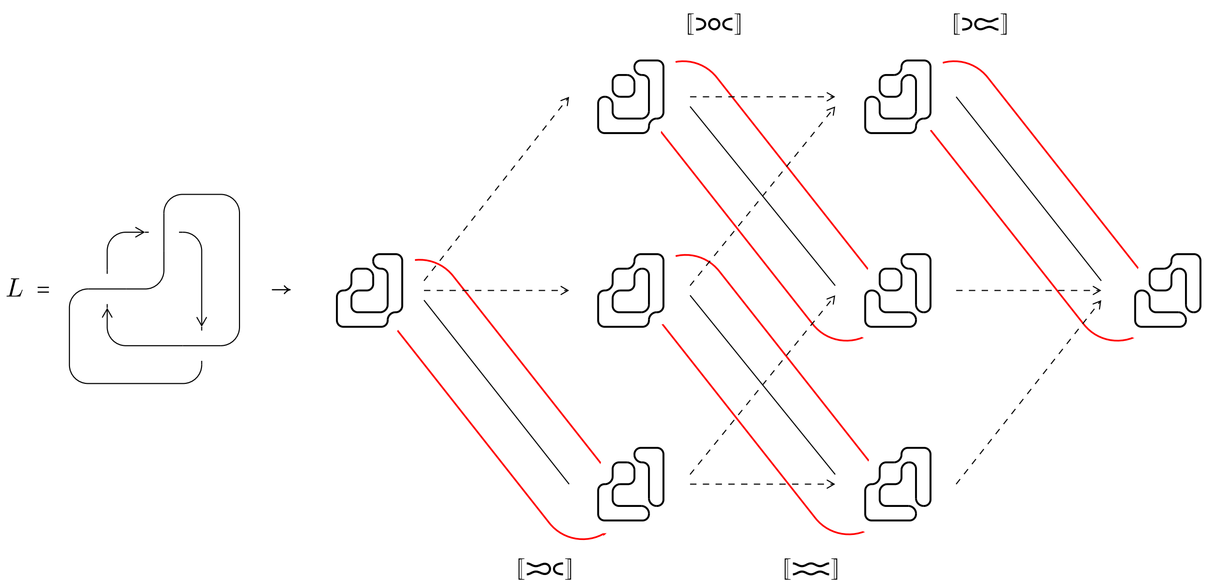
Here we have all elements of $[\![\asymp\!\subset]\!]$ and $[\![\supset\!\circ\!\subset]\!]$, but only elements $\tau\alpha \in [\![\asymp\!\asymp]\!]$ for some $\alpha \in [\![\supset\!\circ\!\subset]\!]_{v_-}\{1\}$. This is a subcomplex of C/C' as the introduction of the map τ, by its definition, ensures that all images of maps are still contained in the diagram. Moreover, as m is bijective (as the special cycle has one dimension and the rest of the diagram is isomorphic to $[\![\asymp\!\subset]\!]$), C''' is acyclic in C/C', meaning that $H(C) = H((C/C')/C''')$. Then $(C/C')/C'''$ is isomorphic to $[\![=]\!]$, as the groups $[\![\asymp\!\subset]\!]$ and $[\![\supset\!\circ\!\subset]\!]$ disappear, leaving $[\![\asymp\!\asymp]\!]$ untouched.

$$
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & & & \\
(C/C')/C''' : & 0 \arrow[ur] \arrow[dr] & & 0 & = & C'' & \cong & [\![=]\!] \\
 & & [\![\asymp\!\asymp]\!]\{1\} \arrow[ur] & & & &
\end{tikzcd}
$$

Therefore $H(C) \cong H([\![=]\!])$, proving that the Khovanov homology is invariant under the $R2$ move.

4.2.1 An example

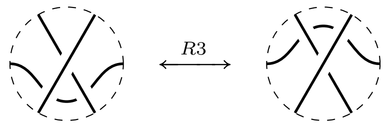
Consider an embedding of the trivial knot.



As previously, the vertices of the diagrams above are circled in the cube of this diagram.

4.3 Invariance under $R3$

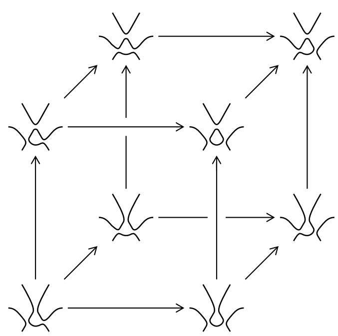
Recall that $R3$ moves a strand to the other side of an intersection. If we consider only a portion of a knot, and leave everything outside the dashed circle constant, then



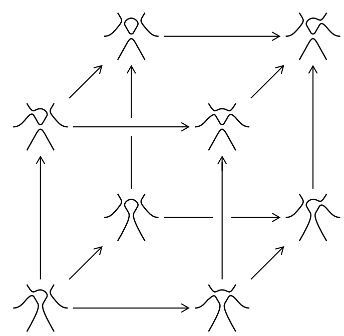
For ease of graphical presentation, the brackets $[\![\cdot]\!]$ are dropped in this section, though whenever a diagram is presented, we are referring to its associated complex.

Consider the cube of resolutions of both of these diagrams.

$\underline{C}$:

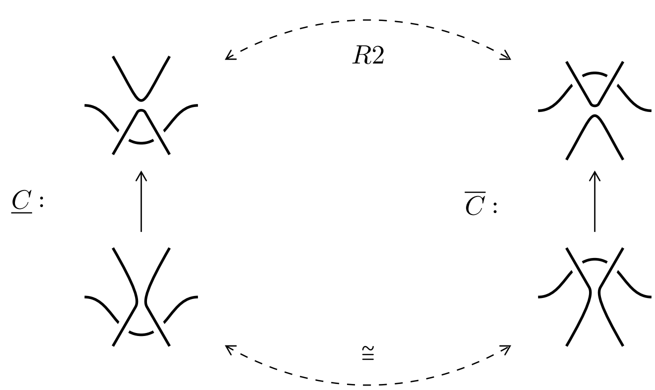


$\overline{C}$:

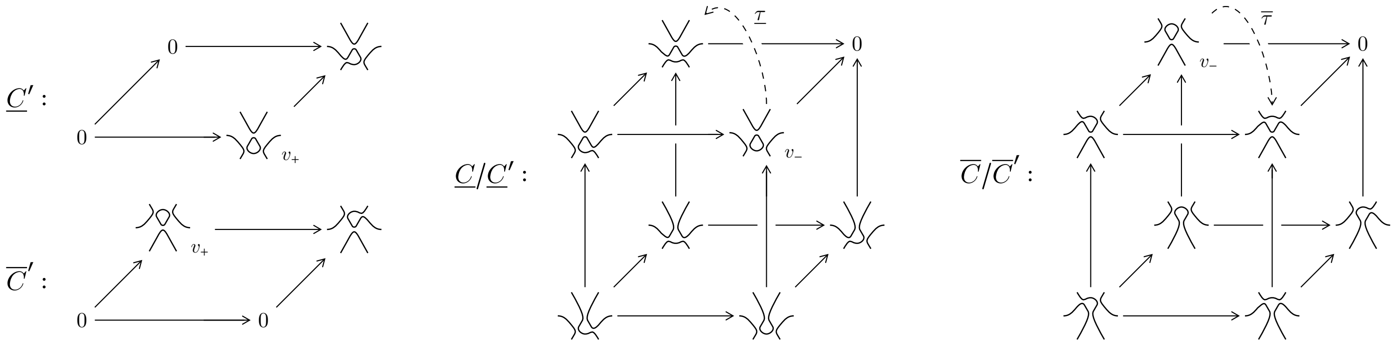


(4.6)

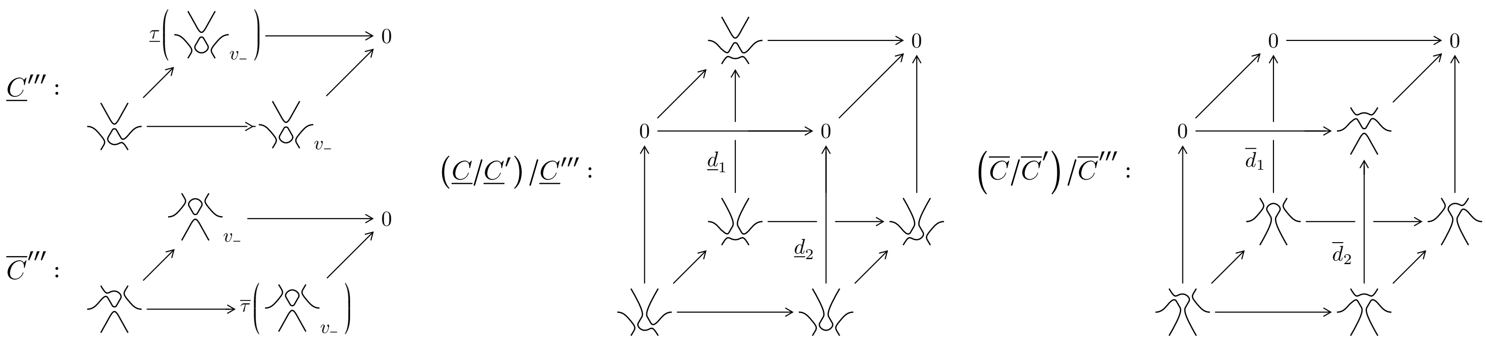
Observe that the bottom layer of $\underline{C}$ has the same diagrams as the bottom layer of $\overline{C}$. Further, the top layer of $\underline{C}$ is related to the top layer of $\overline{C}$ by the $R2$ move. In other words,



Using this observation, the proof will borrow methods from the previous section. In the same manner as for $R2$ in (4.5), define new subcomplexes $\underline{C}'$ and $\overline{C}'$ of $\underline{C}$ and $\overline{C}$, respectively. And again as previously, mod them by $\underline{C}$ and $\overline{C}$, respectively.



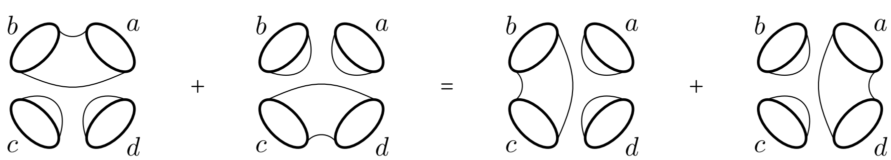
Maps $\underline{\tau}$ and $\overline{\tau}$ have been indicated to construct further subcomplexes $\underline{C}'''$ of $\underline{C}/\underline{C}'$ and $\overline{C}'''$ of $\overline{C}/\overline{C}'$. We then mod the complexes by their respective subcomplexes to arrive at the desired result.



The two complexes $(\overline{C}/\overline{C}')/\overline{C}'''$ and $(\underline{C}/\underline{C}')/\underline{C}'''$ are isomorphic, as their bottom layers are identical, and $\underline{d}_2 \cong \overline{\tau} \circ \overline{d}_1$ and $\overline{d}_1 \cong \underline{\tau} \circ \underline{d}_2$. The checking of these final isomorphisms is left to the curious reader.

5 Invariance (topological) of the Khovanov homology

This section proceeds as the previous section, but with topological instead of algebraic arguments. Before we begin, we give one of the fundamental properties of cobordisms, namely the 4Tu property:

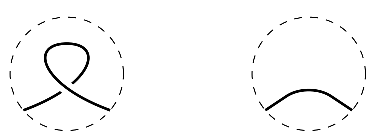


(5.1)

This property holds for a cobordism with a ball taken out that has four sections, as marked, going into it.

5.1 Invariance under $R1$

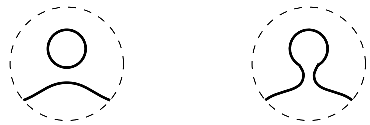
Let D be the diagram of L with a twist, and D' the diagram of L without a twist. The goal of this section is to show that the Khovanov homology is the same for both D and D'.



(5.2)

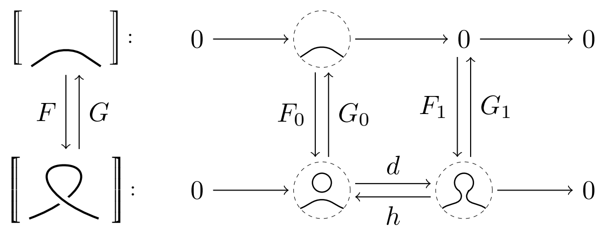
D $\qquad$ D'

Let D_0 be the diagram of L with a 0-resolution at the crossing made by $R1$, and D_1 the diagram of L with a 1-resolution at the crossing made by R_1.



D_0 D_1

We wish to relate the two complexes $[\![\;]\!]$ and $[\![\;]\!]$. We do this by showing that the diagram in (5.3) is commutative, where $F : [\![\;]\!] \to [\![\;]\!]$ is given by $F = (F_0, F_1)$ and $G : [\![\;]\!] \to [\![\;]\!]$ by $G = (G_0, G_1)$. The diagram has chain groups of the same degree in the same columns.



(5.3)

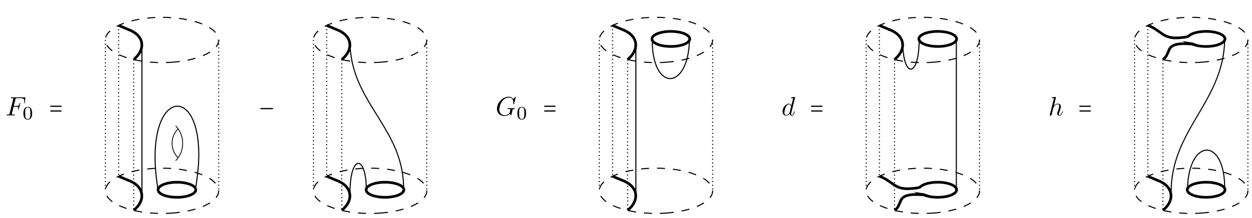
The maps F_1 and G_1 are the zero maps.The map h is a homotopy that comes from the following diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & C^{\bullet-1}(D) \arrow[r] & C^{\bullet}(D) \arrow[r] & C^{\bullet+1}(D) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{\bullet-1}(D) \arrow[r] \arrow[u, shift left, "FG"] \arrow[u, shift right, "\mathrm{id}"'] & C^{\bullet}(D) \arrow[r] \arrow[u, shift left, "FG"] \arrow[u, shift right, "\mathrm{id}"'] \arrow[ul, "h"] & C^{\bullet+1}(D) \arrow[r] \arrow[u, shift left, "FG"] \arrow[u, shift right, "\mathrm{id}"'] \arrow[ul, "h"] & \cdots
\end{tikzcd}
\tag{5.4}
$$

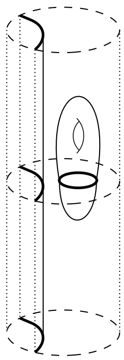
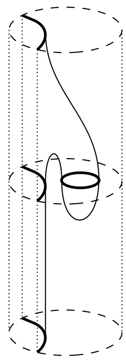
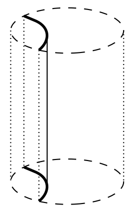
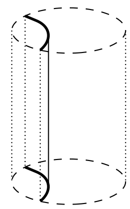
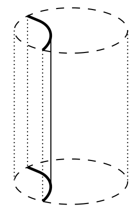
The following identities need to be proven for the diagram in (5.3) to commute, which will in turn prove invariance under $R1$. Note that the second equation in the first column is a statement about the commutativity of the diagram in (5.4).

$$
\begin{aligned}
G_0F_0 &= I_{D'} & \qquad G_0h &= 0 \\
F_0G_0 + hd &= I_{D_0} & \qquad dF_0 &= 0
\end{aligned}
$$

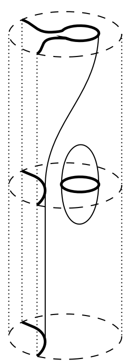
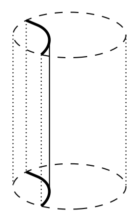
Let us first define the maps that we intend to use. They are given only for the region of the diagram under consideration, with the rest of the diagram having the identity map applied to it. The action of the map moves from the top of the cylinder to the bottom.



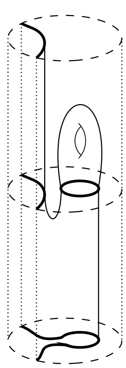
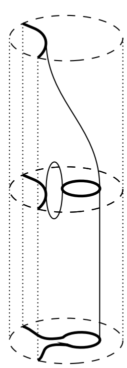
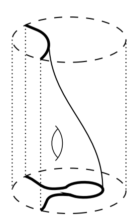
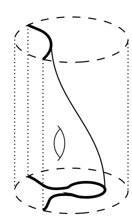
First we prove the simplest equalities:

$G_0F_0 =$  $-$  ≈ 2  $-$  $=$  $= I_{D'}$

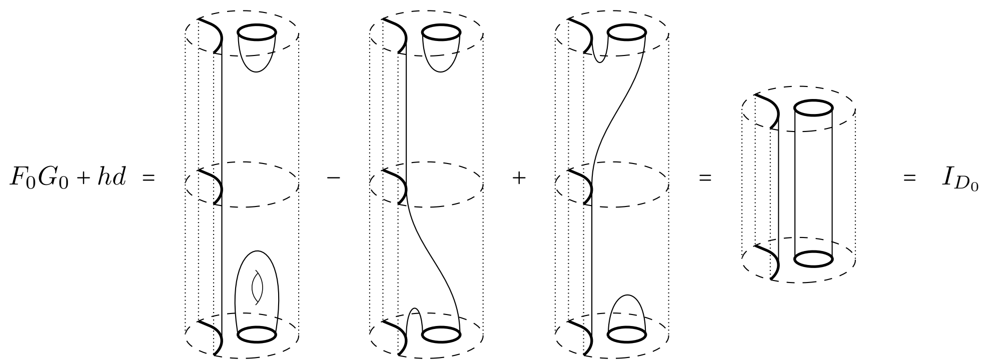
The coefficient 2 appears in front of one of the terms as a surface of genus 1, unattached to the top or bottom of the cobordism, contributes a factor of 2. The reasoning behind this is explained in Appendix A.

$G_0h =$  $= 0 \cdot$  $= 0$

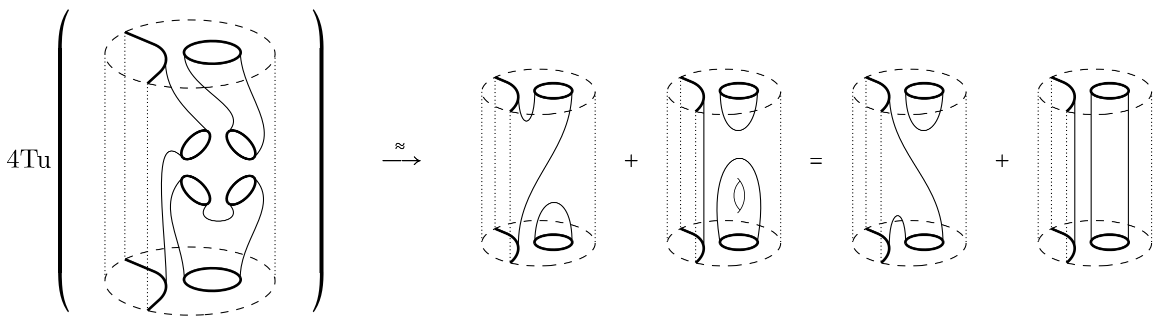
The disjoint surface of genus 0 contributes a factor of 0, so the term vanishes.

$dF_0 =$  $-$  $=$  $-$  $= 0$

This calculation follows from isotopy. We are left with proving $F_0 G_0 + hd = I_{D_0}$, which we claim follows from the application of the $4Tu$ relation from (5.1). The identity we wish to show holds is:



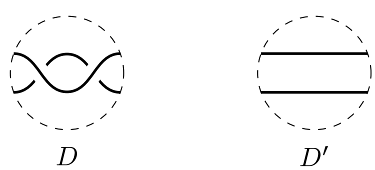
The $4Tu$ relation applied to the following cobordism is precisely that relation.



Therefore, topologically, the Khovanov homology is invariant under the $R1$ move.

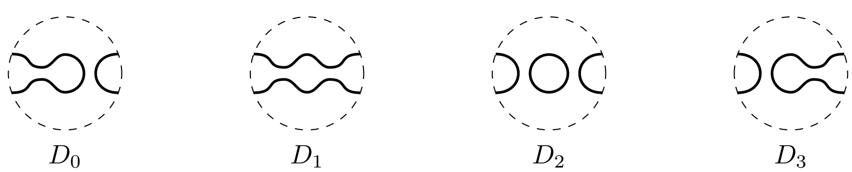
5.2 Invariance under $R2$

In this section we will use many of the previously used tactics. Begin with the same link L and two different embeddings, D and D'.



(5.5)

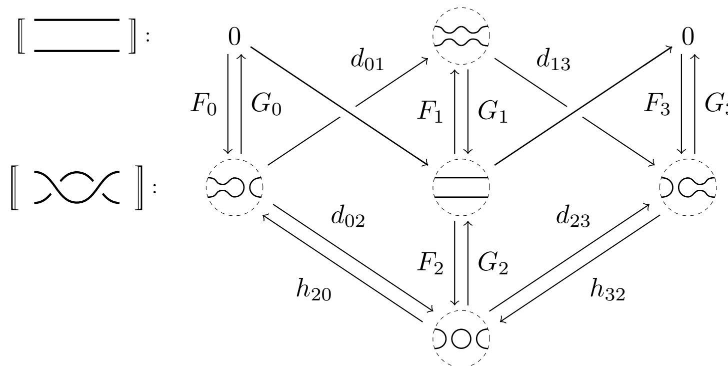
The diagram D has 4 different smoothings, as labelled below.



(5.6)

The first on the left has two 0-resolutions and no 1-resolutions, the middle two have one of each type of resolution, and the last one has two 1-resolutions and no 0-resolutions; the degree shifts should be noted.

We employ the same complex presented in (4.4) to relate it to the chain complex of D'.

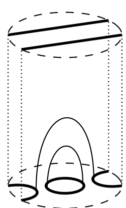
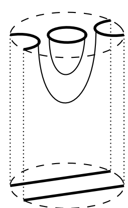
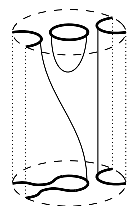
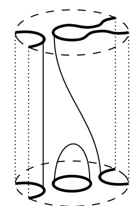


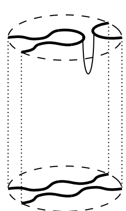
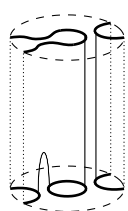
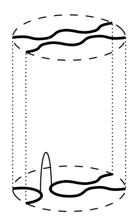
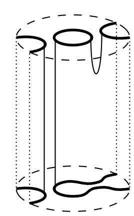
(5.7)

As previously in (5.3), each column has the same degree shift. Similar associations are made, most notably that F_0, F_3, G_0, G_3 are all the zero map, and F_1, G_1 are the identity. The main maps are given by $F = (F_0, F_1, F_2, F_3)$ and $G = (G_0, G_1, G_2, G_3)$. The identities that need to be proved are:

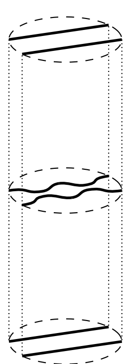
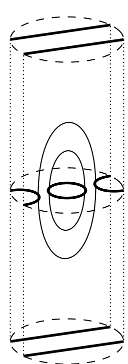
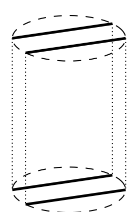
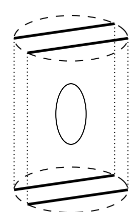
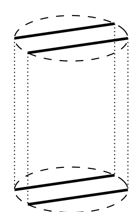
$$GF = G_1F_1 + G_2F_2 = I_{D'} \qquad Gd = G_1d_{01} + G_2d_{02} = 0$$
$$F_2G_2 - h_{32}d_{23} - d_{02}h_{20} = I_{D_2} \qquad dF = d_{13}F_1 + d_{23}F_2 = 0$$

Analogously to the previous section, the maps h_{20} and h_{32} arise from a map $h : C^\bullet(D) \to C^{\bullet-1}(D)$, as in the diagram (5.4). Let us now define the non-trivial maps.

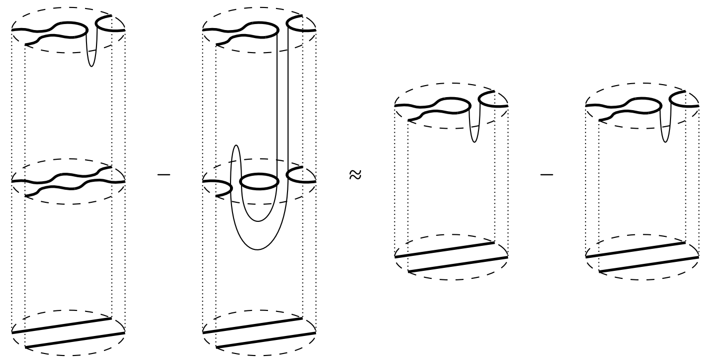
$F_2 =$  $G_2 = -$  $h_{20} = -$  $h_{32} = -$ 

$d_{01} =$  $d_{02} =$  $d_{13} = -$  $d_{23} =$ 

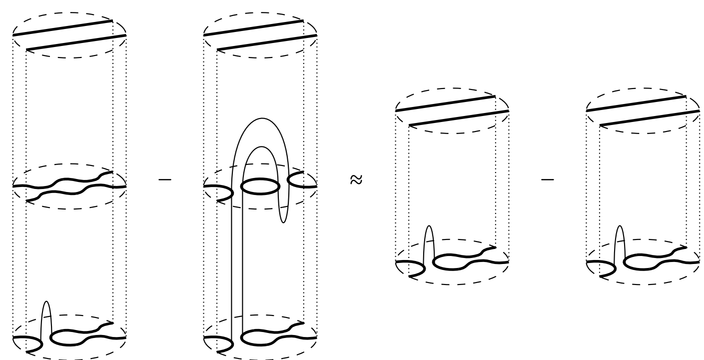
We begin with the simplest identities, involving G, F, and the boundary map d.

$G_1F_1 + G_2F_2 =$  $-$  $\approx$  $- \; 0 \cdot$  $=$  $= \mathrm{id}_{D'}$

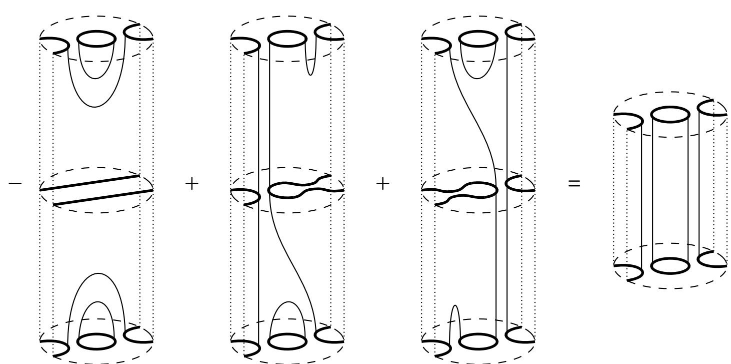
Here we have again interpreted a manifold of genus zero as a factor of zero.

