Abstract tensor systems and diagrammatic representations

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Abstract

The diagrammatic tensor calculus used by Roger Penrose (most notably in [7]) is introduced without a solid mathematical grounding. We will attempt to derive the tools of such a system, but in a broader setting. We show that Penrose's work comes from the diagrammisation of the symmetric algebra. Lie algebra representations and their extensions to knot theory are also discussed.

Contents

<u>*Note:*</u> This work relies heavily upon the text of Chapter 12 of a draft of "An Introduction to Quantum and Vassiliev Invariants of Knots," by David M.R. Jackson and Iain Moffatt, a yet-unpublished book at the time of writing.

1 Abstract tensors and derived structures

Our approach is to begin with the common tensor structure, generalize to abstract tensors (where association with a unique vector space is relaxed), and finally to interpret the abstract tensors as directed graphs of a specific sort - we shall call these tensor diagrams. Let us begin with some notational remarks.

1.1 Abstract tensor notation

Let V be an n-dimensional vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and V^* its dual vector space, with dual basis $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$. We use the convention of representing a vector space V tensored with itself k times by $(V)^{\otimes k} \doteq \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$.

Definition 1.1.1. *Abstract tensor.* An abstract tensor is an element $\mathbf{w} \in W = V^{\otimes r_1} \otimes V^{*\otimes s_1} \otimes \cdots \otimes V^{\otimes r_m} \otimes V^{*\otimes s_m}$, which is a tensor product of the spaces V, V^* in some fixed combination and order. The arrangement of the indeces is part of abstract index notation, and identifies the order of the tensored vector spaces. The following presentation of such a general tensor should be seen as being on a single line.

$$\mathbf{w} = \sum_{\substack{a_1,\ldots,a_{r_1} \\ \vdots \\ g_1,\ldots,g_{s_m}}} \alpha^{a_1,\ldots,a_{r_1}}{}_{b_1,\ldots,b_{s_1}}{}^{\cdots}{}_{\cdots}{}^{f_1,\ldots,f_{r_m}}{}_{g_1,\ldots,g_{s_m}}$$

$$\mathbf{v}_{a_1} \otimes \cdots \otimes \mathbf{v}_{a_{r_1}} \otimes \mathbf{v}^{b_1} \otimes \cdots \otimes \mathbf{v}^{b_{s_1}} \otimes \cdots\cdots \otimes \mathbf{v}_{f_1} \otimes \cdots \otimes \mathbf{v}_{f_{r_m}} \otimes \mathbf{v}^{g_1} \otimes \cdots \otimes \mathbf{v}^{g_{s_m}} \quad (1.1)$$

such that $r_1, \ldots, r_m, s_1, \ldots, s_m \in \mathbb{N} \cup \{0\}$.

Low-dimensional tensors are quite common in mathematics, and they have been given common names:

A tensor with no upper nor lower indeces is termed a *scalar*
A tensor with one index (upper or lower) is termed a *vector*
A tensor with two indeces (upper or lower or any combination of the two) is termed a *matrix*

Conventions. Because of this lengthy presentation, even for simple tensors, we use the indexed coefficient in the summand of $\mathbf{w}$ equivalently as $\mathbf{w}$, noting that one is easily recoverable from the other. That is,

$$\begin{aligned} \mathbf{w} &\doteq \alpha^{a_1,\ldots,a_{r_1}}{}_{b_1,\ldots,b_{s_1}}{}^{\cdots}{}_{\cdots}{}^{f_1,\ldots,f_{r_m}}{}_{g_1,\ldots,g_{s_m}} \mathbf{v}_{a_1} \otimes \cdots \otimes \mathbf{v}^{g_{s_m}} \\ &\doteq \alpha^{a_1,\ldots,a_{r_1}}{}_{b_1,\ldots,b_{s_1}}{}^{\cdots}{}_{\cdots}{}^{f_1,\ldots,f_{r_m}}{}_{g_1,\ldots,g_{s_m}} \end{aligned}$$

Here we incorporate Enistein's shorthand of eliminating the summation symbol. For longer calculations desirous of a more thorough account, we will bring the summation symbol back. It is understood that the summation takes place over all repeated indeces. Removing the basis tensors which (potentially) repeat indeces, will pose no problem, as we then sum over all indeces appearing in the upper and lower positions of the abstract tensor symbol, taking care to note multiplicities. In **2.1** we prove that this causes no confusion.

We also follow a symbol convention. Tensors will be represented by Greek letters $(\alpha, \beta, \ldots)$ and tensor indeces will be given by (possibly indexed) Latin letters $(a, b, \ldots)$. An exception is scalars, which we will sometimes represent by $\lambda, \lambda_1, \lambda_2, \ldots$. In general, no one specific symbol is associated to one specific object, but again we make an exception for the Kronecker-delta symbol (and tensor) δ, and the Lie bracket tensor γ, each of which we use with a specific meaning.

Any scalars discussed are assumed to be over a common field, usually (but not necessarily) $\mathbb{C}$. Any field may be used.

1.2 Some basic operations

Let us consider some abstract tensor operations. Together they make up what is known as a *tensor calculus*, which is essentially the formalization of index manipulation. All these operations can be derived from the proper exhaustive definition of an abstract tensor and abstract index notation.

Reindexation. Tensor indeces may be relabelled as desired, as long as the label is not already in use by another index of the tensor. For example,

$$\alpha_{ab} = \alpha_{ab}\ \mathbf{v}^a \otimes \mathbf{v}^b = \alpha_{az}\ \mathbf{v}^a \otimes \mathbf{v}^z = \alpha_{az}$$

Scalar multiplication. Tensors may be multiplied by scalars over our field, as well as grouped together by scalars. For example, given a scalar λ,

$$\lambda\ \alpha_{ab} = \lambda \left(\alpha_{ab}\ \mathbf{v}^a \otimes \mathbf{v}^b\right)$$

Addition. Addition of tensors need not occur over a common space, as this is just a formal sum. If tensors are of the same type, then it may be possible to combine them, though we will not do that, as information is lost by doing so. Examples of general addition and a special case are

$$\begin{aligned} \beta_{cd}{}^{e} + \chi^{gh}{}_{i} &= \beta_{cd}{}^{e}\ \mathbf{v}^c \otimes \mathbf{v}^d \otimes \mathbf{v}_e + \chi^{gh}{}_{i}\ \mathbf{v}_g \otimes \mathbf{v}_h \otimes \mathbf{v}^i \\ \alpha_{ab} + \alpha_{ab} &= 2\alpha_{ab}\ \mathbf{v}^a \otimes \mathbf{v}^b \end{aligned}$$

Tensor product. The tensor product $\otimes$ also does not necessarily occur over a common space, but even if it does, there is no special simplification. The indeces between the tensors are required to not coincide. For example,

$$\alpha_{ab}\ \otimes\ \beta^{cd}{}_{e} = \alpha_{ab}\ \beta^{cd}{}_{e}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \otimes \mathbf{v}_d \otimes \mathbf{v}^e$$

Contraction. We may identify one upper index with one lower index by the contraction operation, thereby assigning them the same label. We use the operator symbol $\kappa_{i,j}$ to denote identification of the ith and jth indeces of ρ. For example,

$$\kappa_{2,4}\left(\alpha_{ab}\ \beta^{cd}{}_{e}\right) = \alpha_{ab}\ \beta^{cb}{}_{e} = \alpha_{ab}\ \beta^{cb}{}_{e}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \otimes \mathbf{v}_b \otimes \mathbf{v}^e$$

The general idea of operations within a tensor calculus is that we do not have much leeway in simplification. By moving to abstract tensor systems, we introduce commutativity of the product, among other reductions. Accompanying abstract tensors is another system that diagrammises abstract tensors in an effective way. This identification will be bijective by construction. We shall now describe this system.

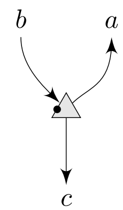
1.3 Tensor diagrams

Definition 1.3.1. *Primitive tensor diagram.* A primitive tensor diagram is a directed graph with $n \geqslant 1$ vertices and 1 connected component such that

- $\cdot$ $n-1$ vertices have degree 1
- $\cdot$ every edge is oriented in one of two directions
- $\cdot$ every edge is given a unique label
- $\cdot$ there is a predefined clockwise ordering of the edges

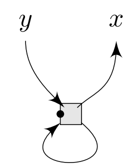
The vertex with the largest degree is the *distinguished vertex* of the graph. If $n = 1$, the distinguished vertex is the sole vertex; for $n = 2$ a decision is made; and for $n \geqslant 3$ it is the only vertex with degree $\geqslant 2$.

The primitive tensor diagram below has 4 vertices, with edges arranged b, a, c, and is identified by $\rightsquigarrow$ to the abstract tensor $\alpha_b{}^{ac}$.



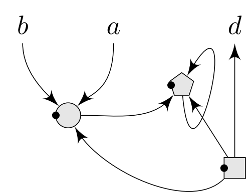
$$\rightsquigarrow \quad \alpha_b{}^{ac} \tag{1.2}$$

The distinguished vertex is denoted by a symbol graphically larger than the other nodes. To fix the ordering of the edges, we place the *anchor* symbol $\bullet$ in the space that divides the first from the last ordered edge. The first edge in the ordering is then directly in front of, with respect to the clockwise direction, the anchor. Primitive tensor diagrams are allowed to have loops, in which case we remove the label of the edge.



$$\rightsquigarrow \quad \beta_y{}^{xz}{}_z \tag{1.3}$$

Definition 1.3.2. *Compound tensor diagram.* A compound tensor diagram is a graph with 1 connected component resulting from identifying edges of different primitve tensor diagrams with opposite orientation. Edges that are identified between distinguished vertices also have their labels identified, and thus removed in the diagram. For example,



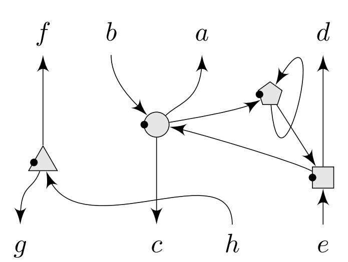
$$\rightsquigarrow \quad \alpha_{ba}{}^{g}{}_{h}\, \beta^{idh}\, \chi_{\ell i}{}^{\ell}{}_{g} \tag{1.4}$$

Definition 1.3.3. *Tensor diagram.* A tensor diagram is a directed graph such that every connected component is either

· a primitive tensor diagram or

· a compound tensor diagram

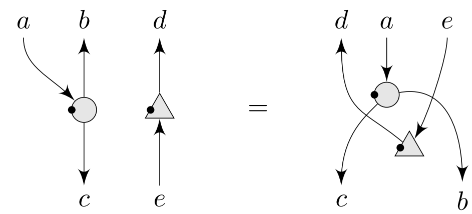
An example of a tensor diagram with 2 connected components is



$$\rightsquigarrow \quad \alpha_b{}^{ak}{}_j{}^c\, \beta^j{}_i{}^d{}_e\, \gamma^f{}_h{}^g\, \chi_\ell{}^{i\ell}{}_k \tag{1.5}$$

Note that the choice of product order of the abstract tensor on the right-hand side is not important, as the product is commutative (see **2.1**). The graphical shape of each distinguished vertex only indicates that all tensors in a given diagram are not the same. We need not keep with, say, denoting α by a circle, and indeed we shall not. Exceptions are made for tensors that will be central to our analysis, as previously mentioned.

The graphical arrangement of the edges and vertices is not important, as long as the order in which the edges connect to the distinguished vertex is respected. Therefore, we may place distinguished vertices and labelled edges wherever we wish in a diagram. In other words, applying smooth maps to the diagram does not change the tensors it diagrammises. For example,



(1.6)

We prefer to keep edge labels arranged in two rows, across the top and bottom of the diagram, though the arrangement is arbitrary. This adherence to a certain aspect of presentation is kept throughout for ease of reading.

Remark 1.3.4. With regards to notation, as already used, we employ the following symbols:

$\leftrightsquigarrow$	identification between an abstract tensor and a tensor diagram
$\simeq$	equivalence of abstract tensors and vector spaces under isomorphism
$=$	equality of scalars, abstract tensors, vector spaces, or tensor diagrams

For the symbol $\leftrightsquigarrow$, we claim that there is a 1-1 correspondence between abstract tensors and tensor diagrams. This is clear from the following directives.

· Go around the boundary of every distinguished vertex, starting from the anchor, in a clockwise direction, marking edges with unique labels, noting whether edges are directed toward or away from it.

· The labels of the edges directed toward the distinguished vertex become subscripts, and the ones directed away become superscripts.

· Edges that have both ends incident with a distinguished vertex have their labels identified with each other.

· The ordering of the abstract tensors does not matter, as abstract tensor systems are commutative.

Abstract tensor systems are commutative in the sense that there exist isomorphisms between $V \otimes V^*$ and $V^* \otimes V$, as well as between $V \otimes V \otimes V^*$ and $V \otimes V^* \otimes V$, and so on, as there is no change in dimension. Thus for a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$ of V and dual basis $\{\mathbf{v}^1, \ldots, \mathbf{v}^6\}$ of the dual space V^* with $\lambda_1, \lambda_2 \in \mathbb{C}$, we write

$$\lambda_1\ \mathbf{v}_1 \otimes \mathbf{v}^3 \simeq \lambda_1\ \mathbf{v}^3 \otimes \mathbf{v}_1 \qquad \text{and} \qquad \lambda_2\ \mathbf{v}_4 \otimes \mathbf{v}_6 \otimes \mathbf{v}^1 \simeq \lambda_2\ \mathbf{v}_4 \otimes \mathbf{v}^1 \otimes \mathbf{v}_6$$

It is important to note that this use of isomorphisms is allowable only if V is finite-dimensional, which it will always be for our discussion. We will occupy ourselves in large part with the formulation of similar isomorphisms into equalities between tensor diagrams.

Proposition 1.3.5. *Twist isomorphism.* Let us define for later use, the simplest isomorphism. Let U be an n-dimensional vector space with basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and V be an m-dimensional vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, and $\lambda \in \mathbb{C}$. Then there exists a map T such that

$$\begin{array}{rrcl} T: & U \otimes V & \to & V \otimes U \\ & \lambda\ \mathbf{u}_i \otimes \mathbf{v}_j & \mapsto & T(\lambda)\ \mathbf{v}_j \otimes \mathbf{u}_i \end{array}$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. This map is generated by identifying the basis $\{\mathbf{u}_1 \otimes \mathbf{v}_1, \ldots, \mathbf{u}_n \otimes \mathbf{v}_m\}$ of $U \otimes V$ to $\{\mathbf{v}_1 \otimes \mathbf{u}_1, \ldots, \mathbf{v}_m \otimes \mathbf{u}_n\}$, which is the basis of $V \otimes U$. Since T is a bijective morphism, we can find its inverse map $T^{-1} : V \otimes U \to U \otimes V$.

2 A diagrammised abstract tensor system

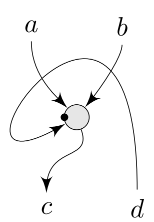
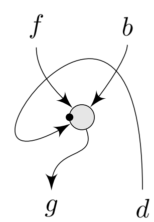
An abstract tensor system is a set of abstract tensors along with operations acting on the abstract tensors.

2.1 Generation

If the elements in a diagrammatic tensor system are precisely the elements which arise through the operations below acting on a given set of S of diagrams, then we say that S generates the diagrammatic tensor system.

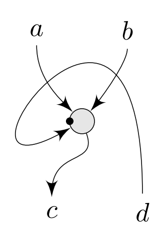
Reindexation. By reassigning different symbols as indeces, we are not changing the value of the tensor, only its presentation. Reindexation is basically renaming with a new symbol that does not already appear as an index, thus any new labels assigned are arbitrary.

$$\begin{aligned} \alpha_{ab}{}^{c}{}_{d} &= \alpha_{ab}{}^{c}{}_{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \otimes \mathbf{v}^d \\ &= \alpha_{fb}{}^{g}{}_{d}\ \mathbf{v}^f \otimes \mathbf{v}^b \otimes \mathbf{v}_g \otimes \mathbf{v}^d \\ &= \alpha_{fb}{}^{g}{}_{d} \end{aligned}$$

$\alpha_{ab}{}^{c}{}_{d} \quad \rightsquigarrow$  $=$  $\rightsquigarrow \quad \alpha_{fb}{}^{g}{}_{d}$ (2.1)

Scalar multiplication. Scalar multiplication of the abstract tensor is scalar multiplication of the formal tensor. In the following tensor diagram, the scalar $\lambda \in \mathbb{C}$ is placed to the right of the distinguished vertex it affects. This can always be done in a manner that does not introduce confusion, as edges may be moved around in the diagram.

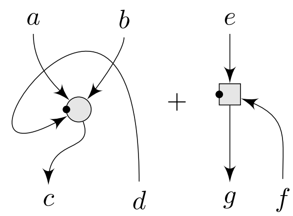
$$\lambda\ \alpha_{ab}{}^{c}{}_{d} = \lambda \alpha_{ab}{}^{c}{}_{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \otimes \mathbf{v}^d$$

$\lambda\ \alpha_{ab}{}^{c}{}_{d} \quad \rightsquigarrow \quad \lambda$  (2.2)

Addition. As a formal operation, addition does not require elements of the same type.

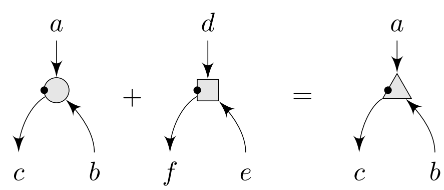
$$\alpha_{ab}{}^{c}{}_{d} + \beta_{ef}{}^{g} = \alpha_{ab}{}^{c}{}_{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \otimes \mathbf{v}^d \ + \ \beta_{ef}{}^{g}\ \mathbf{v}^e \otimes \mathbf{v}^f \otimes \mathbf{v}_g$$

$$\alpha_{ab}{}^{c}{}_{d} + \beta_{ef}{}^{g} \quad \rightsquigarrow \tag{2.3}$$



Every tensor diagram that represents a term in the sum should be separated from the the other tensor diagrams. If the types of the two tensors are the same, then we may combine them. This loses the identity associated with the abstract tensor that we began with, but does simplify presentation. For example,

$$\begin{aligned}
\gamma_{ab}{}^{c} + \beta_{de}{}^{f} &= \gamma_{ab}{}^{c}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c\ + \beta_{de}{}^{f}\ \mathbf{v}^d \otimes \mathbf{v}^e \otimes \mathbf{v}_f \\
&= \gamma_{ab}{}^{c}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c\ + \beta_{ab}{}^{c}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c && (2.4) \\
&= (\gamma_{ab}{}^{c}\ +\ \beta_{ab}{}^{c})\, \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \\
&= \chi_{ab}{}^{c}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c && (2.5) \\
&= \chi_{ab}{}^{c}
\end{aligned}$$

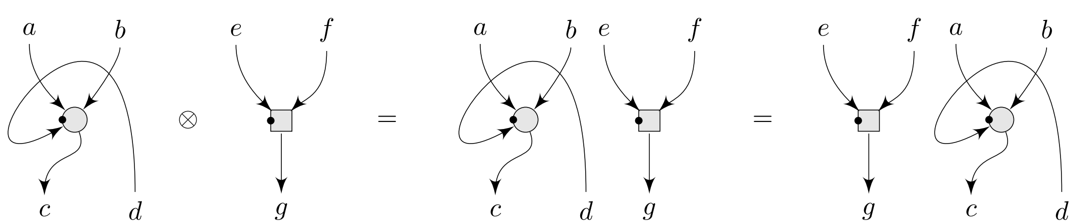


(2.6)

By index renaming we get (2.4), and by addition (2.5). Here we have that the sum of two tensors in a given space is another (unknown) tensor in that same space, which is what (2.6) diagrammises - note the changed final shape. Thus addition with scalar multiplication forms a vector space of abstract tensors.

Tensor product. The operation $\otimes : V \times V \to V \otimes V$ involves joining two tensors into a single tensor. For clarity, we require that the two separate tensors do not have any common indeces, easily ensured by reindexation. Any scalars that the tensors might have are multiplied together.

$$\alpha_{ab}{}^{c}{}_{d} \otimes \beta_{ef}{}^{g} \ = \ \alpha_{ab}{}^{c}{}_{d}\, \beta_{ef}{}^{g} = \beta_{ef}{}^{g}\ \alpha_{ab}{}^{c}{}_{d}$$



(2.7)

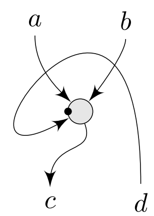
We cannot distinguish exactly which vector space the above element is in, so instead we view it as being in both $(V^*)^{\otimes 2} \otimes V \otimes (V^*)^{\otimes 3} \otimes V$ and $(V^*)^{\otimes 2} \otimes V \otimes (V^*)^{\otimes 2} \otimes V \otimes V^*$ concurrently.

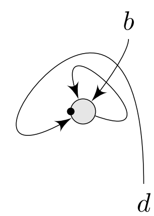
The product operation is distributive over addition. We will further omit the $\otimes$ symbol, when tensor multiplication is understood.

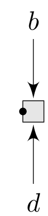
$$\theta^{h}{}_{i}{}^{j} \otimes \left(\alpha_{ab}{}^{c}{}_{d} + \beta_{ef}{}^{g}\right) = \theta^{h}{}_{i}{}^{j} \otimes \alpha_{ab}{}^{c}{}_{d} + \theta^{h}{}_{i}{}^{j} \otimes \beta_{ef}{}^{g}$$

Contraction. We define an operation that sets equal the labels of an upper-lower pair of indeces. For a tensor ρ, the contraction $\kappa_{i,j}(\rho)$ will set the ith and jth indeces equal to each other. Exactly one of the indeces to be contracted must be a lower index, and exactly one must be an upper index. After identification, we may move to a lower-dimensional vector space, recalling that this move will lose us information.

$$\begin{aligned}
\kappa_{(1,3)}\left(\alpha_{\ ab}{}^{c}{}_{d}\right) &= \kappa_{(1,3)}\left(\alpha_{\ ab}{}^{c}{}_{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \otimes \mathbf{v}^d\right) \\
&= \alpha_{\ ab}{}^{a}{}_{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_a \otimes \mathbf{v}^d \\
&= \beta_{\ bd}\ \mathbf{v}^b \otimes \mathbf{v}^d \\
&= \beta_{\ bd}
\end{aligned} \tag{2.8}$$

if $\alpha_{ab}{}^{c}{}_{d}$ $\rightsquigarrow$ 

then $\kappa_{1,3}\left(\alpha_{ab}{}^{c}{}_{d}\right)$ $\rightsquigarrow$ 

= 

(2.9)

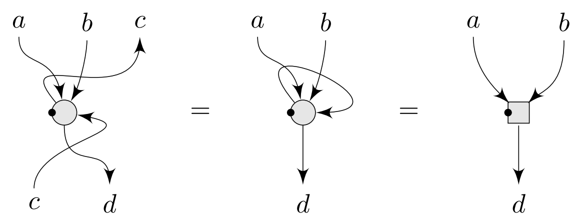
When applying the contraction map to a tensor that is the product of other tensors, care must be taken, as the indeces indicated in the subscript of κ refer to the order of indeces in the argument. Thus

$$\kappa_{(4,7)}\left(\alpha_{ab}{}^{c}{}_{d}\ \beta_{ef}{}^{g}\right) = \alpha_{ab}{}^{c}{}_{d}\ \beta_{ef}{}^{d} \qquad\qquad \kappa_{(4,7)}\left(\beta_{ef}{}^{g}\ \alpha_{ab}{}^{c}{}_{d}\right) = \beta_{ef}{}^{g}\ \alpha_{ab}{}^{c}{}_{a}$$

Through this operation we may reduce the rank of the tensor by two. Identical upper-lower indeces are termed *dummy* indeces. Thus we may consider the tensor in a lower dimensional space.

$$\begin{aligned}
\beta^{c}{}_{abc}{}^{d} &= \beta^{c}{}_{abc}{}^{d}\ \mathbf{v}_c \otimes \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}^c \otimes \mathbf{v}_d \\
&= \beta^{1}{}_{ab1}{}^{d}\ \mathbf{v}_1 \otimes \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}^1 \otimes \mathbf{v}_d + \cdots + \beta^{n}{}_{abn}{}^{d}\ \mathbf{v}_n \otimes \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}^n \otimes \mathbf{v}_d && (2.10) \\
&\simeq \beta^{1}{}_{1ab}{}^{d}\ \mathbf{v}_1 \otimes \mathbf{v}^1 \otimes \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_d + \cdots + \beta^{n}{}_{nab}{}^{d}\ \mathbf{v}_n \otimes \mathbf{v}^n \otimes \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_d && (2.11) \\
&= \gamma_{\ ab}{}^{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_d + \cdots + \chi_{\ ab}{}^{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_d \\
&= \phi_{\ ab}{}^{d}\ \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_d \\
&= \phi_{\ ab}{}^{d}
\end{aligned}$$

The original tensor is an element in $V \otimes (V^*)^{\otimes 3} \otimes V$. By expanding in (2.10) over c, the repeated index in each term does not contribute to the type of that term. In (2.11) we use $\simeq$ as we are applying the T isomorphism from **1.3.5**. This also allows us to find appropriate tensors $\gamma, \ldots, \chi \in (V^*)^{\otimes 3} \otimes V$ to put in place of the previous ones. This calculation is diagrammised by

 (2.12)

The identification of the identical edges is natural, though we may leave the tensor as is with a looped edge, as such loops are worth considering when analyzing knots with the aid of these diagrams. So we see that we may insert and remove an upper-lower pair of identical indeces in any desired positions of an abstract tensor.

Diagrammisation of abstract tensors. The diagrammatic tensor system is generated by equivalences of abstract tensors and tensor operations with tensor diagrams.

Elements of this system are termed $T(V)$-diagrams.

2.2 Tensor concepts

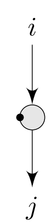
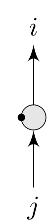
Here we discuss some notions unique to tensor diagrams.

The Kronecker delta tensor. Generally it is defined as a function on two variables.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

We will view it as a tensor. Penrose in [7] uses this symbol without taking into account the order of the indeces. Since their order in this exposition is at times paramount, we will make no such simplification. We will have

$$\delta_i{}^j = \delta_i{}^j \ \mathbf{v}^i \otimes \mathbf{v}_j \qquad\qquad \delta^i{}_j = \delta^i{}_j \ \mathbf{v}_i \otimes \mathbf{v}^j$$

$\delta_i{}^j \quad \leftrightsquigarrow$  $\qquad \delta^i{}_j \quad \leftrightsquigarrow$  (2.13)

When the indeces take on concrete values, we allow the symbol to be evaluated as a scalar value. For example,

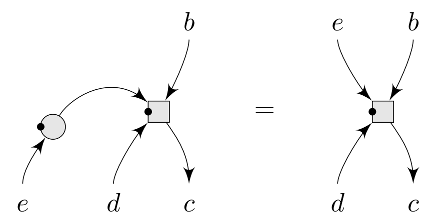
$$\delta_1{}^2 = \delta_3{}^1 = \delta^1{}_2 = 0 \quad \text{and} \quad \delta_1{}^1 = \delta_2{}^2 = \delta^1{}_1 = 1$$

This provides a formal way of relabelling indeces. If we wish to relabel the lower index a in a given tensor

α to another index, say e, we apply the tensor product to α and $\delta^a{}_e$. For example,

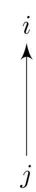
$$
\begin{aligned}
\alpha_{ab}{}^c{}_d\ \delta^a{}_e &= \sum_a \alpha_{ab}{}^c{}_d\ \delta^a{}_e \\
&= \sum_{\substack{a \\ a\neq e}} \alpha_{ab}{}^c{}_d\ \delta^a{}_e + \sum_{\substack{a \\ a=e}} \alpha_{ab}{}^c{}_d\ \delta^a{}_e \qquad (2.14)\\
&= \sum_{\substack{a \\ a=e}} \alpha_{ab}{}^c{}_d\ \delta^a{}_e \qquad (2.15)\\
&= \alpha_{1b}{}^c{}_d\ \delta^1{}_1 + \alpha_{2b}{}^c{}_d\ \delta^2{}_2 + \cdots + \alpha_{nb}{}^c{}_d\ \delta^n{}_n \\
&= \alpha_{1b}{}^c{}_d + \alpha_{2b}{}^c{}_d + \cdots + \alpha_{nb}{}^c{}_d \\
&= \alpha_{eb}{}^c{}_d
\end{aligned}
$$

As Kronecker's delta tensor may be evaluated as a scalar, a rearrangment of the sum in (2.14) shows that most of the terms reduce to zero in (2.15), resulting in the same tensor that we began with, now relabelled. In diagrammatic presentation,



(2.16)

The above indicates that we may depict the Kronecker delta tensor as a line, i.e. a graph with a single edge but no distinguished vertex. Diagrammatically,

$\delta_i{}^j \quad \leftrightsquigarrow$  $\qquad \delta^i{}_j \quad \leftrightsquigarrow$  (2.17)

This interpretation is useful in manipulating diagrams. It should also be noted that a delta-tensor may be split up into more delta tensors, into as many as desired:

$$\delta_i{}^j = \delta_i{}^k\ \delta_k{}^j = \delta_i{}^k\ \delta_k{}^\ell\ \delta_\ell{}^j = \delta_i{}^k\ \delta_k{}^m\ \delta_m{}^\ell\ \delta_\ell{}^j = \ldots$$

Here we have only discussed the Kronecker delta symbol with an upper-lower pair, so it might be tempting to consider the symbol with just an upper pair or a lower pair, δ_{ij} and δ^{ij}. We put off this temptation to when the need for such a symbol will necessitate its use (in **3.6**).

Equality. The equality relation makes a statement that two tensors are identical. When declaring equality, a claim is made about he scalars also. For example,

$$\alpha_{ab}{}^a{}_d = \beta_{bd} \tag{2.18}$$

This is equivalently stated as

$$\alpha_{ab}{}^a{}_d \mathbf{v}^b \otimes \mathbf{v}^d \otimes = \beta_{bd} \mathbf{v}^b \otimes \mathbf{v}^d$$

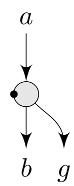
Fully expanded, we have

$$\alpha_{ab}{}^{a}{}_{d} = \sum_{a,b,d} \alpha_{ab}{}^{a}{}_{d}\ \mathbf{v}^b \otimes \mathbf{v}^d \qquad\qquad \beta_{bd} = \sum_{b,d} \beta_{bd}\ \mathbf{v}^b \otimes \mathbf{v}^d$$

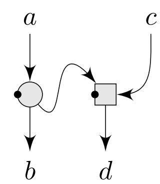
Then (2.18) asserts that, once b and d are fixed, the following equation holds for the scalars.

$$\sum_a \alpha_{ab}{}^{a}{}_{d} = \beta_{bd}$$

Subconfigurations. This term applies to diagrams generated by our system. Given a tensor ρ, isolating a subset of the distinguished vertices and all their associated legs, we get a subconfiguration of the original tensor. For example,



is a subconfiguration of



3 Representations of algebras

The phrase "*the algebra is generated by*" will mean that the tensor operations, described in the previous section, with the included list of tensors, constrained by any included restrictions, bring forth every tensor in the algebra under discussion. Recall first some basic facts about the nature and construction of algebras.

Definition 3.0.1. *Algebra.* Let V be a vector space over a field $\mathbb{F}$. Define a map $p : V \otimes V \to V$ to be a product on V. Note that p is linear in each of its arguments. Then the pair $\mathfrak{A} = (V, p)$ is termed an algebra of the vector space V endowed with p.

Definition 3.0.2. *Tensor algebra.* Given a vector space V over a field $\mathbb{F}$, the tensor algebra $\mathfrak{T}(V)$ over V is defined to be the algebra of the vector space

$$T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

endowed with an appropriate product. Every element of this algebra is strictly within some $V^{\otimes n}$ for $n \in \mathbb{N} \cup \{0\}$, so the product map is given by $p : V^{\otimes p} \otimes V^{\otimes q} \to V^{\otimes (p+q)}$. We may also have a tensor algebra $\mathfrak{T}(\mathfrak{A})$ over an algebra $\mathfrak{A}$, in which case the same construction, with $\mathfrak{A}$ instead of V, holds. The product on the tensor algebra in this case would be the multilinear equvalent of the algebra product of $\mathfrak{A}$.

Proposition 3.0.3. *Vector space isomorphism.* Let U be an n-dimensional vector space with basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and the dual space U^* with dual basis $\{\mathbf{u}^1, \ldots, \mathbf{u}^n\}$. Let V be an m-dimensional vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, and the dual space V^* with dual basis $\{\mathbf{v}^1, \ldots, \mathbf{v}^m\}$. Then there exists an isomorphism described by

$$\begin{array}{rrcl} T_{U,V} : & \mathrm{Hom}(U,V) & \to & U^* \otimes V \\ & \mathsf{f} & \mapsto & f_k{}^j\ \mathbf{u}^k \otimes \mathbf{v}_j \end{array}$$

The object f is a map, and $f_k{}^j$ represents the scalar coefficient of the tensor in the image. The action of the map f is given by:

$$\begin{array}{rrcl} \mathsf{f} : & U & \to & V \\ & \mathbf{u}_k & \mapsto & f_k{}^j\ \mathbf{v}_j \end{array}$$

It is important to note that the coefficients in the images of f and $T_{U,V}$ are identical. This gives us a way to interpret maps between vector spaces as elements in a tensor product of vector spaces.

3.1 The symmetric algebra

Definition 3.1.1. *Symmetric algebra.* The symmetric group, or permutation group of order n is denoted by $\mathfrak{S}_n$. The symmetric algebra is defined, over a vector space V over a field $\mathbb{F}$, to be the space

$$S(V) \doteq \bigoplus_{k \geqslant 0} S^{(k)}(V) = \bigoplus_{k \geqslant 0} \frac{\mathfrak{T}^{(k)}(V)}{\mathfrak{S}_k}$$

where $\mathfrak{T}^{(k)}(V)$ is the tensor algebra of the k-dimensional vector space $T(V)$ endowed with the tensor product. The quotient of this space by the symmetric group allows us to reduce the structure of the algebra in the following manner.

Proposition 3.1.2. Every abstract tensor, and equivalently, every tensor diagram, describes a unique element of $S(V)$.

Consider an abstract tensor. If it has a primitive diagram, the identification is clear. Suppose we have a compound diagram. As noted previously, a tensor that is the product of two tensors cannot be identified with certainty as belonging to a single space, thus to a single element of $T(V)$ (because of commutativity of the product). In $\mathfrak{S}(V)$ a tensor appears in all possible permutations of its primitive parts, therefore mudulo this group, they all describe the same element in $S(V)$.

That being said, in this algebra the order of spaces in the construction of a given tensor is not important. For example, consider the abstract tensor

$$\beta_{ab}{}^{c} = \beta_{ab}{}^{c}\, \mathbf{v}^a \otimes \mathbf{v}^b \otimes \mathbf{v}_c \in T^{(3)}(V)$$

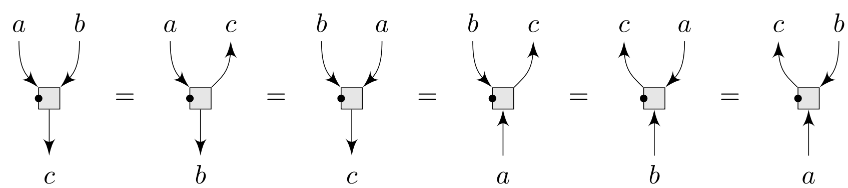
In the symmetric group we find as permutations all of the cycles

$$\begin{matrix} (a\ b\ c) & (b\ a\ c) & (c\ a\ b) \\ (a\ c\ b) & (b\ c\ a) & (c\ b\ a) \end{matrix}$$

Therefore we have as a string of equalities

$$\beta_{ab}{}^{c} = \beta_{a}{}^{c}{}_{b} = \beta_{ba}{}^{c} = \beta_{b}{}^{c}{}_{a} = \beta^{c}{}_{ab} = \beta^{c}{}_{ba}$$

and as tensor diagrams,



(3.1)

Recall that the labels have no significance past being placeholders. So in (3.1) we have a tensor with equality among all the permutations of its edges. This leads to the conclusion that the anchor $\bullet$ and the imposed order of vector spaces can be disregarded. Therefore we may remove anchors of tensors and labels of edges in the diagrammisation of the symmetric algebra, keeping only the direction in which each edge points, even disregarding the edge labels. Therefore in $S(V)$,

$$V^* \otimes V^* \otimes V = V^* \otimes V \otimes V^* \otimes = V \otimes V^* \otimes V^*$$

This allows us to diagrammise this algebra.

Diagrammisation of the symmetric algebra $S(V)$. The symmetric algebra is generated by unlabelled $T(V)$-diagrams with no anchors.

The diagrammatic tensor calculus that Roger Penrose uses (in [6],[7],[8]) is an excellent template for such a diagrammisation, as his system makes no use of anchors, but fully adopts the other tensor operations presented in the previous section. In his tensor diagrams, directed edges are distinguished by where they are located on the diagram, and those that join two distinguished vertices have no defining features.

3.2 Lie algebras

The diagrammisation of Lie algebras provides some motivation for Vassiliev invariants in knot theory (see [4]), as we shall partly discover. A Lie algebra $(\mathfrak{g}, [\ ,\ ])$ is an algebra $\mathfrak{g}$ associated with a bilinear map $[\ ,\ ]$. We will denote this pair, and sometimes the vector space, simply by $\mathfrak{g}$. This will hopefully not cause any confusion as it will be made clear what object we are referring to at each instance.

Definition 3.2.1. *Lie bracket.* The bilinear map $[\ ,\ ] \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, termed the *Lie bracket*, is defined by

$$\begin{array}{rccc} [\cdot,\cdot]: & \mathfrak{g} \otimes \mathfrak{g} & \to & \mathfrak{g} \\ & x \otimes y & \mapsto & [x, y] \end{array}$$

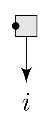
satisfying the following relations, for all $x, y, z \in \mathfrak{g}$.

$$[x, y] = -[y, x] \tag{R1}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \tag{R2}$$

The relation (R1) is known as the *anti-symmetric* relation, and (R2) is the *Jacobi* relation.

