Moduli spaces of Morse functions for persistence

AATRN online seminar

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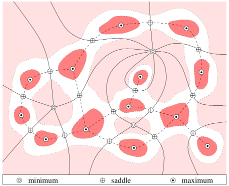
Slides at: `jlazovskis.com/talks`

Based on [CCF+] "Moduli spaces of Morse functions for persistence", JACT 2020.

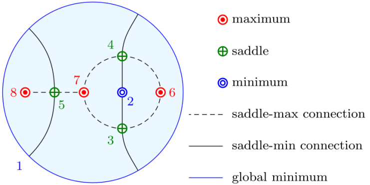
Motivation.

- What do functions that have the same barcode have in common?
- Does the decomposition of Morse–Smale functions help?



← Edelsbrunner, Harer, Zomorodian: "Hierarchical Morse Complexes for Piecewise Linear 2-manifolds," SoCG 2001.

CCF+, 2020. →



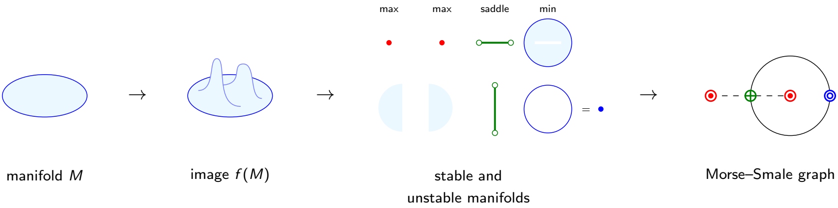
Plan.

1. Deep dive into invariants on $\mathbf{S}^2$
2. Compare MS functions by their quadrangle decomposition
3. Compare embeddings of $\mathbf{S}^2$ in $\mathbf{R}^3$ by their level sets

Background 1: Decomposing functions

For (M, g_M) a nice manifold, let $f\colon M \to \mathbf{R}$ be smooth.

- f is (excellent) Morse if all critical points are (distinct) non-degenerate.
- f is Morse–Smale if it is Morse and the gradient ∇f generates transversally intersecting manifolds, the intersections of which are cells.



For M 2-dimensional, the faces of this graph always have 4 edges and critical values around the faces always follow a certain order (Edelsbrunner–Harer–Zomorodian 2001).

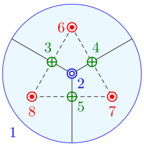
Approach: Play with the combinatorics of the Morse–Smale graph for $M = \mathbf{S}^2$.

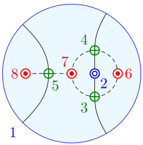
Let $f, g \colon M \to \mathbf{R}$ be Morse with slicings $f_0 < \cdots < f_n$ and $g_0 < \cdots < g_m$, respectively. In order of increasing coarseness, f and g are:

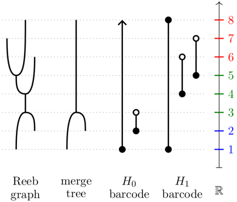
- geometrically equivalent if there exist orientation-preserving diffeos $r \colon M \to M$, $\ell \colon \mathbf{R} \to \mathbf{R}$ such that $\ell \circ f = g \circ r$;
- topologically equivalent if $n = m$ and $f^{-1}(-\infty, f_i]$ is diffeomorphic to $g^{-1}(-\infty, g_i]$ for all i, via orientation-preserving diffeos;
- homologically equivalent if $n = m$ and $f^{-1}(-\infty, f_i]$ has the same Betti numbers as $g^{-1}(-\infty, g_i]$ for all i

Nicolaescu: On $\mathbf{S}^2$, Reeb graph isomorphism is geometric equivalence.

CCF+: On $\mathbf{S}^2$, graph equivalence is finer than geometric equivalence.



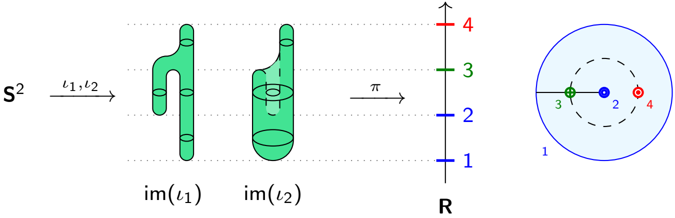




Enriching invariants

However, even graph equivalence does not capture everything.

- A Morse–Smale function $\mathbf{S}^2 \to \mathbf{R}$ may have several embeddings in $\mathbf{R}^3$ that are diffeomorphic, but are still heuristically "different":



- Factor f as $f\colon \mathbf{S}^2 \xrightarrow{\iota_{1,2}} \mathbf{R}^3 \xrightarrow{\pi} \mathbf{R}$, for ι_1, ι_2 smooth embeddings and $\pi\colon \mathbf{R}^3 \to \mathbf{R}$ the projection onto a fixed axis.
- Consider the preimage $\iota(f^{-1}(z))$ as nested circles, for z a regular value.

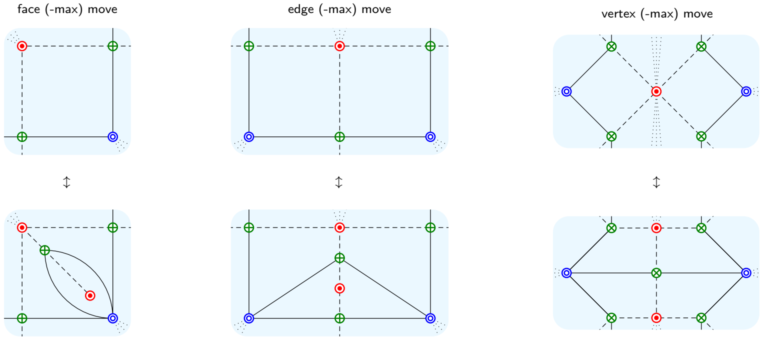
The compositions $\mathbf{S}^2 \xrightarrow{\iota_1} \mathbf{R}^3 \xrightarrow{\pi} \mathbf{R}$ and $\mathbf{S}^2 \xrightarrow{\iota_2} \mathbf{R}^3 \xrightarrow{\pi} \mathbf{R}$ are:

- height equivalent if $\iota_2 = \iota_1 \circ \varphi$ for some level-set preserving homeo $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$;
- poset equivalent if they are height equivalent and φ induces an isomorphism of nesting posets on all level sets.

Generating all functions on $\mathbf{S}^2$

Cerf: Any two Morse functions on M are connected by a path in the space of all smooth functions on M, with finitely many non-Morse points along this path.

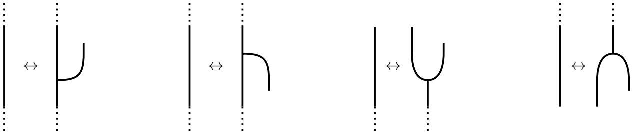
CCF+: For $M = \mathbf{S}^2$, every such path can be considered as a sequence of one of three types of local changes to the Morse–Smale graph.



- Adding critical points must occur in pairs of adjacent indices (Euler char).
- Saddle can appear in face, on edge, or at existing vertex.
- Saddles always have degree 4. Faces always have same sequence of vertex types.
- Connections of new saddle determine type of move (vertex move in general).

Face, vertex, edge moves

Each move adds / removes a branch from the Reeb graph of f. Known as "elementary deformations" of the Reeb graph (Di Fabio–Landi 2016).



Face-max, edge-max   Face-min, edge-min   Vertex-max   Vertex-min

Playground for space of all Morse–Smale functions: `github.com/zhou325/VIS-MSVF`

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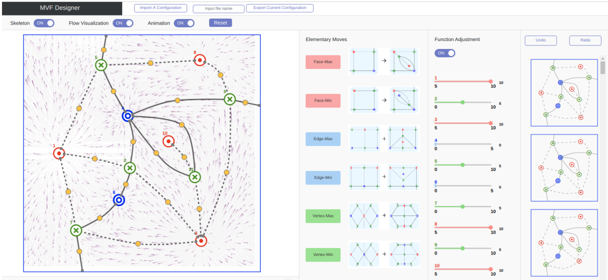
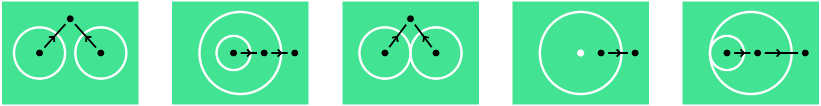


The nesting poset for level sets: $\mathbf{S}^2$ embedded in $\mathbf{R}^3$

For every $z \in \mathbf{R}$, the preimage $\iota(f^{-1}(z)) \subseteq \pi^{-1}(z) = \mathbf{R}^2$ is:

- a union of circles for z regular, and
- a union of circles and $\mathbf{S}^1 \vee \mathbf{S}^1$ or $*$, for z critical.



Instead of an order on the circles in $\iota(f^{-1}(z))$, define an order on the connected components of $\pi^{-1}(z) - \iota(f^{-1}(z)) =: X_z$, that is, on $\pi_0(X_z)$.

1. Label Jordan curves $\gamma_1, \dots, \gamma_n \in \iota(f^{-1}(z))$
2. Set $p_0 \in \pi_0(X_z)$ to be unbounded component
3. Set $p_i \in \pi_0(X_z)$ to be the component whose "exterior" boundary is γ_i

Definition: For $p_i, p_i \in \pi_0(X_z)$, let $p_i \leqslant p_j$ whenever

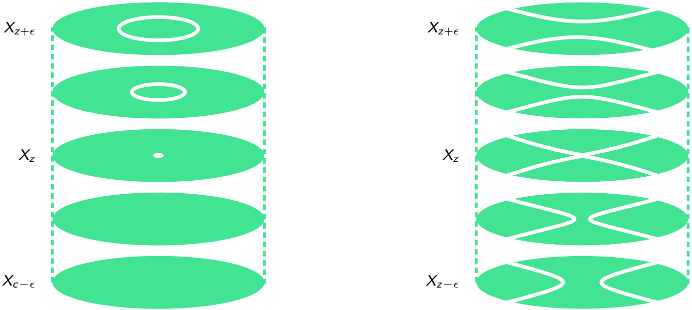
- $\text{int}(\gamma_i) \subseteq \text{int}(\gamma_j)$, or
- $\mathbf{R}^2 \setminus \text{int}(\gamma_j)$ is unbounded.

This is a partial order on $\pi_0(X_z)$, so we call $(\pi_0(X_z), \leqslant)$ the nesting poset.

Relations among nesting posets

Motivation: How is the natural poset structure for z, z' related?
Intuition: Use topology to motivate maps between posets.

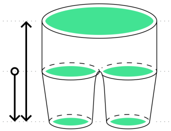
Clear for regular values and max/min, but ambiguous for saddles:



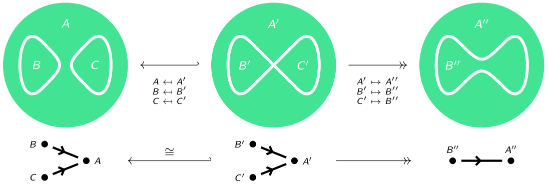
Resolution: Consider the larger picture at saddles.

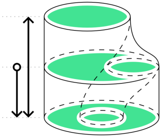
Saddle points: Nesting / non-nesting, merging/ splitting

CCF+: Canonical choices can be always be made based on the type of saddle.

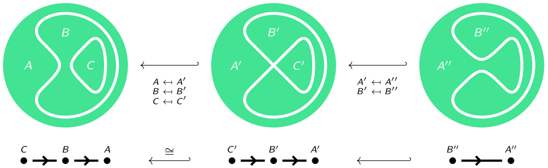


non-nesting saddle





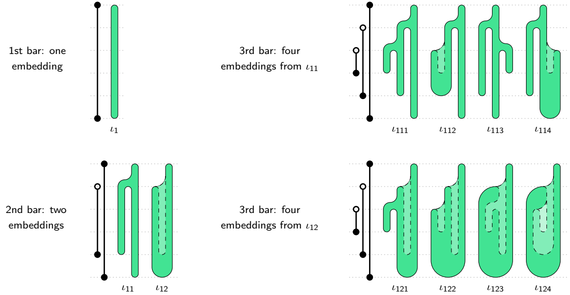
nesting saddle



Extensions

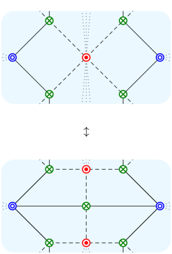
- **Enriched barcode:** We showed the barcode is a zigzag of posets. This can be generalized to a zigzag of algebras (the algebra of intervals of a poset)

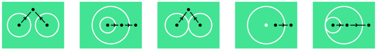
- **Counting preimages of a barcode:** Every open end in a barcode / fork in a Reeb graph is a choice of nesting or non-nesting saddle.



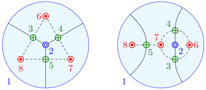
  - Are some choices forced / double counted by (non-)nested pairs of bars?
  - What happens when there is more than one maximum?

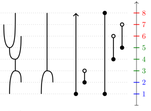
- **Setting change:** Heavy use was made of the nice properties of S^2.
  - How does our analysis work for surfaces with different Euler characteristic?
  - Non-orientable surfaces, other orientable manifolds?
  - Can 2-dim manfiolds be nested in preimages on M 3-dimensional?

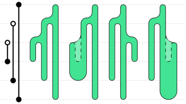


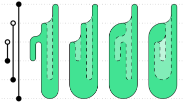


Thank you for your attention!









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