$G_1 d_{01} + G_2 d_{02} =$  $= 0$

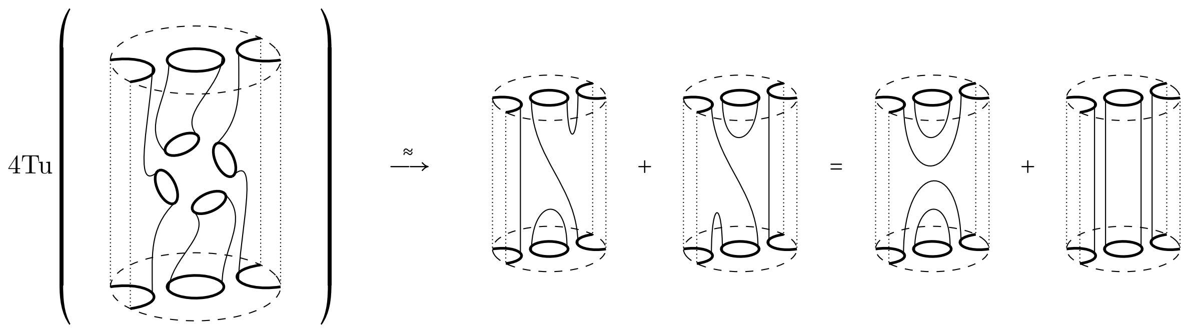
This argument, as well as the following one, is done by a simple isotopy.

$d_{13} F_1 + d_{23} F_2 =$  $= 0$

The final equation remains, for which we will need to apply the $4Tu$-relation. Let us cosider what we intend to prove first:

$F_2 G_2 - h_{32} d_{23} - d_{02} h_{20} =$  $= \mathrm{id}_{D_2}$

Here we apply the $4Tu$-relation as given in (5.1), but we switch the ends marked a and b in every diagram of (5.1) to make the presentation clearer. The resulting equation is exactly the one given above.



Therefore, topologically, the Khovanov homology is invariant under the $R2$ move.

5.3 Invariance under $R3$

In this section we we will introduce auxiliary definitions and lemmae to simplify the main proof, due to the large size of the complexes involved.

Definition 5.1. *(cone of a complex)* Let (A, d_0) and (B, d_1) be two cochain complexes with maps $\Psi^r : (A^r, d_0^r) \to (B^r, d_1^r)$. Define the cone of Ψ to be the complex $\Gamma(\Psi)$ with cochain spaces $A^{r+1} \oplus B^r$ and differential $d_* = \begin{pmatrix} -d_0^{r+1} & 0 \\ \Psi^{r+1} & d_1^r \end{pmatrix}$, as depicted below.

$$\begin{tikzcd}
\cdots \arrow[r, "-d_0^{r-2}"] & A^{r-1} \arrow[r, "-d_0^{r-1}"] \arrow[d, "\Psi^{r-1}"] \arrow[dr, phantom, "\oplus"] & A^r \arrow[r, "-d_0^r"] \arrow[d, "\Psi^r"] \arrow[dr, phantom, "\oplus"] & A^{r+1} \arrow[r, "-d_0^{r+1}"] \arrow[d, "\Psi^{r+1}"] \arrow[dr, phantom, "\oplus"] & A^{r+2} \arrow[r, "-d_0^{r+2}"] \arrow[d, "\Psi^{r+2}"] & \cdots \\
\cdots \arrow[r, "d_1^{r-2}"'] & B^{r-1} \arrow[r, "d_1^{r-1}"'] & B^r \arrow[r, "d_1^r"'] & B^{r+1} \arrow[r, "d_1^{r+1}"'] & B^{r+2} \arrow[r, "d_1^{r+2}"'] & \cdots
\end{tikzcd}$$

ψ^{r-1} $\qquad$ ψ^r $\qquad$ ψ^{r+1}

Next, we consider two lemmae that together will imply invariance of the Khovanov homology under $R3$ without the lengthy cobordism calculations we dealt with previously to prove $R1$ and $R2$. The proofs presented are sketches, and are not meant to be complete.

Lemma 5.2.

$$[\![\times]\!] = \Gamma\left([\![)(]\!] \to [\![\asymp]\!]\right)[-1] \qquad \text{and} \qquad [\![\times]\!] = \Gamma\left([\![\asymp]\!] \to [\![)(]\!]\right)$$

<u>*Proof:*</u> We wil only inspect the first equation. Consider the cochain complex that results in splitting $[\![\times]\!]$ into its constituent chain complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & {[\![)(]\!]}^{r-1} \arrow[r] \arrow[d] & {[\![)(]\!]}^{r} \arrow[r] \arrow[d] & {[\![)(]\!]}^{r+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & {[\![\asymp]\!]}^{r-1} \arrow[r] & {[\![\asymp]\!]}^{r} \arrow[r] & {[\![\asymp]\!]}^{r+1} \arrow[r] & \cdots
\end{tikzcd}$$

Then $\Gamma([\![\times]\!])$ is the complex with cochain spaces $\Gamma([\![\times]\!])^r = [\![)(]\!]^{r-1} \oplus [\![\asymp]\!]^r$. ■

For the next lemma, we introduce a new idea.

Definition 5.3. *(strong deformation retract)* Given two complexes A, B and a morphism $G : A \to B$ between them, the map G is termed a strong deformation retract if there exists a morphism $F : B \to A$ and a homotopy $h : A^{\bullet} \to A^{\bullet -1}$ such that the following conditions are satisfied:

$$GF = \mathrm{id}_B \qquad\qquad FG = \mathrm{id}_A - dh - hd \qquad\qquad hF = 0 \tag{5.8}$$

where $d : A^{\bullet} \to A^{\bullet+1}$ is the boundary operator for A. In this case, the map F is termed the inclusion of the strong deformation retract.