In this section we will discuss the n-dimensional vector space V as $\mathfrak{g}$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and its dual space V^* as the dual Lie algebra $\mathfrak{g}^*$ with dual basis $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$. An element in the vector space $\mathfrak{g}$ may be represented by $\sum_i \alpha^i \mathbf{v}_i$ as a tensor. Then we associate the abstract tensor α^i with $\sum_i \alpha^i \mathbf{v}_i$, and diagrammise this element by

$\alpha_i \quad \leftrightsquigarrow$ 

(3.2)

Analogously, any element in the vector space $\mathfrak{g}^*$ may be represented in the form $\sum_i \alpha_i \mathbf{v}^i$ for some scalars $\alpha_1, \ldots, \alpha_n$. Then we associate the tensor α_i with $\sum_i \alpha_i \mathbf{v}^i$, and diagrammise this element by

$\alpha^i \quad \leftrightsquigarrow$ 

(3.3)

As $[\cdot,\cdot] \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$, for some scalars β^k we have

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_k \beta^k\ \mathbf{v}_k \tag{3.4}$$

$$[\mathbf{v}_i, [\mathbf{v}_j, \mathbf{v}_k]] = [\mathbf{v}_i, \sum_\ell \beta^\ell\ \mathbf{v}_\ell] = \sum_\ell \beta^\ell\ [\mathbf{v}_i, \mathbf{v}_\ell] = \sum_{\ell,m} \beta^\ell\ \kappa^m\ \mathbf{v}_m \tag{3.5}$$

It seems that the constants β^k should somehow depend on $\mathbf{v}_i$ and $\mathbf{v}_j$, otherwise they would be the same no matter what tensors we put in, a counter-intuitive result. Recall the vector space isomorphism **3.0.3**. We wish to apply it to (3.4) and (3.5), so considering their ambient spaces, we see that $T_{\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}\big([\mathbf{v}_i,\mathbf{v}_j]\big) \in (\mathfrak{g}\otimes\mathfrak{g})^* \otimes \mathfrak{g} \simeq \mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}$. Applying $T_{\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}$ to (3.4),

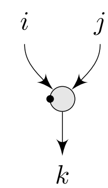
$$T_{\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}\big([\mathbf{v}_i,\mathbf{v}_j]\big) = \sum_{i,j,k} {\gamma_{ij}}^k\ \mathbf{v}^i\otimes\mathbf{v}^j\otimes\mathbf{v}_k$$

Now the scalar shows the contribution of the indeces i and j. Similarly we have $T_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}\big([\mathbf{v}_i,\mathbf{v}_j,\mathbf{v}_k]\big) \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})^*\otimes\mathfrak{g} \simeq \mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}$, and applied to (3.5),

$$T_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}\big([\mathbf{v}_i,[\mathbf{v}_j,\mathbf{v}_k]]\big) = \sum_{i,j,k,m} {\gamma_{i\ell}}^m{\gamma_{jk}}^\ell\ \mathbf{v}^i\otimes\mathbf{v}^j\otimes\mathbf{v}^k\otimes\mathbf{v}_m$$

Now we know enough to diagrammise the Lie bracket. This is done as

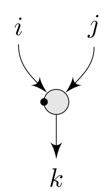
$$[\cdot,\cdot] \quad = \quad {\gamma_{ij}}^k \quad \rightsquigarrow \tag{3.6}$$



From now on we will represent the Lie bracket as the abstract tensor ${\gamma_{ij}}^k$ and with a circular diagram in its tensor diagrammisation. We can now restate the relations (R1) and (R2) in terms of abstract tensors and diagrams, rather than maps.

$$\begin{aligned} {\gamma_{ij}}^k + {\gamma_{ji}}^k &= 0 && \text{restatement of (R1)} \\ {\gamma_{i\ell}}^m{\gamma_{jk}}^\ell + {\gamma_{k\ell}}^m{\gamma_{ij}}^\ell + {\gamma_{j\ell}}^m{\gamma_{ki}}^\ell &= 0 && \text{restatement of (R2)} \end{aligned}$$

The diagrammisation of these tensors follows immediately. The anti-symmetric relation (R1) becomes



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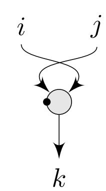
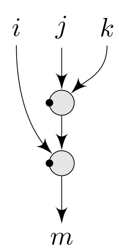


$$= \quad 0 \tag{3.7}$$

And the Jacobi relation (R2) becomes



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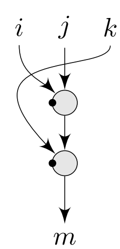
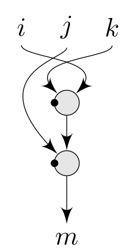
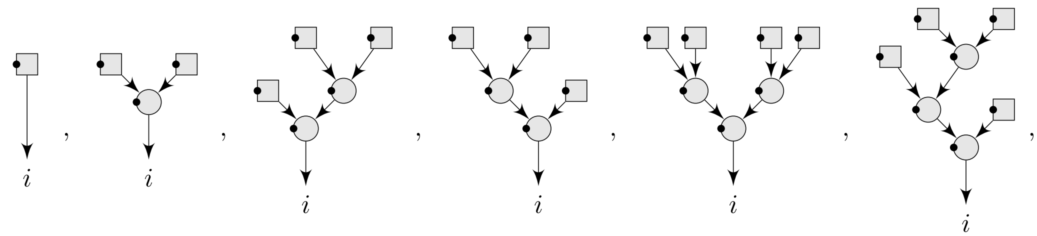


$+$



$$= \quad 0 \tag{3.8}$$

So far we have only inspected how the Lie bracket operates on primitive tensors, so now we generalize for arbitrary tensors ρ,κ to consider $[\rho,\kappa]$. But how do we get to an "arbitrary" tensor? Let us consider the simplest of elements in this algebra. These are presented in (3.6) and (3.2). Thus we consider only tensor diagrams directly arising from these abstract tensors. While it is possible to produce other very simple tensors through the product operation (and with information loss), we would then no longer be able to diagrammise them with the generator tensor. In order of increasing complexity, they would be



(3.9)

These diagrams correspond to the abstract tensors

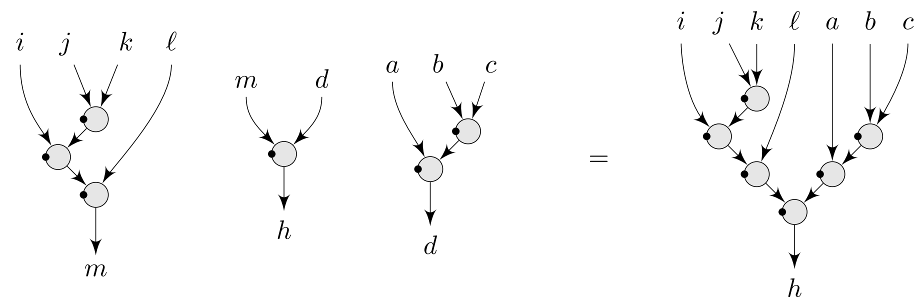
$$\mathbf{v}_i \quad , \quad [\mathbf{v}_a, \mathbf{v}_b] \quad , \quad [\mathbf{v}_a, [\mathbf{v}_b, \mathbf{v}_c]] \quad , \quad [[\mathbf{v}_a, \mathbf{v}_b], \mathbf{v}_c] \quad , \quad [[\mathbf{v}_a, \mathbf{v}_b], [\mathbf{v}_c, \mathbf{v}_d]] \quad , \quad [[\mathbf{v}_a, [\mathbf{v}_b, \mathbf{v}_c]], \mathbf{v}_d]$$

So, given n copies of $\mathfrak{g}$, i.e. $\mathfrak{g}^{\otimes n}$, there is a certain number of ways that the Lie bracket may be composed with itself to yield a particular tensor from this space. In general, given two specific nesting orders of Lie brackets $\rho = [\cdots] : \mathfrak{g}^{\otimes p} \to \mathfrak{g}$ and $\kappa = [\cdots] : \mathfrak{g}^{\otimes q} \to \mathfrak{g}$, we will have that

$$[\rho, \kappa] = [[\cdots], [\cdots]] : \mathfrak{g}^{\otimes (p+q)} \to \mathfrak{g}$$

For example, such a combination could be given by:

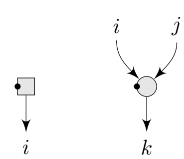
$[\rho, \kappa] \quad \rightsquigarrow$



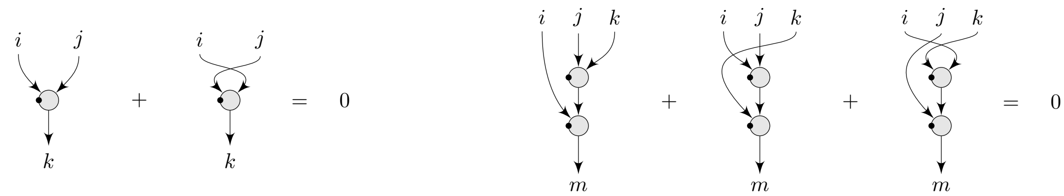
(3.10)

The pattern has now become clear. It is important to notice that in the diagrammisation of the Lie algebra $\mathfrak{g}$ by the generator (3.2) and the Lie bracket (3.6), every primitive tensor has only a single labelled edge directed away from the tensor. With that in mind, we may now characterize this diagrammisation.

Diagrammisation of the Lie algebra $\mathfrak{g}$. The elements of $\mathfrak{g}$ are generated by



Every diagram has exactly one labelled edge directed away from a distinguished vertex. The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)



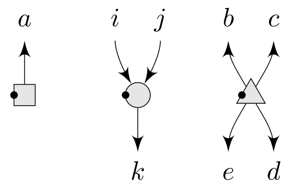
The Lie bracket is useful for reducing valence of a tensor diagram, as it is the only tensor diagram with edges of both types. For example, consider the abstract tensor

$$\alpha^{a}\ \beta^{bcde}$$

To associate it diagrammatically with a Lie bracket, we apply the contraction operation.

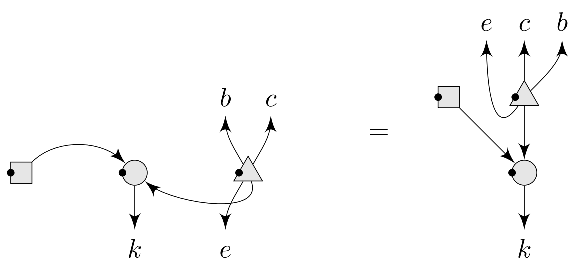
$$\kappa_{4,7}\left(\kappa_{1,6}\left(\alpha^{a}\ \beta^{bcde} \otimes \gamma_{ij}{}^{k}\right)\right) = \alpha^{a}\ \beta^{bcde}\gamma_{ab}{}^{k}$$

We start with the tensor diagram



(3.11)

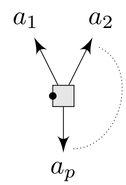
Identifying labels, contracting them, and rearranging the whole diagram yields the following result, showing that the Lie bracket has in effect turned an upper index into a lower index.



(3.12)

3.3 The tensor algebra $\mathfrak{T}(\mathfrak{g})$

From **3.1.1** we have that every element in $\mathfrak{T}^{(k)}(V)$ is a linear combination of elements in $V^{\otimes k}$. Any particular element of $V^{\otimes k}$ is then given by the abstract tensor $\alpha^{a_1 \cdots a_p} = \alpha^{a_1 \cdots a_p}\ \mathbf{v}_{a_1} \otimes \cdots \otimes \mathbf{v}_{a_p}$ for all $p \in \mathbb{N}$. This is diagrammised by



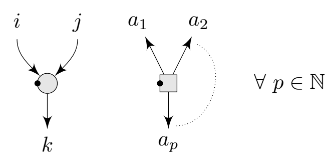
(3.13)

The labels and corresponding edges $a_3, \ldots, a_{p-1}$ have been omitted and replaced by a dotted line. Because of the ambient space, all edges are directed away from the distinguished vertex. Note that for $p = 1$, this is equivalent to the Lie algebra generator (3.2).

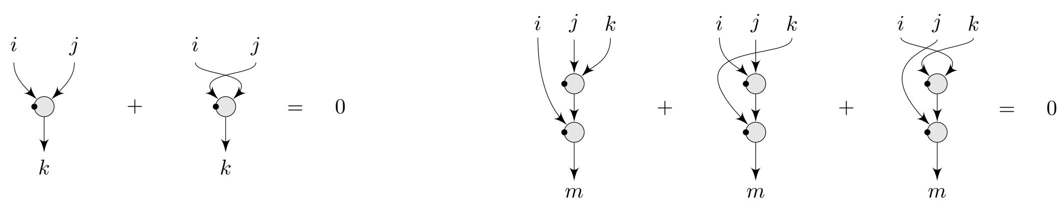
By **3.1.1**, the tensor algebra is to be accompanied by a product, and in this case we have multiplication. In the general setting, Multiplication of two different tensors is their disjoint union in a diagram (the task of connecting them is undertaken by the contraction operation).

Since we have already analyzed a Lie algebra $\mathfrak{g}$, by taking the tensor algebra of $\mathfrak{g}$ the structure is unchaged, and only new fundamental tensors are introduced. In contrast with the Lie algebra $\mathfrak{g}$, it is not true that there is exactly one labeled edge directed outward in each primitive tensor diagram, due to the introduction of these tensors.

Diagrammisation of the tensor algebra $\mathfrak{T}(\mathfrak{g})$. The elements of $\mathfrak{T}(\mathfrak{g})$ are generated by



The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)

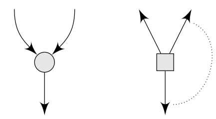


Elements of this system are termed $\mathfrak{T}(V)$-diagrams.

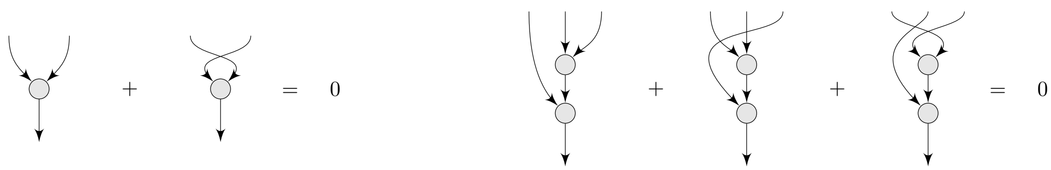
3.4 The symmetric Lie algebra $S(\mathfrak{g})$

As noted in (3.1), use of the symmetric algebra as the quotient in a quotient space removes the labels and anchors but preserves edge orientation. Therefore we make the direct analogy to the simpler symmetric algebra and the previous section, while keeping ideas from the Lie algebra.

Diagrammisation of the symmetric algebra $S(\mathfrak{g})$. The elements of $S(\mathfrak{g})$ are generated by



The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)



3.5 The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$

This algebra gives an interpretation of the Lie bracket. This allows for simplification, while still retaining the basic structure of $\mathfrak{g}$. Given a Lie algebra $\mathfrak{g}$ with bracket operator $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ and product $\cdot : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$ is defined as

$$\mathfrak{U}(\mathfrak{g}) \mathrel{\hat{=}} \frac{\mathfrak{T}(\mathfrak{g})}{[x,y] = x\cdot y - y\cdot x} = \bigoplus_{k\geqslant 0} \mathfrak{g}^{\otimes k} \Big/ [x,y]$$

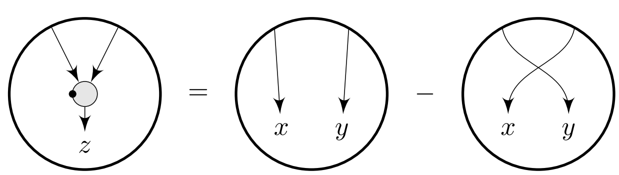
Our next task is to find a proper way to diagrammise $x\cdot y - y\cdot x$. Based on our analysis so far, it is natural to choose the tensor product $\otimes$ as the product $\cdot$. Recall that in **2.1** a commutative product was constructed, such that for tensors ρ,κ we have $\rho\otimes\kappa = \kappa\otimes\rho$. Although then

$$[\rho,\kappa] = \rho\otimes\kappa - \kappa\otimes\rho = \rho\otimes\kappa - \rho\otimes\kappa = 0$$

Such a statement would render useless the potential algebra structure. However, the Lie bracket operates not on an arbitrary pair of tensors, but on a pair of tensors in the vector space $\mathfrak{g}^{\otimes 1}$. Diagrammatically speaking, this is equivalent to a pair of labeled edges directed away from their respective distinguished vertices.

So for $\rho,\kappa \in \mathfrak{T}(\mathfrak{g})$. the expression $[\rho,\kappa]$ is better interpreted as $[x,y]$, where x is an outward edge of the diagram of ρ and y is an outward edge of the diagram of κ, and z is the outward directed edge of the Lie bracket diagram. In view of this, it is natural to interpret the introduced definition of $[x,y]$ as

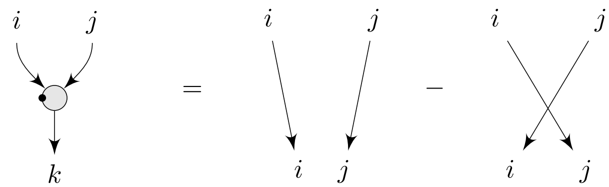
$$[x,y] \quad = \quad xy - yx \quad \rightsquigarrow$$



(3.14)

We have isolated a section of the diagram $[\rho,\kappa]$ on the left-hand side, observing only the outward edges x, y and how they connect to ${\gamma_{xy}}^{z}$. Everything else outside this circle stays the same in both terms on the right-hand side, which is a linear combination of two tensors. Essentially, decomposition of the Lie bracket produces two copies of the original diagram, one with interchanged labels.

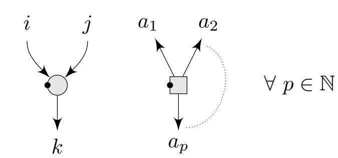
If instead of tensors, we consider just the Lie bracket operator and its diagram, we get the equality



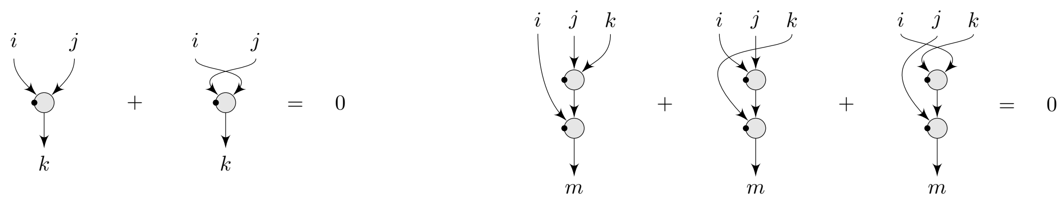
(3.15)

This allows for a reinterpretation of diagrams, as we may reduce or increase the number of trivalent vertices in a given graph. Now we may diagrammise this algebra.

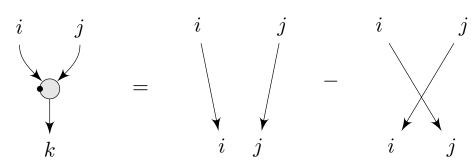
Diagrammisation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. The elements of $\mathfrak{U}(\mathfrak{g})$ are generated by



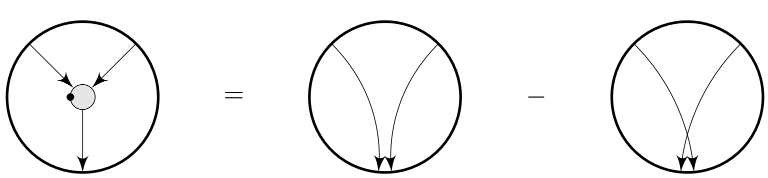
The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)



as well as an interpretation of the Lie bracket diagram (3.15)

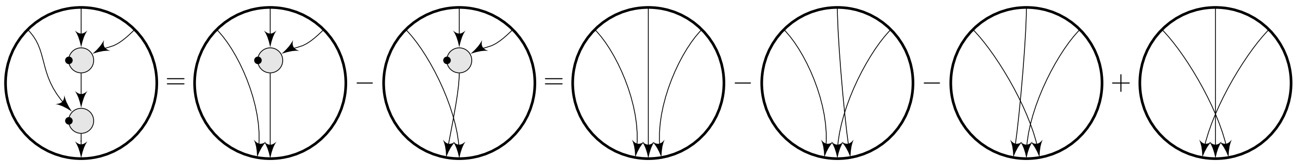


The Lie bracket decomposition may be represented in another similar manner, by ordering the labels along a line, or fixing them along a circle. Labels are then removed from the diagrams, as their order has been fixed.

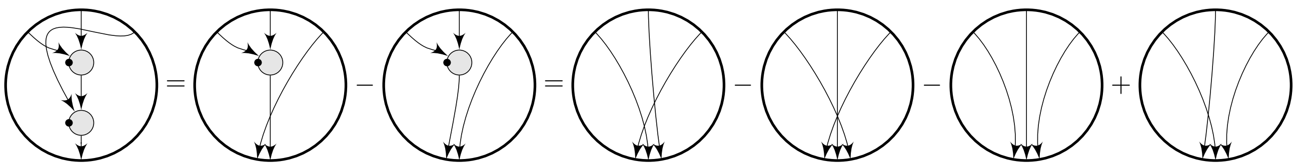


(3.16)

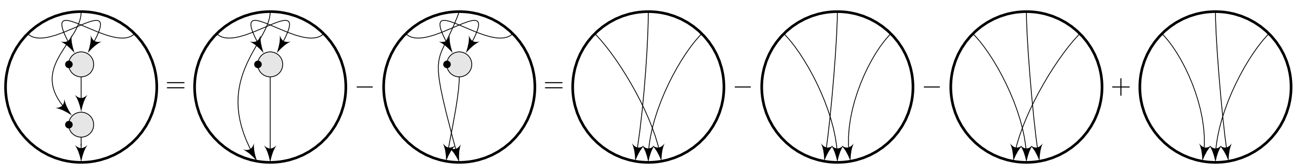
For tensors having the Lie bracket as a subconfiguration, the same decomposition applies. Representing them on such circles is also analogous, but care must be taken to add / remove nodes and edges in an identical manner for all diagrams in an equality - no added edges may end in the shortest section between edges on the diagram above. If, for example, we wish to decompose the Jacobi relation, we may describe the terms in fixed circular diagrams and expand by (3.16). For each term separately,



(3.17)



(3.18)



(3.19)

Combining the three equations results in complete reduction to zero on the right-hand side, showing the Jacobi relation is a true statement.

3.6 The metrized Lie algebra

This algebra has a rich structure, with room for more extensions than expanded upon here.

3.6.1 Diagrammisation with a bilinear form

Here we shall introduce an abstract metric through a type of distance function. The metrized Lie algebra $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a Lie algebra $\mathfrak{g}$ accompanied by a non-degenerate bilinear form $\langle\ ,\ \rangle$. See [3] for a more extensive description of bilinear forms and what may be accomplished with them. If $\mathfrak{g}$ is over a field $\mathbb{F}$, then formally

$$\begin{array}{rrcl} \langle\ ,\ \rangle : & \mathfrak{g} \otimes \mathfrak{g} & \to & \mathbb{F} \\ & (\mathbf{v}_i, \mathbf{v}_j) & \mapsto & f \end{array}$$

We would like the image f of the input tensors $\mathbf{v}_i$ and $\mathbf{v}_j$ to depend upon i and j. Considering the bilinear map by itself and recalling the vector space isomorphism we previously had in **3.0.3**,

$$\langle\ ,\ \rangle \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{F}) \cong (\mathfrak{g} \otimes \mathfrak{g})^* \cong \mathfrak{g}^* \otimes \mathfrak{g}^*$$

Thus we have for some scalars h_{ij}

$$\langle \cdot, \cdot \rangle = h_{ij}\ \mathbf{v}^i \otimes \mathbf{v}^j \tag{3.20}$$

From this we may diagrammize the bilinear map. We assign to it a unqiue tensor diagram shape.

$$\langle \cdot , \cdot \rangle \quad = \quad h_{ij} \quad \leftrightsquigarrow \tag{3.21}$$

To procure an analogous tensor with inward rather than outward directed edges, we note that by the same vector space isomorphism, we have $\langle\ ,\ \rangle \in \mathfrak{g}^* \otimes \mathfrak{g}^* \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^*)$, giving the map, for the same scalars h_{ij} as above

$$\begin{array}{rccc} \langle\ ,\ \rangle : & \mathfrak{g} & \to & \mathfrak{g}^* \\ & \mathbf{v}_i & \mapsto & h_{ij}\mathbf{v}^j \end{array} \tag{3.22}$$

Since we have a non-degenerate bilinear form $\langle\ ,\ \rangle$, it is invertible. Therefore there exists a map

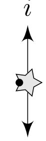
$$\langle\ ,\ \rangle^{-1} \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}$$

As such we may characterize it, using the letter h for continuity, by

$$\langle \cdot , \cdot \rangle^{-1} = h^{ij}\ \mathbf{v}_i \otimes \mathbf{v}_j$$

From this we may diagrammize the inverted bilinear map.

$$\langle \cdot , \cdot \rangle^{-1} \quad = \quad h^{ij} \quad \leftrightsquigarrow \tag{3.23}$$



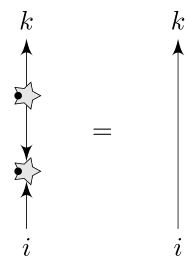
We shall call these the h-tensors. Since $\langle\ ,\ \rangle^{-1}$ is a homomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}$, we have another description with the same scalars h^{ij} of this map

$$\begin{array}{rccc} \langle\ ,\ \rangle^{-1} : & \mathfrak{g}^* & \to & \mathfrak{g} \\ & \mathbf{v}^i & \mapsto & h^{ij}\mathbf{v}_j \end{array} \tag{3.24}$$

These two maps are related through the Kronecker delta symbol, which we encountered earlier in **2.2**. Consider

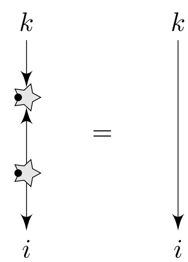
$$\begin{aligned} \mathbf{v}_i &= \mathrm{id}(\mathbf{v}_i) \\ &= \langle\ ,\ \rangle^{-1} \circ \langle\ ,\ \rangle(\mathbf{v}_i) \\ &= \langle\ ,\ \rangle^{-1}\left(h_{ij}\mathbf{v}^j\right) \\ &= h_{ij}\langle\ ,\ \rangle^{-1}\left(\mathbf{v}^j\right) \\ &= h_{ij}h^{jk}\ \mathbf{v}_k \end{aligned}$$

Comparing coefficients, it is immediate that as scalars, $h_{ij}h^{jk} = \delta_i{}^k$. Implementing the structure from **2.2** with this tensor, we now have that



(3.25)

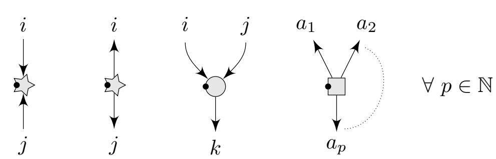
Similarly performing an analogous series of moves on $\mathbf{v}^i$, we find that $h^{ij}h_{jk} = \delta^i{}_k$, and diagrammised,



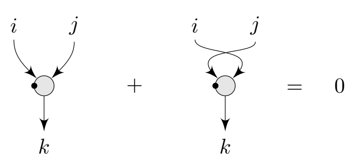
(3.26)

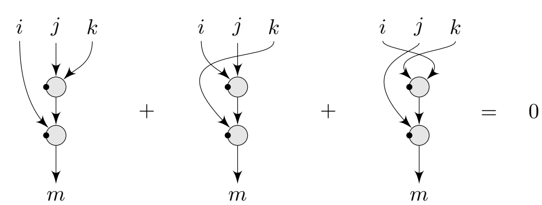
While (3.25) and (3.26) seem similar, it should be noted that the placement of the anchor differs. To combine the diagrammisations, it should be that the placement of the basepoint be irrelevant. No such a simplification can be made yet. We proceed with the diagrammisation of this algebra.

Diagrammisation of the metrized Lie algebra. The elements of $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ are generated by

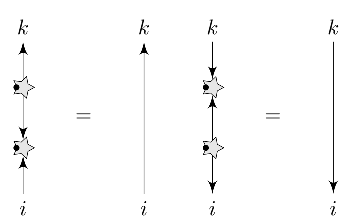


The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)





as well as two relations on the h-tensors, (3.25) and (3.26)



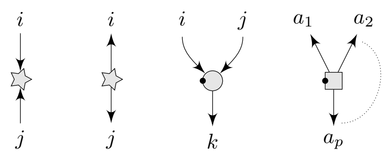
3.6.2 Diagrammisation with a symmetric bilinear form

If the bilinear form is symmetric, i.e. $\langle a, b\rangle = \langle b, a\rangle$ for any appropriate arguments a, b, then the anchor on h_{ij} may be removed, as

$$\rightsquigarrow \quad \langle i, j\rangle \quad = \quad \langle j, i\rangle \quad \rightsquigarrow \tag{3.27}$$

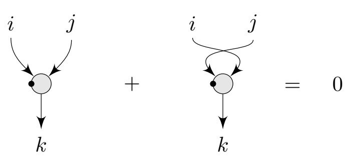
Therefore $h_{ij} = h_{ji}$ and analogously $h^{ij} = h^{ji}$. This allows us to remove one of the h-tensor relations from the diagrammisation of this algebra.

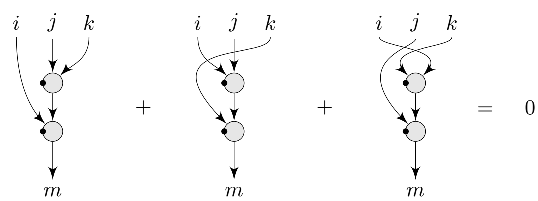
Diagrammisation of the metrized Lie algebra with a symmetric bilinear form. The elements of $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ are generated by



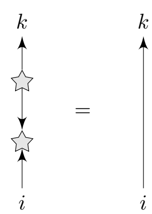
$\forall\ p \in \mathbb{N}$

The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)





as well the h-tensor relation



To view the Lie bracket tensor also without an anchor, it is necessary to assume that we have an orthonormal basis, at least by our approach.

3.6.3 Diagrammisation with a symmetric bilinear form and an orthonormal basis

Consider an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. This will allow us to demonstrate that the diagrammisation of the Lie bracket is invariant under movement of the anchor - that $\gamma_{ij}^{\ jk} = \gamma^{k}_{\ ij} = \gamma_{j\ i}^{\ k}$. Begin by simplifying some tensor expressions.

$$
\begin{aligned}
{\gamma_{ij}}^{k}\ h_{\ell k} &= {\gamma_{ij}}^{k}\ h_{k\ell} && (3.28)\\
&= \sum_{i,j,k,\ell} {\gamma_{ij}}^{k}\ h_{k\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_k \otimes \mathbf{e}^k \otimes \mathbf{e}^\ell \\
&= \sum_{i,j,\ell} \Big({\gamma_{ij}}^{1}\ h_{1\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_1 \otimes \mathbf{e}^1 \otimes \mathbf{e}^\ell + \cdots + {\gamma_{ij}}^{n}\ h_{n\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_n \otimes \mathbf{e}^n \otimes \mathbf{e}^\ell \Big) \\
&= \sum_{i,j,\ell} \Big({\gamma_{ij}}^{1}\ h_{1\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell + \cdots + {\gamma_{ij}}^{n}\ h_{n\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \Big) \\
&= \sum_{i,j} \Big({\gamma_{ij}}^{1}\ h_{11}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^1 + \cdots + {\gamma_{ij}}^{n}\ h_{nn}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^n \Big) && (3.29)\\
&= \sum_{i,j} \Big({\gamma_{ij}}^{1}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^1 + \cdots + {\gamma_{ij}}^{n}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^n \Big) \\
&= \sum_{i,j,\ell} {\gamma_{ij}}^{\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \\
&= \sum_{i,j,\ell} \gamma_{ij\ell}\ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell && (3.30)\\
&= \gamma_{ij\ell} && (3.31)
\end{aligned}
$$

In (3.28) we use the fact that the bilinear form is symmetric, and in (3.29) we expanded over ℓ to show that many terms reduce to zero. For (3.30) we made the switch of upper index ℓ to lower index ℓ to accomodate the fact that the tensor is in $(\mathfrak{g}^*)^{\otimes 3}$ now. The equality in (3.30) also means that, as scalars, ${\gamma_{ij}}^{k} = \gamma_{ij\ell}$ for $k = \ell$, which we use in (3.32) below. We continue in the same manner as above.

$$
\begin{aligned}
\gamma_{ij\ell}\ h^{jk} &= \gamma_{ij\ell}\ h^{kj} \\
&= h^{kj}\ \gamma_{ij\ell} \\
&= -h^{kj}\ \gamma_{ji\ell} && (3.32)\\
&= -\sum_{i,j,k,\ell} h^{kj}\ \gamma_{ji\ell}\ \mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}^j \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell \\
&= -\sum_{i,k,\ell} \big(h^{k1}\ \gamma_{1i\ell}\ \mathbf{e}_k \otimes \mathbf{e}_1 \otimes \mathbf{e}^1 \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell + \cdots + h^{kn}\ \gamma_{ni\ell}\ \mathbf{e}_k \otimes \mathbf{e}_n \otimes \mathbf{e}^n \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell \big) \\
&= -\sum_{i,k,\ell} \big(h^{k1}\ \gamma_{1i\ell}\ \mathbf{e}_k \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell + \cdots + h^{kn}\ \gamma_{ni\ell}\ \mathbf{e}_k \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell \big) \\
&= -\sum_{i,\ell} \big(h^{11}\ \gamma_{1i\ell}\ \mathbf{e}_1 \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell + \cdots + h^{nn}\ \gamma_{ni\ell}\ \mathbf{e}_n \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell \big) && (3.33)\\
&= -\sum_{i,\ell} \big(\gamma_{1i\ell}\ \mathbf{e}_1 \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell + \cdots + \gamma_{ni\ell}\ \mathbf{e}_n \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell \big) \\
&= -\sum_{i,k,\ell} \gamma_{ki\ell}\ \mathbf{e}_k \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell && (3.34)\\
&= -\sum_{i,k,\ell} {\gamma^{k}}_{i\ell}\ \mathbf{e}_k \otimes \mathbf{e}^i \otimes \mathbf{e}^\ell && (3.35)\\
&= -{\gamma^{k}}_{i\ell} \\
&= {\gamma_i}^{k}{}_{\ell} && (3.36)
\end{aligned}
$$

Equation (3.32) uses the fact that ${\gamma_{ij}}^k = \gamma_{ij\ell}$ as scalars for $k = \ell$, and the anti-symmetric relation. The same change is made in (3.34) as above, to accomodate for the change of space, keeping in mind that scalars have not changed. That is, as scalars $-{\gamma^k}_{i\ell} = \gamma_{ij\ell} = -\gamma_{ji\ell}$ for $j = k$, which allows us to reorder the indeces in (3.36), continuing from the properties of the relabelled Lie bracket above. In (3.33) the same change is made as in (3.29).

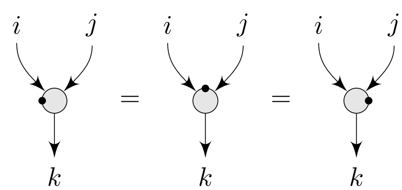
Now we combine the above results, and apply the Kronecker delta, as a tensor.

$$
\begin{aligned}
{\gamma_{ij}}^k &= {\gamma_{ij}}^k \ {\delta_\ell}^\ell \\
&= {\gamma_{ij}}^k \ {\delta_\ell}^i \ {\delta_i}^\ell \\
&= {\gamma_{ij}}^k \ {\delta_\ell}^j \ {\delta_j}^i \ {\delta_i}^\ell \\
&= {\delta_j}^i \ {\gamma_{ij}}^k \ {\delta_\ell}^j \ {\delta_i}^\ell \\
&= {\delta_j}^i \ {\gamma_{ij}}^k \ h_{\ell k} \ h^{kj} \ {\delta_i}^\ell \\
&= {\delta_j}^i \ \gamma_{ij\ell} \ h^{kj} \ {\delta_i}^\ell && (3.37) \\
&= {\delta_j}^i \ \gamma_{ij\ell} \ h^{jk} \ {\delta_i}^\ell \\
&= {\delta_j}^i \ {\gamma_i}^k{}_\ell \ {\delta_i}^\ell && (3.38) \\
&= {\gamma_j}^k{}_\ell \ {\delta_i}^\ell \\
&= {\gamma_j}^k{}_i
\end{aligned}
$$

In (3.37) we used the conclusion from (3.31), and in (3.38) we used (3.36). The rest comes from manipulation with the Kronecker delta (label reassignment) and the h-tensor (edge redirectioning). A similar argument, given in full in Appendix **A**, shows that ${\gamma_{ij}}^k = {\gamma^k}_{ij}$, leading to the conclusion that

$$
{\gamma_{ij}}^k = {\gamma_j}^k{}_i = {\gamma^k}_{ij} \tag{3.39}
$$

This may be diagrammatically expressed as



(3.40)

This gives us a justification for removing the anchor completely from the diagram of the Lie bracket, as diagrammised further below.

Now we consider other benefits of having an orthonormal basis. It would seem natural for the h-tensors switch the direction of edges, as they are related to the Kronecker-delta tensor. This intuition turns out to be correct. First note that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_j, \mathbf{e}_i \rangle = \delta_{ij}$, where δ_{ij} the Kronecker-delta symbol. From (3.20) we have the identification

$$
\langle \cdot, \cdot \rangle = h_{ij} \ \mathbf{e}^i \otimes \mathbf{e}^i \tag{3.41}
$$

In the case the form has an orthonormal basis, it follows that $h_{ij} = \delta_{ij}$, where δ_{ij} is now the Kronecker-delta tensor. That is, reinterpreting (3.22) and applying the previous definition, we have

$$
\begin{array}{rccl}
\langle \ , \ \rangle : & \mathfrak{g} & \to & \mathfrak{g}^* \\
& \mathbf{e}_i & \mapsto & \delta_{ij} \ \mathbf{e}^i = \mathbf{e}^i
\end{array} \tag{3.42}
$$

This yields a map that changes direction of tensor component. Indeed, given an arbitrary tensor and an arbitrary index from it, joining it with the appropriate h-tensor:

$$\alpha^{a_1,\ldots,a_{r_1}}{}_{b_1,\ldots,b_{s_1}}{}^{\cdots}{}_{\cdots}{}^{\cdots}{}_{d}{}^{\cdots}{}_{\cdots}{}^{f_1,\ldots,f_{r_m}}{}_{g_1,\ldots,g_{s_m}}\ h^{dk} = \alpha^{a_1,\ldots,a_{r_1}}{}_{b_1,\ldots,b_{s_1}}{}^{\cdots}{}_{\cdots}{}^{k}{}^{\cdots}{}_{\cdots}{}^{f_1,\ldots,f_{r_m}}{}_{g_1,\ldots,g_{s_m}} \tag{3.43}$$

Instead of proving this in the general case, we will show the mechanism in an example. First recall that from **1.3.5**, we already have an isomorphism $T : U \otimes V \to V \otimes U$ for finite-dimensional vector spaces U, V. Let $U = \mathfrak{g}$ and $V = \mathfrak{g}^*$, giving maps

$$\begin{array}{rrcl} T: & \mathfrak{g}\otimes\mathfrak{g}^* & \to & \mathfrak{g}^*\otimes\mathfrak{g} \\ & \lambda\ \mathbf{e}_i\otimes\mathbf{e}^j & \mapsto & T(\lambda)\ \mathbf{e}^j\otimes\mathbf{e}_i \end{array} \qquad \begin{array}{rrcl} T^{-1}: & \mathfrak{g}^*\otimes\mathfrak{g} & \to & \mathfrak{g}\otimes\mathfrak{g}^* \\ & \lambda\ \mathbf{e}^i\otimes\mathbf{e}_j & \mapsto & T^{-1}(\lambda)\ \mathbf{e}_j\otimes\mathbf{e}^i \end{array} \tag{3.44}$$

By making the correspondence in basis vectors between $\mathfrak{g}$ and the dual basis of $\mathfrak{g}^*$, the inverse map may also be viewed as

$$\begin{array}{rrcl} T^{-1}: & (\mathfrak{g}^*)^*\otimes(\mathfrak{g}^*) & \to & (\mathfrak{g}^*)\otimes(\mathfrak{g}^*)^* \\ & \lambda\ \mathbf{e}_i\otimes\mathbf{e}^j & \mapsto & T^{-1}(\lambda)\ \mathbf{e}^j\otimes\mathbf{e}_i \end{array} \tag{3.45}$$

So then T is an involution, i.e. $T^{-1} = T$ and so $T^2 = \mathrm{id}\otimes\mathrm{id}$. Now consider the tensor $\alpha_a{}^{bc}{}_d\ h_{bg}$, which is equivalent to $\alpha_{ag}{}^c{}_d$. To prove this we use the long notation of a tensor.

