# Higher structures for topological data analysis

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**Abstract:** "Higher" is meant in an algebraic way - instead of spaces, use maps into spaces; instead of maps, use sequences of maps. I will describe a partial order on simplicial complexes motivated by persistent homology, and use it as a gateway to simplicial sets. This allows for a consistent description of persistent homology as its underlying data changes.

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## 0.1 PH / TDA

*Concepts 1:*

- finite metric space $(P, d)$
- topological space $(X, \{U_i\})$
  - simplicial complex: pair $(V, S)$, where $V$ is a set and $S \subseteq P(V)$ is closed under subsets
  - simplicial map: $f\colon C \to D$ is two maps $f_V\colon V_C \to V_D$ and $f_S\colon S_C \to S_D$, with $f_S(\sigma) = \{f_V(v) : v \in \sigma\}$
    - *Example:* sc, simp map, non simp map
- group $G$
  - homology

*Example:* points in $\mathbf{R}^2$, distance among them $\to$ simplicial complex at different radii (filtr) $\to$ $d$-homology groups

**Motivation:** How does homology change as the finite metric space changes?

## 0.2 Partial orders

**Def:** A *poset* is a pair $(A, \geqslant)$, where $A$ is a set and $\geqslant$ is a reflexive, antisymmetric, transitive relation on $A$.

A *relation* on $A$ is a map $\geqslant\colon A \times A \to \{0, 1\}$.

Let $\mathsf{SC}$ be the set of simplicial complexes, and $[\mathsf{SC}] := \mathsf{SC}/_{\cong}$ their isomorphism classes.
Let $\geqslant$ be the following relation on $\mathsf{SC}$:

$$C \geqslant D \quad \Longleftrightarrow \quad \left(\begin{array}{c}\text{there exists a simplicial map } C \to D \\ \text{that is surjective on vertices}\end{array}\right)$$

*Example of partial order.*
**Motivation:** Topological changes can be tracked when points merge, not split.

**Lemma:** This is a partial order on $[\mathsf{SC}]$.

**Remark:** Every poset $(A, \geqslant)$ has a topology, with a basis of open sets $U_a = \{b \in A : b \geqslant a\}$.
This is the *Alexandrov*, or *upset* topology.

## 0.3 Stratifications

Fix $n, N \in \mathbf{N}$.
**Def:** The *configuration space* of $n$ points in $\mathbf{R}^N$ is $\mathrm{Conf}_n(\mathbf{R}^N) = \{P \subseteq \mathbf{R}^n : |P| = n\}$.

This is a topological space with the *Hausdorff metric*

$$d_H(P, q) = \max\left\{\min_r \left\{Q \subseteq \bigcup_{p \in P} B(p, r)\right\}, \min_r \left\{P \subseteq \bigcup_{q \in Q} B(q, r)\right\}\right\}$$

*Example of distance.*

This is our topological space of all finite metric spaces. Note that $\dim(\mathrm{Conf}_n(\mathbf{R}^N)) = nN$.

**Def:** Let $\mathrm{VR}\colon \mathrm{Conf}_n(\mathbf{R}^N) \times \mathbf{R}_{\geqslant 0} \to \mathsf{SC}$ be given by $(P, r) \mapsto (P, \{\sigma : d(u, v) \leqslant r \ \forall\ u, v \in \sigma\})$.
The *Vietoris–Rips* map. Also can do the *Čech* map.

**Thm:** The map [VR]: $\mathrm{Conf}_n(\mathbf{R}^N) \to ([\mathsf{SC}], \geqslant)$ is continuous.

**Def:** A continous map $f\colon X \to (A, \geqslant)$ from a topological space to a poset is a *stratification*.
The preimages $X_a := \{x \in X : f(x) = a\}$ are *strata*.

We are interested in nice stratifications.
*Example:* Strata of 2-d slice. Draw simplicial maps. Not nice. How to simplify?

Back to poset - one simplicial complex in each stratum.

## 0.4 Simplicial sets

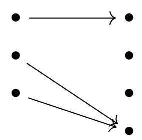
**Def:** A *category* contains *objects* and *morphisms* between the objects, satisfying:

- if $X$ is an object, there is a morphism $\mathbf{1}\colon X \to X$
- if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then there is a morphism $g \circ f\colon X \to Z$

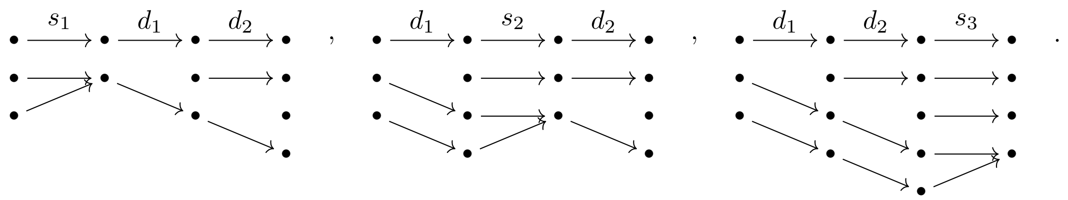
**Def:** Let $\Delta$ be the category whose objects are finite ordered sets $[n] = (0, 1, \dots, n)$ and whose morphisms are monotonic (order-preserving) maps $[m] \to [n]$.

**Remark:** Every morphism in $\Delta$ is a composition of *coface maps* $s_i : [n] \to [n-1]$, which hit $i$ twice, and *codegeneracy maps* $d_i : [n] \to [n+1]$, which skip $i$. This composition is not necessarily unique.

**Example:** Observe that the morphism $[2] \to [3]$ given by



can be decomposed into any one of the compositions




**Def:** A *simplicial set* is a contravariant functor $\Delta \to \mathrm{Set}$.
**Example:** The *nerve* of a category $\mathcal{C}$, denoted $N(\mathcal{C})$, is a simplicial set with:

- $[0] \mapsto \mathrm{Obj}(\mathcal{C})$
- $[1] \mapsto \mathrm{Mor}(\mathcal{C})$
- $[n] \mapsto$ (sequences of $n$ composable morphisms)

The *face map* $N(s_i)$ goes from $(n-1)$-seq to $n$-seq. Inserts $\mathbf{1}$ at $i$th obj: $\cdots \to X_i \xrightarrow{\mathbf{1}} X_i \to \cdots$.
The *degeneracy map* $N(d_i)$ goes from $(n+1)$-seq to $n$-seq. Composes maps around $i$th obj: $\cdots \to X_{i-1} \xrightarrow{g \circ f} X_{i+1} \to \cdots$.

We are interested in the simplicial set $N([\mathsf{SC}])$.
**Implications:**

- PH is equivalently:
  - a subsimplicial set of $N([\mathsf{SC}])$, with $0 \mapsto \{$simplicial complexes in filtr$\}$ and $1 \mapsto \{$inclusions among them$\}$
  - an $n$-chain, or an $n$-simplex (element of $N([\mathsf{SC}])_n$) in $N([\mathsf{SC}])$
- Path in $\mathrm{Conf}_n(\mathbf{R}^N)$ is "roofs" among $n$-chains: