

The logarithmic spiral and recurrence relations

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1 Abstract

A spiral is heuristically a very simple figure, but how simple is the formal construction and approximation of such a form? In this paper we discuss a unique type of spiral and two separate geometric methods for its approximation, one unresolvable and the other conclusive. Instead of inspecting the unique properties of the spiral, we first design curves with certain qualities from which the spiral naturally arises. We conclude with applications to word problems and some interesting extensions.

2 Introduction

The variable n always refers to the number of sides of the regular polygon at the center of each construction. Most calculations are done in the (geometrical) base case where $n = 3$. Extensions to arbitrary n are made where convenient, and overall the computations change little.

2.1 First construction

Place three points on the plane, equidistant from the origin (a fixed distance r , the *radius*) and equidistant from each other, with one vertex on the positive horizontal axis. Fix a *difference factor* d such that $d > 0$. The most interesting patterns appear when $d \in (0, 1)$.

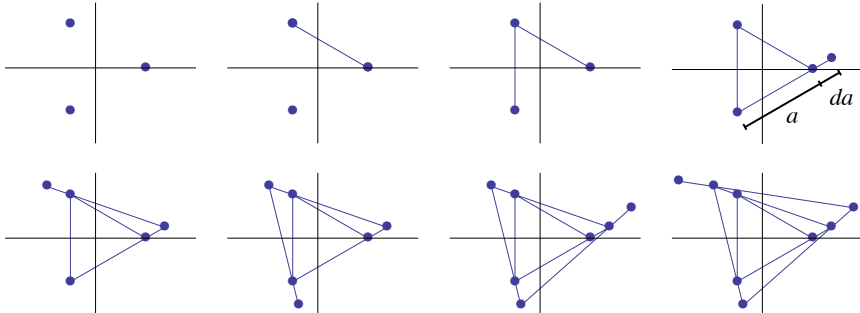


Figure 1: Initialization of the construction

Starting with the point on the horizontal axis, draw a line segment to the next point counter-clockwise and then from the second point to the third point. Begin another line segment from the

third back through the first point, noting the distance, a . End this segment when its total length reaches $(1 + d)a$, and add the endpoint (the fourth point) to the progression. Repeat this process: set a to be the distance from the fourth to the second point, and draw a line segment from the fourth through the second point of length $(1 + d)a$. Continue by starting at the i th point and going through the $(i - 2)$ th point (where $2 := n - 1$) to set the location of the $(i + 1)$ th point.

As d approaches zero, the progression of points emanating from each corner of the the original shape (termed *rays*) appears to approach a fixed curve.

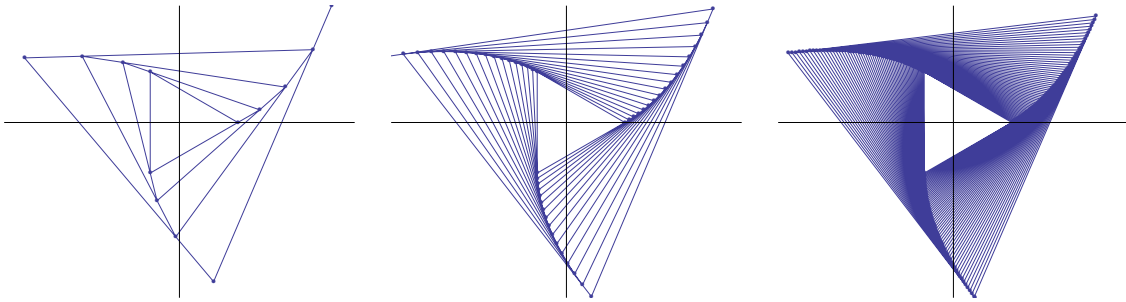


Figure 2: At $d = 1/4, 1/20, 1/100$ and 13, 50, 183 iterations

For $n = 3$, Figure 2 shows that there is no strong deviation from the limit curve. For larger n , the limiting spiral is evident only for the first few iterations (more as d goes to 0), and the behavior of the rays becomes difficult to describe. Further, for all $d > 0$ the construction is never bounded.

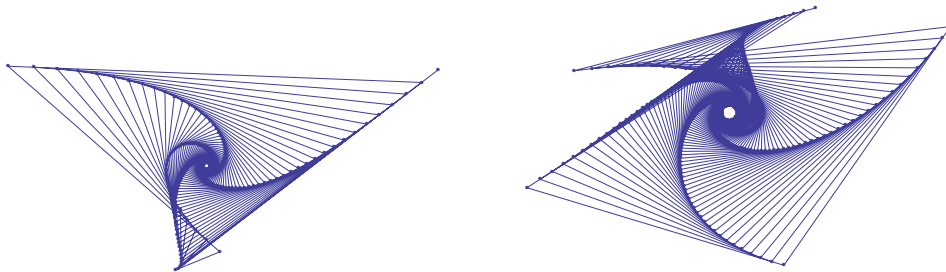


Figure 3: At $n = 4$ and $n = 5$

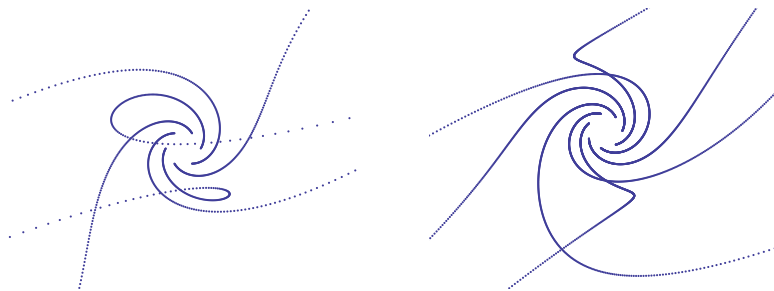


Figure 4: At $n = 6$ and $n = 7$ (line segments are dropped to ease graphical analysis)

This erratic behavior comes from the irregular distances between points in each of the n rays of the figure, which has magnified effects further in the progression. As d approaches 0, an increasing number of points become stationary.



Figure 5: Convergence for $n = 8$ as d ranges from 10^{-1} to 10^{-5}

2.2 Formalization

In this section we conclude that it is difficult, using generating functions, to find a closed form function that produces the vertices of this progression. The use of polar coordinates could be more suited for this problem, but the issue of factoring an n th degree polynomial (described below) arises in any situation.

Below, (x_i, y_i) represents the Cartesian coordinates of the i th point, with $(x_0, y_0) = (1, 0)$ for all n . The first n points are simply vertices of a regular n -gon centered at the origin. All other points are placed as described above in the recurrence relation. For a fixed n :

$$(x_i, y_i) = \begin{cases} \left(r \cos\left(i\frac{2\pi}{n}\right), r \sin\left(i\frac{2\pi}{n}\right) \right) & \text{if } 0 \leq i \leq n-1 \\ (x_{i-n}, y_{i-n}) + d((x_{i-n}, y_{i-n}) - (x_{i-1}, y_{i-1})) & \text{if } i \geq n \end{cases}$$

For simplicity, we now consider only the x -coordinates:

$$x_i + dx_{i-1} - (1+d)x_{i-n} = \begin{cases} r & i = 0 \\ r \cos\left(\frac{2\pi}{n}\right) + dr & i = 1 \\ r \cos\left(2\frac{2\pi}{n}\right) + dr \cos\left(\frac{2\pi}{n}\right) & i = 2 \\ \vdots & \vdots \\ r \cos\left((n-1)\frac{2\pi}{n}\right) + dr \cos\left((n-2)\frac{2\pi}{n}\right) & i = n-1 \\ 0 & i \geq n \end{cases}$$

$$\begin{aligned} \implies \sum_{i=0}^{\infty} (x_i + dx_{i-1} - (1+d)x_{i-n}) z^i &= r + \sum_{i=1}^{n-1} \left(r \cos\left(i\frac{2\pi}{n}\right) + dr \cos\left((i-1)\frac{2\pi}{n}\right) \right) z^i \\ \sum_{i=0}^{\infty} x_i z^i + dz \sum_{i=0}^{\infty} x_i z^i - (1+d) z^n \sum_{i=0}^{\infty} x_i z^i &= r + \sum_{i=1}^{n-1} \left(r \cos\left(i\frac{2\pi}{n}\right) + dr \cos\left((i-1)\frac{2\pi}{n}\right) \right) z^i \\ \sum_{i=0}^{\infty} x_i z^i &= \frac{r + \sum_{i=1}^{n-1} \left(r \cos\left(i\frac{2\pi}{n}\right) + dr \cos\left((i-1)\frac{2\pi}{n}\right) \right) z^i}{1 + dz - (1+d) z^n} \quad (1) \end{aligned}$$

To express the coefficients of this generating function, it is necessary to apply partial fractions to the right-hand side of (1), or equivalently, factor the n th degree polynomial $1 + dz - (1 + d)z^n$, which is difficult, especially if n and d are arbitrary.

So instead of this approach, we use a simpler problem, by observing the regularization pattern. As $d \rightarrow 0$, the distance between the i th and $(i + 1)$ th vertices in every ray approaches a constant value – zero. Thus we construct a sequence of numbers where this difference is always constant, not only in the limiting case. Graphically, this is exhibited by regular n -gons of increasing radius.

3 Simplified construction

Begin with three points as above, connecting them. Fix a *difference angle* ϕ such that $\phi > 0$. To avoid repetition, let $\phi \in (0, \frac{2\pi}{3})$, where $3 := n$.

Around the first 3-gon circumscribe another regular 3-gon, so that the angle of inclination of the vertex closest (counter-clockwise) to the positive x -axis is ϕ . Around this 3-gon circumscribe another regular 3-gon, so that the difference in angles of inclination of the previously discussed vertex and the one closest (again CCW) to it on the third 3-gon is ϕ . Continue *ad indefinitum*. This process of circumscribing defines unique 3-gons.

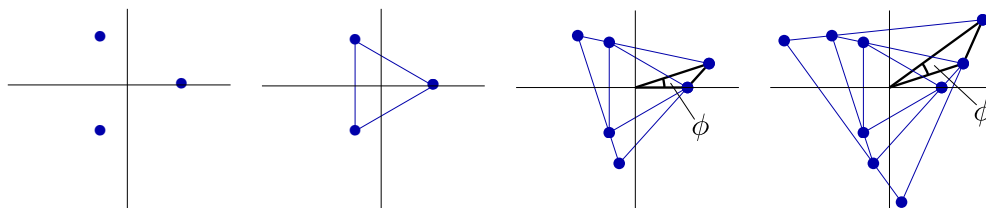


Figure 6: Initialization of the simplified construction

Dropping two altitudes from the vertices CCW nearest to each other on consecutive 3-gons, and using one of the triangles thus formed (emphasized in Figure 6 in the last two frames) and the law of sines, a polar recurrence relation giving the coordinates of each vertex can be created for each ray. The two legs of this triangle that have an angle of ϕ between them are the i th and $(i + 1)$ th radius. Numbering the triangles in order of increasing size with the original triangle being the 0th triangle, we have

$$\theta_i = \begin{cases} j \frac{2\pi}{3} & \text{if } i = 0 \\ \theta_{i-1} + \phi & \text{if } i \geq 1 \end{cases}, \quad j = 0, 1, 2$$

$$r_i = \begin{cases} r & \text{if } i = 0 \\ \sin\left(\pi - \frac{\pi}{6} - \phi\right) \frac{r_{i-1}}{\sin\left(\frac{\pi}{6}\right)} = 2 \sin\left(\frac{\pi}{6} + \phi\right) r_{i-1} & \text{if } i \geq 1 \end{cases}$$

for j the index of the rays, with $j = 0$ for the *principal ray* (the one beginning on the horizontal axis) and proceeding counter-clockwise. Here and below we are considering only a single ray of vertices in the progression. With the same methods as above,

$$r_i - 2 \sin\left(\frac{\pi}{6} + \phi\right) r_{i-1} = \begin{cases} r & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

$$\sum_{i=0}^{\infty} \left(r_i - 2 \sin\left(\frac{\pi}{6} + \phi\right) r_{i-1} \right) z^i = r$$

$$\sum_{i=0}^{\infty} r_i z^i - 2 \sin\left(\frac{\pi}{6} + \phi\right) z \sum_{i=0}^{\infty} r_i z^i = r$$

$$\sum_{i=0}^{\infty} r_i z^i = \frac{r}{1 - 2 \sin\left(\frac{\pi}{6} + \phi\right) z}$$

This directly implies that

$$r_i = r \left(2 \sin\left(\frac{\pi}{6} + \phi\right) \right)^i \quad (2)$$

As for the angles, the pattern is seen directly:

$$\theta_i = i\phi + j \frac{2\pi}{3}, \quad j = 0, 1, 2 \quad (3)$$

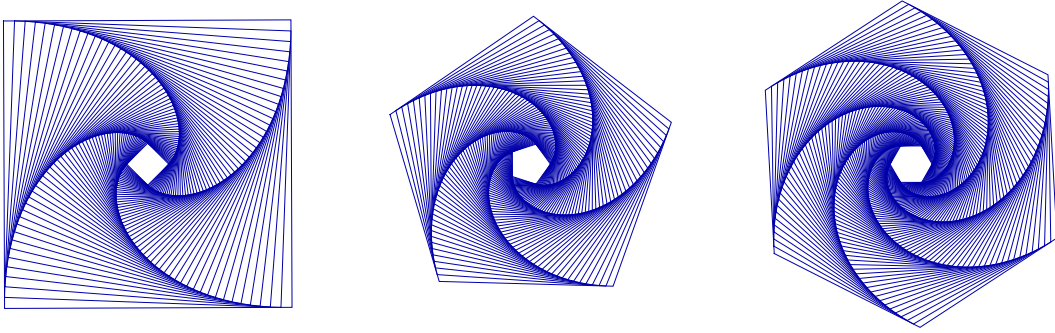


Figure 7: Patterns for $n = 4, 5, 6$, also termed *whirls*.

3.1 Limiting behavior

Taking the limit as $\phi \rightarrow 0$, we will find the curve that both this and the original construction resemble in their respective limits. First we develop a polar function for the radius in terms of the angle, by first analyzing the principal ray.

Instead of deciding the radius and angle (θ) based on the i th ordered triangle, we make the angle independent and everything else a function of the angle, as there is a clear bijection between θ_i and i , once θ_0 is fixed. Modifying and joining (2) and (3), we have

$$\begin{aligned}
\theta &= i(\theta)\phi \\
r(\theta) &= r \left(2 \sin \left(\frac{\pi}{6} + \phi \right) \right)^{i(\theta)} \\
&= r \left(2 \sin \left(\frac{\pi}{6} + \phi \right) \right)^{\frac{\theta}{\phi}} \\
&= r \cdot \exp \left\{ \frac{\theta}{\phi} \log \left(2 \sin \left(\frac{\pi}{6} + \phi \right) \right) \right\}
\end{aligned} \tag{4}$$

A polar function of the form $r(\theta) = ae^{b\theta}$ as in (4) is termed a *logarithmic spiral*, or *equiangular spiral*, because the tangential angle at all θ is constant.

Now we take the limit as $\phi \rightarrow 0$. Doing so, we make use of l'Hôpital's rule, since a limit of the form $\frac{0}{0}$ is encountered, and take the derivative, as the functions are differentiable.

$$\begin{aligned}
\lim_{\phi \rightarrow 0} [r(\theta)] &= r \lim_{\phi \rightarrow 0} \left[\exp \left\{ \frac{\theta}{\phi} \log \left(2 \sin \left(\frac{\pi}{6} + \phi \right) \right) \right\} \right] \\
&= r \lim_{\phi \rightarrow 0} \left[\exp \left\{ \theta \frac{2 \cos \left(\frac{\pi}{6} + \phi \right)}{2 \sin \left(\frac{\pi}{6} + \phi \right)} \right\} \right] \\
&= r e^{\sqrt{3}\theta}
\end{aligned}$$

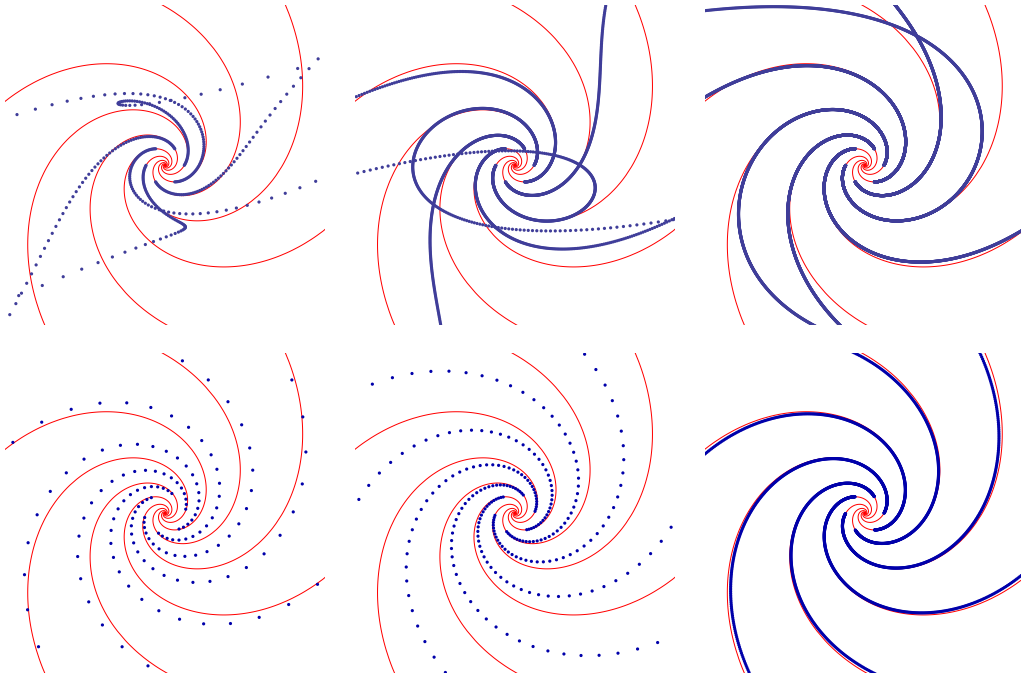


Figure 8: Convergence for $n = 6$ for the original sequence (top) and simplified sequence

For the other rays, a shift of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ in the argument is necessary. For arbitrary n , the above process can be repeated analogously (by using $\frac{(n-2)\pi}{2n}$ instead of $\frac{\pi}{6}$ in the primary recurrence relation, representing half of an interior angle of a regular n -gon), giving

$$\begin{aligned} r(\theta) &= re^{\cot(\frac{(n-2)\pi}{2n})(\theta+j\frac{2\pi}{n})} \\ &= re^{\tan(\frac{\pi}{n})(\theta+j\frac{2\pi}{n})}, \quad j = 0, 1, \dots, n-1 \end{aligned} \tag{5}$$

4 Applications

To conclude this paper, we broaden the scope of the above findings.

4.1 Puzzles

The *Mice problem* applies the logarithmic spiral to its solution: A regular n -gon of fixed radius has one mouse at each vertex. Each mouse then moves toward its closest neighbor in a counter-clockwise direction, and all at the same constant speed. How long does it take for the mice to meet?

The paths traced out by the mice are exactly the limiting logarithmic spirals described above. Moreover, the reverse of the simplified polygon progression shows the progress of the little creatures. Though instead of creating ever-larger polygons, use ever-smaller polygons, placing every next one inside the previous one.

4.2 Extensions to art

In the interests of making mathematical art, the described recurrences can be slightly adjusted, as demonstrated below, by (for example) making them quadratic or quartic, adding terms, or changing the depth of the recurrence.

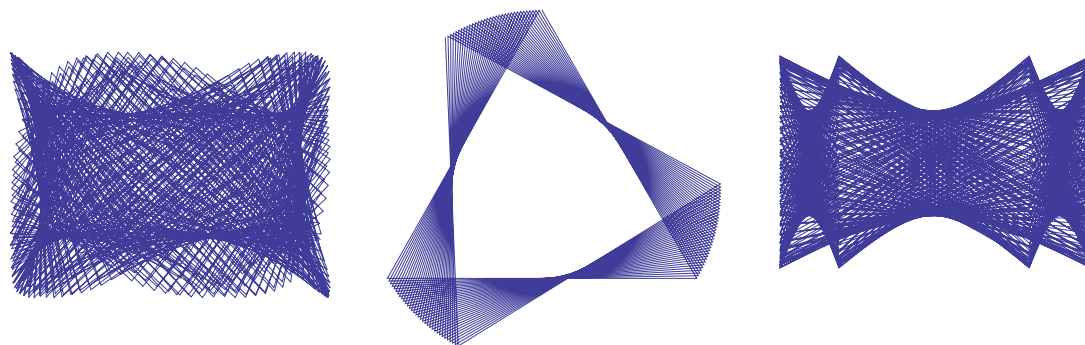


Figure 9: Variations of the original progression with $n = 5, 3, 5$

The exact recursive formulae for producing the images in Figure 9 (from left to right - (6), (7), (8)), beyond the vertices of the regular n -gon at the center, are:

$$(x_i, y_i) = (x_{i-5}, y_{i-3}) - d((x_{i-4}, y_{i-1}) - (x_{i-1}, y_{i-5})) \quad (6)$$

with $d = 1/80$ at 600 iterations

$$(x_i, y_i) = (x_{i-3}, y_{i-3}) + d((x_{i-1}, y_{i-1}) - (x_{i-2}, y_{i-2})) \quad (7)$$

with $d = 1/100$ at 100 iterations

$$(x_i, y_i) = (x_{i-4}, y_{i-3}) - d((x_{i-1}, y_{i-1}) - (x_{i-5}, y_{i-5})) \quad (8)$$

with $d = 1/60$ at 500 iterations

5 Acknowledgements

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6 References

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