$$\begin{aligned} \alpha_a{}^{bc}{}_d\ h_{bg}\ \mathbf{e}^a\otimes\mathbf{e}_b\otimes\mathbf{e}_c\otimes\mathbf{e}^d\otimes\mathbf{e}^b\otimes\mathbf{e}^g &\xmapsto{\mathrm{id}\otimes T\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}} \alpha_a{}^{cb}{}_d\ h_{bg}\ \mathbf{e}^a\otimes\mathbf{e}_c\otimes\mathbf{e}_b\otimes\mathbf{e}^d\otimes\mathbf{e}^b\otimes\mathbf{e}^g \\ &\xmapsto{\mathrm{id}\otimes\mathrm{id}\otimes T\otimes\mathrm{id}\otimes\mathrm{id}} \alpha_a{}^c{}_d{}^b\ h_{bg}\ \mathbf{e}^a\otimes\mathbf{e}_c\otimes\mathbf{e}^d\otimes\mathbf{e}_b\otimes\mathbf{e}^b\otimes\mathbf{e}^g \\ &= \alpha_a{}^c{}_{dg}\ \mathbf{e}^a\otimes\mathbf{e}_c\otimes\mathbf{e}^d\otimes\mathbf{e}^g \\ &\xmapsto{\mathrm{id}\otimes\mathrm{id}\otimes T} \alpha_a{}^c{}_{gd}\ \mathbf{e}^a\otimes\mathbf{e}_c\otimes\mathbf{e}^g\otimes\mathbf{e}^d \\ &\xmapsto{\mathrm{id}\otimes T\otimes\mathrm{id}} \alpha_{ag}{}^c{}_d\ \mathbf{e}^a\otimes\mathbf{e}^g\otimes\mathbf{e}_c\otimes\mathbf{e}^d \end{aligned}$$

The combined composition of the maps applied above reduces to the identity map. Compose the first two maps:

$$(\mathrm{id}\otimes\mathrm{id}\otimes T\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes T\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}) = \mathrm{id}\otimes(\mathrm{id}\otimes T)\circ(T\otimes\mathrm{id})\otimes\mathrm{id}\otimes\mathrm{id}$$

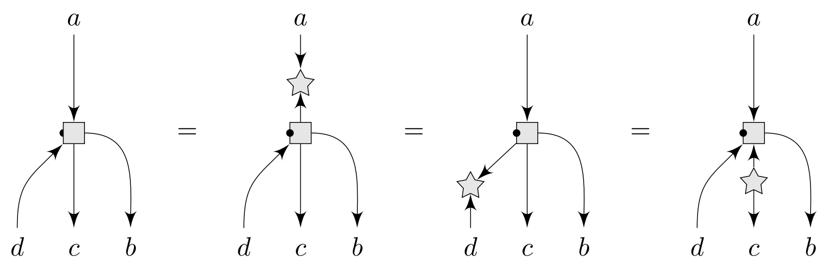
Apply the next map to the dimension-reduced map:

$$\begin{aligned} (\mathrm{id}\otimes\mathrm{id}\otimes T)\circ(\mathrm{id}\otimes(\mathrm{id}\otimes T)\circ(T\otimes\mathrm{id})) &= (\mathrm{id}\otimes T\otimes\mathrm{id})\circ(\mathrm{id}\otimes(\mathrm{id}\otimes T)\circ(\mathrm{id}\otimes T)\circ(T\otimes\mathrm{id})) \\ &= (\mathrm{id}\otimes(\mathrm{id}\otimes T^2)\circ(T\otimes\mathrm{id})) \\ &= (\mathrm{id}\otimes(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\circ(T\otimes\mathrm{id})) \\ &= (\mathrm{id}\otimes(T\otimes\mathrm{id})) \end{aligned}$$

Apply the last map:

$$\begin{aligned} (\mathrm{id}\otimes T\otimes\mathrm{id})\circ(\mathrm{id}\otimes(T\otimes\mathrm{id})) &= (\mathrm{id}\otimes(T\otimes\mathrm{id})\circ(T\otimes\mathrm{id})) \\ &= (\mathrm{id}\otimes(T^2\otimes\mathrm{id})) \\ &= (\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}) \\ &= \mathrm{id}^{\otimes 4} \end{aligned}$$

This manipulation involves, in an arbitrary tensor, moving the index to be changed k positions forward (resulting in k applications of T), reducing degree, then moving k positions backward (k more applications of T). In total, T is applied $2k$ times, and since the order in which it is applied is reversed going back, every pair of maps reduces to the identity map, giving the desired result. Therefore the h-tensors act to simply change the direction of an edge. With diagrams and the corresponding abstract tensors,



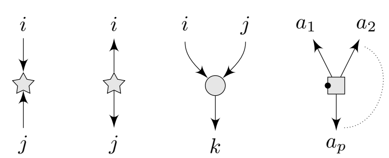
$$\beta_a{}^{bc}{}_d \quad = \quad \beta^{fbc}{}_d\, h_{fa} \quad = \quad \beta_a{}^{abg}\, h_{gd} \quad = \quad \beta_a{}^{b}{}_{id}\, h^{ic} \tag{3.46}$$

In general, through the diagrammatic relations

(3.47)

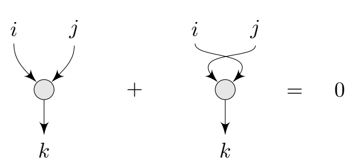
The dotted lines in the indicate an arbitrary edge in the described direction of an arbitrary tensor, keeping the same tensor in each respective equality. We may now diagrammise this algebra.

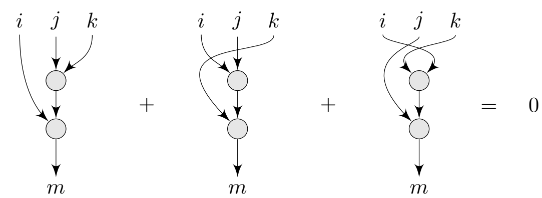
Diagrammisation of the metrized Lie algebra in an orthonormal basis with a symmetric bilinear form. The elements of $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ are generated by



$\forall\, p \in \mathbb{N}$

The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)





as well as the edge-reversing action (3.47) of the h-tensors

3.6.4 Diagrammisation under ad-invariance

Definition 3.6.1. *Adjoint map.* For $x \in \mathfrak{g}$, define the adjoint map ad_x by

$$\begin{array}{rccc} \mathrm{ad}_x : & \mathfrak{g} & \to & \mathfrak{g} \\ & y & \mapsto & [x, y] \end{array}$$

Definition 3.6.2. Ad-*invariance.* The bilinear form $\langle \ , \ \rangle$ is termed ad-invariant if, for $x, y, z \in \mathfrak{g}$, the following equality holds:

$$\langle \mathrm{ad}_x(y), z \rangle = -\langle y, \mathrm{ad}_x(z) \rangle \tag{3.48}$$

Or, equivalently,

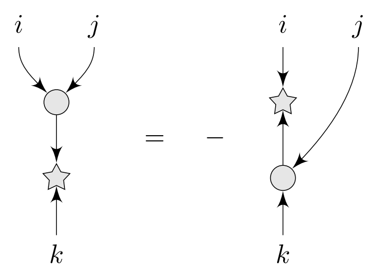
$$\langle [x, y], z \rangle = -\langle y, [x, z] \rangle \tag{3.49}$$

As such, we may view $\langle [\ ,\ ],\ \rangle$ and $\langle\ ,\ [\ ,\ ]\rangle$ as maps, in the sense that

$$\begin{array}{rcccl} \langle [\ ,\ ],\ \rangle : & \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} & \to & \mathbb{F} & \\ & (\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) & \mapsto & \langle [\mathbf{e}_i, \mathbf{e}_j], \mathbf{e}_k \rangle & = \langle \gamma_{ij}{}^{\ell}\, \mathbf{e}_\ell, \mathbf{e}_k \rangle \\ & & & & = \gamma_{ij}{}^{\ell} \langle \mathbf{e}_\ell, \mathbf{e}_k \rangle \\ & & & & = \gamma_{ij}{}^{\ell}\, h_{\ell k} \end{array}$$

$$\begin{array}{rcccl} \langle\ ,\ [\ ,\ ]\rangle : & \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} & \to & \mathbb{F} & \\ & (\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) & \mapsto & \langle \mathbf{e}_i, [\mathbf{e}_j, \mathbf{e}_k] \rangle & = \langle \mathbf{e}_i, \gamma_{jk}{}^{m}\, \mathbf{e}_m \rangle \\ & & & & = \gamma_{jk}{}^{m} \langle \mathbf{e}_i, \mathbf{e}_m \rangle \\ & & & & = \gamma_{jk}{}^{m}\, h_{im} \\ & & & & = \gamma_{jk}{}^{m}\, h_{mi} \end{array}$$

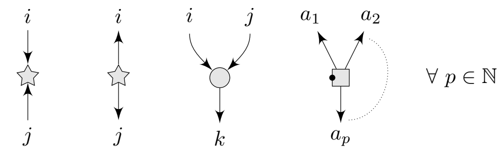
We choose not to reduce this to a delta tensor, but rather keep the h-tensors as they are. This interpretation allows us to diagrammize (3.49). In keeping with the progression of results, we continue to omit anchors for the diagrams of the Lie and h-tensors.



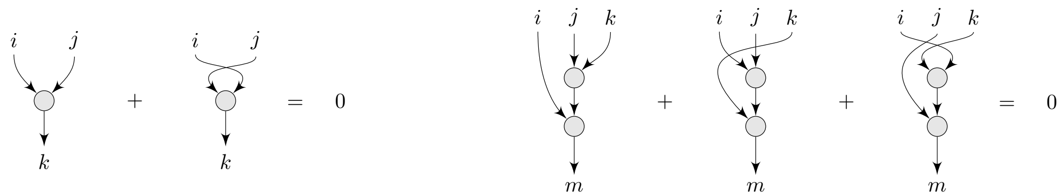
(3.50)

Again, it may seem natural to apply (3.47) to eliminate the h-tensors. We choose not to, as this could potentially cause confusion as to how the Lie bracket diagram operates. Now we diagrammise this algebra.

Diagrammisation of the metrized Lie algebra in an orthonormal basis with an ad-invariant symmetric bilinear form. The elements of $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ are generated by

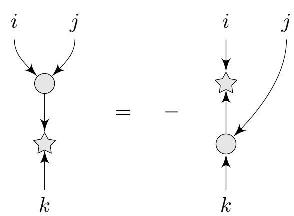


The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)



as well as the edge-reversing action (3.47) of the h-tensors

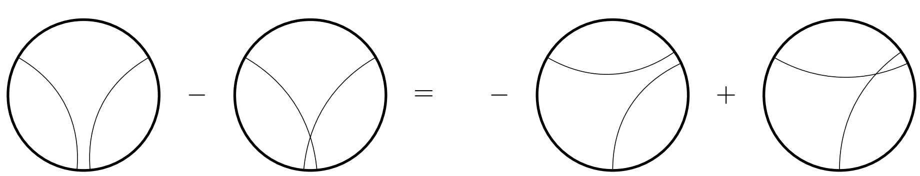
and the ad-invariant relation (3.50)



In the next section, by combining (3.50) with (3.16), we are able to produce a well known relation. This is the 4-term relation used in computation of knot invariants.

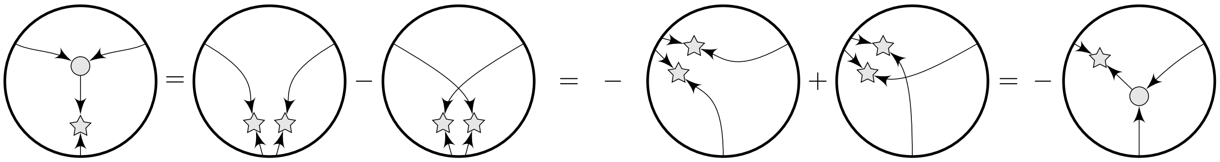
3.7 The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ for a metrized Lie algebra $\mathfrak{g}$

The structure combining the universal enveloping algebra and a metrized Lie algebra with an orthonormal basis and an ad-invariant symmetric bilinear form will be endowed with the Lie bracket decomposition and the ad-invariance relation, which will, employed together, lead nicely into a study of chord diagrams. In knot theory, the 4-term relation is most often given as an equality of chord diagrams,



(3.51)

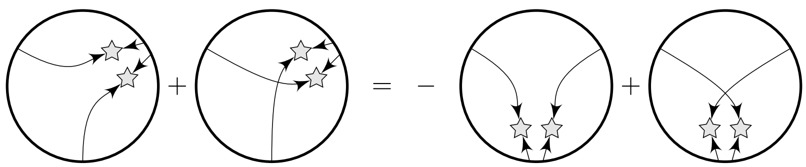
More chords may be added, to each diagram in an identical manner, but none may end in the shortest gaps between chord ends on the boundary circle. Note that there are no directed edges, as the lines in these diagrams do not indicate direction, since direction is counter-clockwise along the circle. The lines indicate which points are to be identified together - they are the singularities of the knot diagram. Now take the relation (3.50) and assign fixed positions to the free ends of edges, embedding each term in a circular diagram. Then (3.16) yields



(3.52)

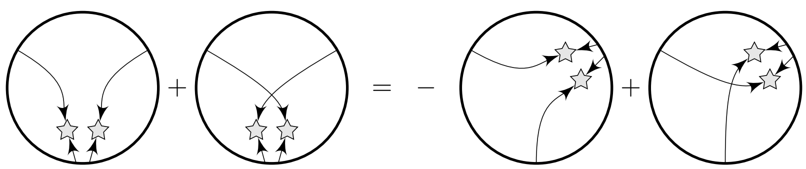
As above, more tensor diagrams may be added identically to the equation, but none may end in the shortest gaps between edge ends, as that could change any diagrams lying outside the restricted circle.