Lemma 5.4. Let A, A', B, B' be cochain complexes related by maps as below.

$$\begin{tikzcd} \hat{A} \arrow[r, shift left, "F_0"] & A \arrow[l, shift left, "G_0"] \arrow[r, "\Psi"] & B \arrow[r, shift left, "F_1"] & \hat{B} \arrow[l, shift left, "G_1"] \end{tikzcd} \tag{5.9}$$

If G_0 is a strong deformation retract with inclusion F_0, then $\Gamma(\Psi)$ is homotopy equivalent to $\Gamma(\Psi F_0)$. Similarly, if G_1 is a strong deformation retract with inclusion F_1, then $\Gamma(\Psi)$ is homotopy equivalent to $\Gamma(F_1 \Psi)$.

Proof: Let $h : A^{\bullet} \to A^{\bullet-1}$ be the homotopy that satisfies the conditions of (5.8). Then the cones $\Gamma(\Psi)$ and $\Gamma(\Psi F_0)$ induce morphisms $\tilde{F}_0, \tilde{G}_0$ and the homotopy $\tilde{h}_0$, given by

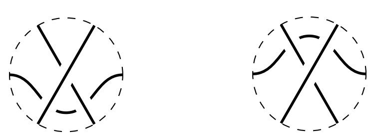
$$\begin{array}{l} \tilde{F}_0 : \Gamma(\Psi F_0) \to \Gamma(\Psi) \\ \tilde{G}_0 : \Gamma(\Psi) \to \Gamma(\Psi F_0) \end{array} \qquad\qquad \tilde{h}_0 : \Gamma(\Psi)^{\bullet} \to \Gamma(\Psi)^{\bullet-1}$$

The rest follows from the considering the diagram below:

$$\begin{tikzcd} \Gamma(\Psi F_0): & \cdots \arrow[r, "\tilde{d}^{r-2}"] & A^r \oplus B^{r-1} \arrow[r, "\tilde{d}^{r-1}"] \arrow[d, shift left, "\tilde{F}_0^{r-1}"] & A^{r+1} \oplus B^r \arrow[r, "\tilde{d}^r"] \arrow[d, shift left, "\tilde{F}_0^r"] & A^{r+2} \oplus B^{r+1} \arrow[r, "\tilde{d}^{r+1}"] \arrow[d, shift left, "\tilde{F}_0^{r+1}"] & \cdots \\ \Gamma(\Psi): & \cdots \arrow[r, shift left, "\tilde{d}^{r-2}"] & \hat{A}^r \oplus B^{r-1} \arrow[l, shift left, "\tilde{h}_0^{r-2}"] \arrow[r, shift left, "\tilde{d}^{r-1}"] \arrow[u, shift left, "\tilde{G}_0^{r-1}"] & \hat{A}^{r+1} \oplus B^r \arrow[l, shift left, "\tilde{h}_0^{r-1}"] \arrow[r, shift left, "\tilde{d}^r"] \arrow[u, shift left, "\tilde{G}_0^r"] & \hat{A}^{r+2} \oplus B^{r+1} \arrow[l, shift left, "\tilde{h}_0^r"] \arrow[r, shift left, "\tilde{d}^{r+1}"] \arrow[u, shift left, "\tilde{G}_0^{r+1}"] & \cdots \arrow[l, shift left, "\tilde{h}_0^{r+1}"] \end{tikzcd}$$

The relentless reader is invited to complete the missing parts of this proof. ∎

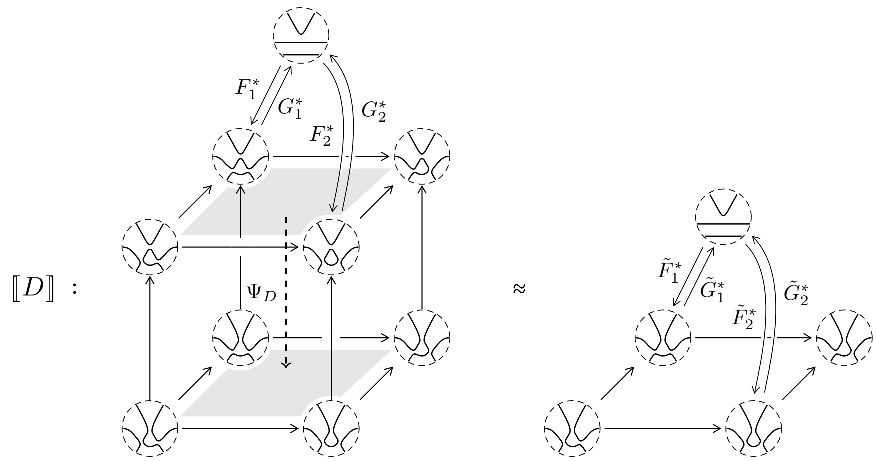
Now we may move to actually proving the invariance of the Khovanov homology under $R3$. We do this by showing that the two complexes associated with both stages of $R3$ are homotopic. Begin with the same link L and two different embeddings, D and D'.

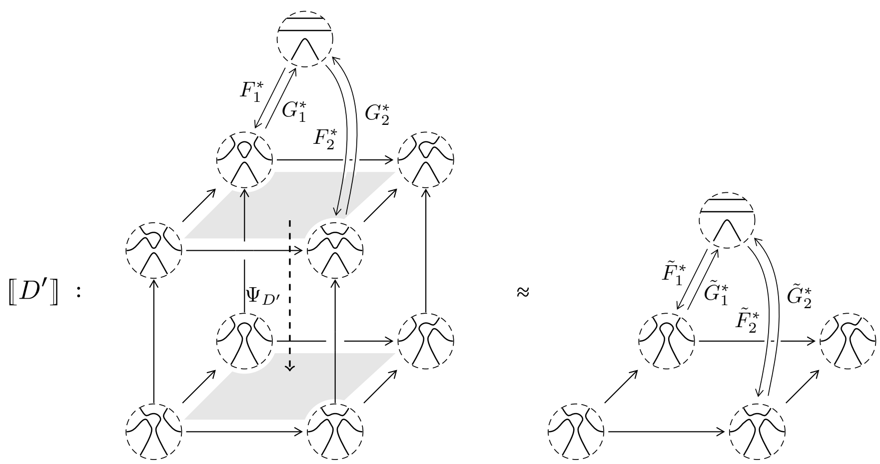


D $\qquad\qquad$ D'

We now take the cubes of resolutions corresponding to D and D', already presented in (4.6), and by Lemma 5.2 associate each to a cone of a two-term complex (each term is indicated below by a gray shaded plane). To apply Lemma 5.4, we first associate the top layer with the bottom layer of the complex of $R2$, seen in

(5.7). Now we apply the lemma and get a reduced complex homotopic to the original one.





Note that the maps F_0 and G_0 from (5.9) are given by $F_0 = (F_1^*, F_2^*)$ and $G_0 = (G_1^*, G_2^*)$, where F_1, F_2, G_1, G_2 are the maps presented in (5.7), from the section on $R2$. The induced maps in the reduced complex are induced analogs of those maps as well.

The two resulting complexes are isotopic, so invariance of the Khovanov homology under $R3$ follows.

6 Concluding remarks

6.1 Reduction to the Jones polynomial

Let us briefly touch on how the Khovanov homology may be reduced to yield the Jones polynomial. Above we constructed cohomology groups $H^n(C(L))$. Then the Jones polynomial of a link L is given by

$$J(L) = \sum_i (-1)^i \mathrm{qdim}(H^i(C(L)))$$

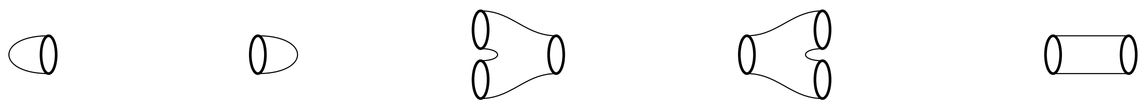
The q-dimension of the cohomology groups is taken over the rationals. Note that this result follows by simply comparing the state sum definition of the Jones polynomial with the definition of the graded Euler characteristic.

6.2 A note on grading

The cohomological groups constructed were graded in two ways, one in the degree shift, and the other by the height shift. There also exists, by [5], a homology theory of groups graded in three ways, whose graded Euler characteristic is revealed to be the HOMFLY-PT polynomial. Unfortunately, a deeper perusal of the relation to this invariant is not undertaken here, and is left to the delighted reader.

A The structure of cobordisms

Let us consider the fundamental cobordisms and their associated actions.



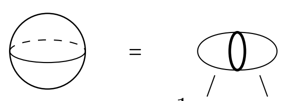
$$
\begin{array}{ccc}
k & \to & V \\
1 & \mapsto & v_+
\end{array}
\qquad
\begin{array}{ccc}
V & \to & k \\
v_+ & \mapsto & 0 \\
v_- & \mapsto & 1
\end{array}
\qquad
\begin{array}{ccc}
V \otimes V & \to & V \\
v_+v_+ & \mapsto & v_+ \\
v_+v_- & \mapsto & v_- \\
v_-v_+ & \mapsto & v_- \\
v_-v_- & \mapsto & 0
\end{array}
\qquad
\begin{array}{ccc}
V & \to & V \otimes V \\
v_+ & \mapsto & v_+v_- + v_-v_+ \\
v_- & \mapsto & v_-v_-
\end{array}
\qquad
\begin{array}{ccc}
V & \to & V \\
v_+ & \mapsto & v_+ \\
v_- & \mapsto & v_-
\end{array}
$$

The first one follows as v_+ is a unit for V. The second one follows from the definition of the counit. The next two follow from the decided-upon definitions in (2.3), and the last one is the identity.

A.1 Isolated surfaces

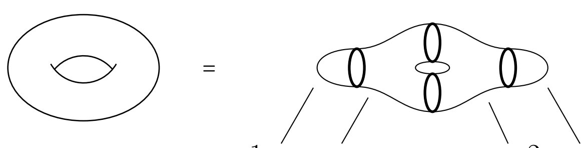
We now consider the effect on a cobordism that contains a compact closed surface separate from the main part of the cobordism.

The effect of an isolated ball on a cobordism is multiplication by 0:



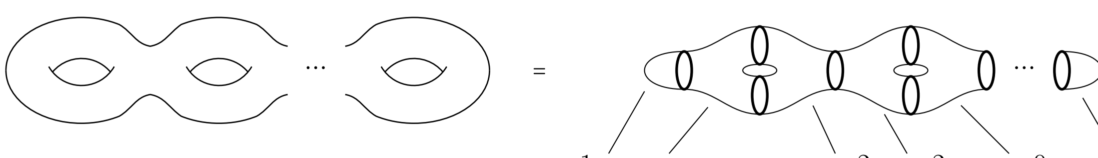
$$1 \mapsto v_+ \mapsto 0$$

The effect of an isolated torus on a cobordism is multiplication by 2:



$$1 \mapsto v_+ \mapsto v_+v_- + v_-v_+ \mapsto 2v_- \mapsto 2$$

The effect of an isolated surface of genus $g \geqslant 2$ on a cobordism is multiplication by 0:



$$1 \mapsto v_+ \mapsto v_+v_- + v_-v_+ \mapsto 2v_- \mapsto 2v_-v_- \mapsto 0 \mapsto \cdots \mapsto 0$$

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