Paths and higher simplices in TDA

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Abstract: This talk will explore the unifying role of simplices. First we show how to simplify persistent homology computations between subsets of the same space with a path between them (joint with David Millman). Then we assemble all simplices respecting a stratification of a metric space into a higher dimensional analogue of the fundamental group to describe a constructible sheaf structure.

0.1 Background

M is a metric space with a metric d_M.
SC is the set of (abstract, finite) simplicial complexes, $[\mathsf{SC}] := \mathsf{SC}_{/\cong}$.

$$\mathrm{Conf}_n(M) := \{P \subseteq M \;:\; |P| = n\} \qquad \mathrm{Ran}(M) := \{P \subseteq M \;:\; 0 < |P| < \infty\}.$$

Distance on these spaces is Hausdorff distance d_H, and topology on them is the metric topology induced by d_H.

$$d_H(P,Q) := \max\left\{ r \;:\; Q \subseteq \bigcup_{p\in P} B(p,r),\; P \subseteq \bigcup_{q\in Q} B(q,r) \right\}$$

Equivalent to the the coarsest topology for which, given any collection $\{U_i \subseteq M\}_i$ of pairwise disjoint open sets,

$$\{P \in \mathrm{Ran}(M) \;:\; P \subseteq \textstyle\bigcup_i U_i,\; P \cap U_i \neq \emptyset \;\forall\, i\} \subseteq \mathrm{Ran}(M)$$

is open. Consider $\check{C}\colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SC}]$ where $\check{C}(P,r)$ is the (isomorphism class of the) simplicial complex on P built by the Čech construction at radius r. Decompose domain by $\check{C}$ or $VR = \mathrm{flag} \circ \check{C}$.

0.2 Paths (1-simplices)

Joint with David Millman.
Take a path $\gamma\colon [0,1] \to \mathrm{Ran}(M)$ and consider the 2-dimensional space $\mathrm{im}(\gamma) \times \mathbf{R}_{\geqslant 0} \subseteq \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$.
Assume:

1. γ is linear, so $\gamma(t) = P(1-t) + Qt$
2. No points split / merge along γ, so $\gamma(t) \in \mathrm{Conf}_n(\mathbf{R}^N)$ for all t

visualization

Suppose we have computed 0-barcode of $P = \gamma(0)$ with the distance filtration.
Does knowing γ help us compute 0-barcode of $Q = \gamma(1)$ faster?

Q1: What is "faster"? The computational complexity of computing a barcode is cubic in the number of simplices, for 0-barcode we have $n + n(n-1)/2 = (n^2+n)/2$ simplices, so the complexity is: $\mathcal{O}(n^6)$

The distance functions are quadratic, there are $n(n-1)/2$ pairwise distance functions, and so at most

$$\sum_{i=1}^{n(n-1)/2-1} 2i = (n^4 - 2n^3 - n^2 + 2n)/4 = \mathcal{O}(n^4)$$

intersections among them.

Q2: What makes a linear path special? Observe:

1. Between these $\approx n^4$ intersections, persistence modules are isomorphic.
2. Around these intersections, only a small part changes.

Q3: How can this be harnessed? Complexity of swapping pair of simplices is at most linear in the number of simplices (Morozov), so $\mathcal{O}(n^2)$. Use Obs.2. to make it constant, because swap is "simple".

0.3 Higher simplices

Define a partial order on [SC].
Map $\check{C}$ is continuous, have stratification.
Category of exit and entrance paths.
classic bundle fibration example, we want to get a new fibration. How?
simplicial sets
sheaves