The choice of ordering the edges starting from the top of the circle is arbitrary, in the 4-term relation as in the above expansion. So a uniform change of initial point of observation will only result in a visual change, with no informational change. So if we look at the inner part of the above equation and rotate it 120 degrees counter-clockwise, we get



(3.53)

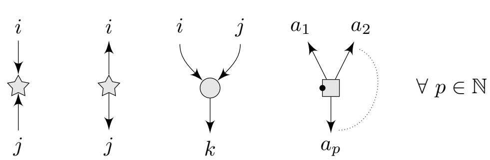
And rearranged,



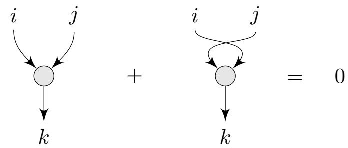
(3.54)

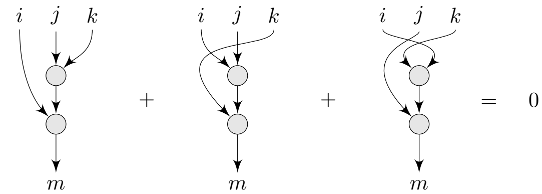
This now can be seen as the original 4-term relation in (3.51). The h-tensors in the diagrams work to eliminate a notion of direction, which is consistent with the fact that chords in chord diagrams indicate only singularities, as mentioned. We may now diagrammise this algebra.

Diagrammisation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ for a metrized Lie algebra $\mathfrak{g}$. The elements of $\mathfrak{U}(\mathfrak{g})$ are generated by



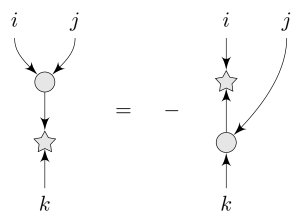
The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)



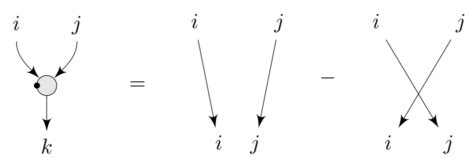


as well as the edge-reversing action (3.47) of the h-tensors

and the ad-invariant relation (3.50)



and an interpretation of the Lie bracket diagram (3.15)

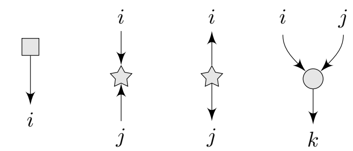


4 Ensuing connections

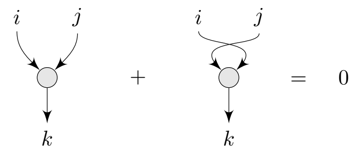
Here $\mathfrak{g}$ will denote a metrized Lie algebra with an orthonormal basis and an ad-invariant symmetric bilinear form. In the previous section we saw the algebra $\mathfrak{U}(\mathfrak{g})$ with an anchor-less Lie tensor and an interesting interpretation of the Lie bracket.

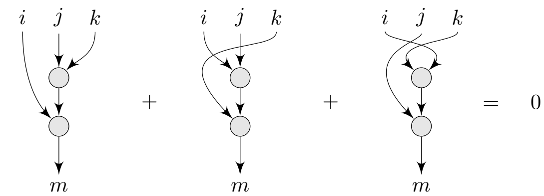
Consider the space $\mathcal{U} \subset \mathfrak{U}(\mathfrak{g})$ of tensors generated by univalent tensors, the h-tensors (representing the metric $\langle\ ,\ \rangle$), and the Lie bracket. Then all diagrams in $\mathcal{U}$ have vertices of degree 1 or 3, termed unitrivalent diagrams. By the relation in (3.15), trivalent vertices may be eliminated from the diagram altogether. It would seem natural to consider this space as a space of chord diagrams, which will lead to the study of Vassiliev invariants. This space may be concisely diagrammised.

Diagrammisation of the set $\mathcal{U} \subset \mathfrak{U}(\mathfrak{g})$. The elements of $\mathcal{U}$ are generated by



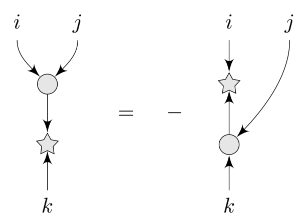
The system is subject to the anti-symmetric relation (R1) and the Jacobi relation (R2)



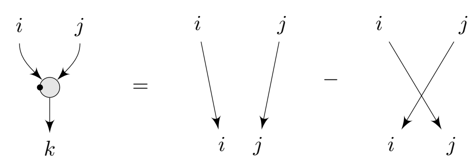


as well as the edge-reversing action (3.47) of the h-tensors

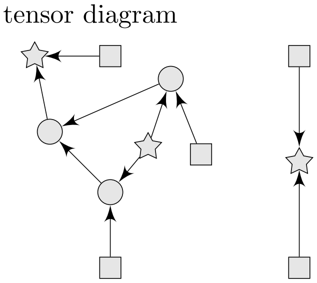
and the ad-invariant relation (3.50)



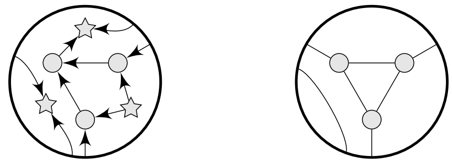
and an interpretation of the Lie bracket diagram (3.15)



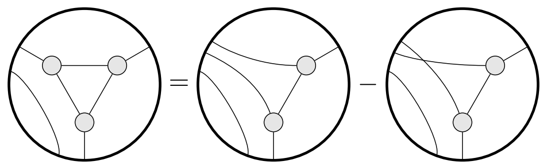
For example, consider the following tensor diagram



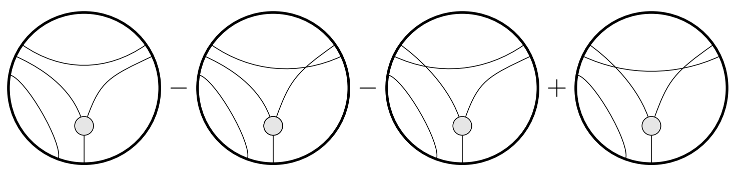
We now interpret it in a circular diagram according to **3.7**, and make some simplifications to express such a diagram with chord diagrams. The below interpretation is not unique, as the univalent vertices may be projected in any desired order on the circular skeleton.



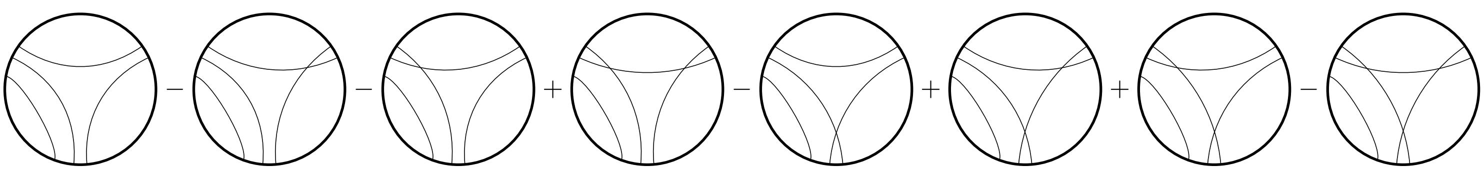
Bivalent vertices (the h-tensors) have disappeared, as a chord with a bivalent vertex has the same structure as the same chord without a bivalent vertex. Directed edges have lost direction, as chords in chord diagrams have no predetermined direction. By applying 3.16, the diagram on the right may be decomposed further.



A repeated application continues decomposition.



A final application gives a complete expression of the tensor diagram in terms of chord diagrams.



The ad-invariant relation in circular diagrams from 3.51 gives the 4-term relation of chord diagrams, again highlighting the similarities. Therefore a clear association may be made between elements of $\mathcal{U}$ and chord diagrams, an association that will not be pursued further here.

A Appendix

In **3.6.3** we showed that ${\gamma_{ij}}^k = {\gamma_j}^k{}_i$, and here we will show that ${\gamma_{ij}}^k = {\gamma^k}_{ij}$. We proceed initially as before.

$$
\begin{aligned}
{\gamma_{ij}}^k \ h_{\ell k} &= {\gamma_{ij}}^k \ h_{k\ell} && \text{(A.1)}\\
&= \sum_{i,j,k,\ell} {\gamma_{ij}}^k \ h_{k\ell} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_k \otimes \mathbf{e}^k \otimes \mathbf{e}^\ell \\
&= \sum_{i,j,\ell} \Big({\gamma_{ij}}^1 \ h_{1\ell} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_1 \otimes \mathbf{e}^1 \otimes \mathbf{e}^\ell + \cdots + {\gamma_{ij}}^n \ h_{n\ell} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}_n \otimes \mathbf{e}^n \otimes \mathbf{e}^\ell \Big) \\
&= \sum_{i,j,\ell} \Big({\gamma_{ij}}^1 \ h_{1\ell} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell + \cdots + {\gamma_{ij}}^n \ h_{n\ell} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \Big) \\
&= \sum_{i,j} \Big({\gamma_{ij}}^1 \ h_{11} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^1 + \cdots + {\gamma_{ij}}^n \ h_{nn} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^n \Big) && \text{(A.2)}\\
&= \sum_{i,j} \Big({\gamma_{ij}}^1 \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^1 + \cdots + {\gamma_{ij}}^n \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^n \Big) \\
&= \sum_{i,j,\ell} {\gamma_{ij}}^\ell \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \\
&= \sum_{i,j,\ell} \gamma_{ij\ell} \ \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell && \text{(A.3)}\\
&= \gamma_{ij\ell} && \text{(A.4)}
\end{aligned}
$$

As previously, (A.1) follows as the bilinear form is symmetric. In (A.2) we expand over ℓ to reduce zero terms, and (A.3) makes the switch of upper to lower index ℓ to accomodate the change of space. As scalars, ${\gamma_{ij}}^k = \gamma_{ij\ell}$ for $k = \ell$ from (A.3).

$$
\begin{aligned}
\gamma_{ij\ell} \ h^{ik} &= h^{ki} \ \gamma_{ij\ell} && \text{(A.5)}\\
&= \sum_{i,j,k,\ell} h^{ki} \ \gamma_{ij\ell} \ \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^j \\
&= \sum_{j,k,\ell} \big(h^{k1} \ \gamma_{1j\ell} \ \mathbf{e}_k \otimes \mathbf{e}_1 \otimes \mathbf{e}^1 \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell + \cdots + h^{kn} \ \gamma_{nj\ell} \ \mathbf{e}_k \otimes \mathbf{e}_n \otimes \mathbf{e}^n \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \big) \\
&= \sum_{j,k,\ell} \big(h^{k1} \ \gamma_{1j\ell} \ \mathbf{e}_k \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell + \cdots + h^{kn} \ \gamma_{nj\ell} \ \mathbf{e}_k \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \big) \\
&= \sum_{j,\ell} \big(h^{11} \ \gamma_{1j\ell} \ \mathbf{e}_1 \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell + \cdots + h^{nn} \ \gamma_{nj\ell} \ \mathbf{e}_n \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \big) \\
&= \sum_{j,\ell} \big(\gamma_{1j\ell} \ \mathbf{e}_1 \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell + \cdots + \gamma_{nj\ell} \ \mathbf{e}_n \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \big) \\
&= \sum_{k,j,\ell} \gamma_{kj\ell} \ \mathbf{e}_k \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \\
&= \sum_{k,j,\ell} {\gamma^k}_{j\ell} \ \mathbf{e}_k \otimes \mathbf{e}^j \otimes \mathbf{e}^\ell \\
&= {\gamma^k}_{j\ell} && \text{(A.6)}
\end{aligned}
$$

In (A.5) we use the properties of the h-tensor, and the rest is as before. Now combine the above results, and apply the Kronecker delta tensor.

$$
\begin{aligned}
{\gamma_{ij}}^{k} &= {\gamma_{ij}}^{k}\ {\delta_\ell}^{\ell} \\
&= {\gamma_{ij}}^{k}\ {\delta_\ell}^{i}\ {\delta_i}^{\ell} \\
&= {\gamma_{ij}}^{k}\ {\delta_\ell}^{i}\ {\delta_i}^{j}\ {\delta_j}^{\ell} \\
&= {\gamma_{ij}}^{k}\ h_{\ell k}\ h^{ki}\ {\delta_i}^{j}\ {\delta_j}^{\ell} \\
&= \gamma_{ij\ell}\ h^{ki}\ {\delta_i}^{j}\ {\delta_j}^{\ell} \\
&= \gamma_{ij\ell}\ h^{ik}\ {\delta_i}^{j}\ {\delta_j}^{\ell} && \text{(A.7)} \\
&= {\gamma^{k}}_{j\ell}\ {\delta_i}^{j}\ {\delta_j}^{\ell} && \text{(A.8)} \\
&= {\gamma^{k}}_{i\ell}\ {\delta_j}^{\ell} \\
&= {\gamma^{k}}_{ij}
\end{aligned}
$$

In (A.7) we used (A.4), and in (A.8) we used (A.6). The rest comes from manipulation with the Kronecker delta and the h-tensor. This completes the argument to prove that

$$
{\gamma_{ij}}^{k} = {\gamma_{j}}^{k}{}_{i} = {\gamma^{k}}_{ij} \qquad \text{(A.9)}
$$

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