Classifying Morse functions for persistent homology

EPFL Applied Topology seminar

9 February 2021

Jānis Lazovskis

Mike Catanzaro
Iowa State University

Justin Curry
University at Albany SUNY

Brittany Fasy
Montana State University

Jānis Lazovskis
University of Latvia

Greg Malen
Duke University

Hans Riess
University of Pennsylvania

Bei Wang
University of Utah

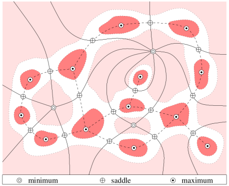
Matt Zabka
NC State University

Slides at: `jlazovskis.com/talks`

Based on [CCF+] "Moduli spaces of Morse functions for persistence", JACT 2020.

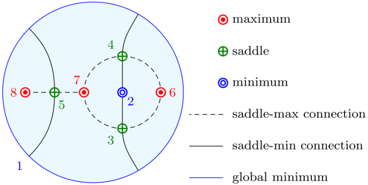
Motivation.

- What do functions that have the same barcode have in common?
- Does the decomposition of Morse–Smale functions help?



← Edelsbrunner, Harer, Zomorodian: "Hierarchical Morse Complexes for Piecewise Linear 2-manifolds," SoCG 2001.

CCF+, 2020. →



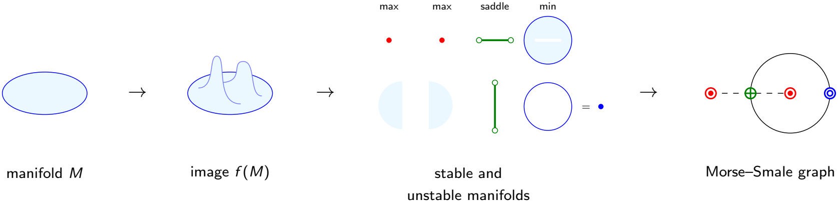
Plan.

1. Background: Invariants on S^2
2. Decomposition: Morse–Smale functions by quadrangles
3. Embeddings: Level sets of S^2 in R^3

Background: Decomposing functions

For (M, g_M) a nice manifold, let $f \colon M \to \mathbb{R}$ be smooth.

- f is (excellent) Morse if all critical points are (distinct) non-degenerate.
- f is Morse–Smale if it is Morse and the gradient ∇f generates transversally intersecting manifolds, the intersections of which are cells.



For M 2-dimensional, the faces of this graph always have 4 edges and critical values around the faces always follow a certain order (Edelsbrunner–Harer–Zomorodian 2001).

Approach: Play with the combinatorics of the Morse–Smale graph for $M = \mathrm{S}^2$.

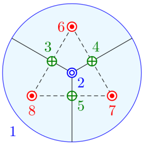
Background: Equivalences of Morse(–Smale) functions

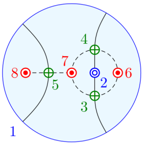
Let $f, g\colon M \to \mathrm{R}$ be Morse with slicings $f_0 < \cdots < f_n$ and $g_0 < \cdots < g_m$, respectively. In order of increasing coarseness, f and g are:

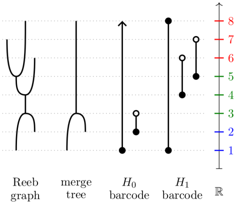
- geometrically equivalent if there exist orientation-preserving diffeos $r\colon M \to M$, $\ell\colon \mathrm{R} \to \mathrm{R}$ such that $\ell \circ f = g \circ r$;
- topologically equivalent if $n = m$ and $f^{-1}(-\infty, f_i]$ is diffeomorphic to $g^{-1}(-\infty, g_i]$ for all i, via orientation-preserving diffeos;
- homologically equivalent if $n = m$ and $f^{-1}(-\infty, f_i]$ has the same Betti numbers as $g^{-1}(-\infty, g_i]$ for all i

Nicolaescu: On S^2, Reeb graph isomorphism is geometric equivalence.

CCF+: On S^2, graph equivalence is finer than geometric equivalence.

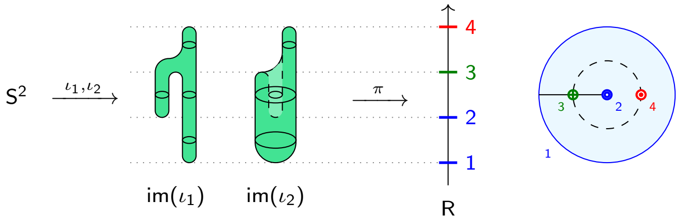






However, even graph equivalence does not capture everything.

- A Morse–Smale function $S^2 \to R$ may have several embeddings in R^3 that are diffeomorphic, but are still heuristically "different":



- Factor f as $f\colon S^2 \xrightarrow{\iota_{1,2}} R^3 \xrightarrow{\pi} R$, for ι_1, ι_2 smooth embeddings and $\pi\colon R^3 \to R$ the projection onto a fixed axis.
- Consider the preimage $\iota(f^{-1}(z))$ as nested circles, for z a regular value.

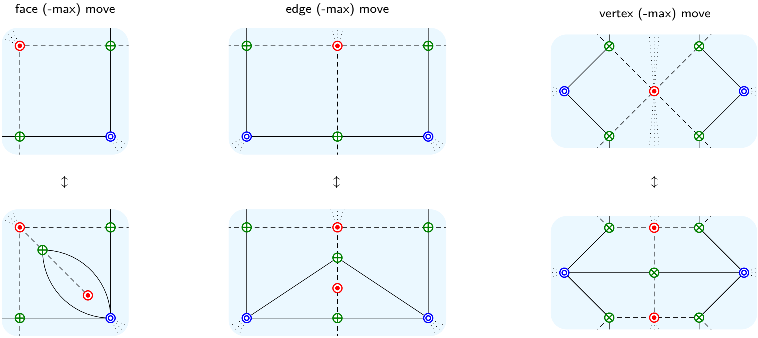
The compositions $S^2 \xrightarrow{\iota_1} R^3 \xrightarrow{\pi} R$ and $S^2 \xrightarrow{\iota_2} R^3 \xrightarrow{\pi} R$ are:

- height equivalent if $\iota_2 = \iota_1 \circ \varphi$ for some level-set preserving homeo $\varphi\colon R^3 \to R^3$;
- poset equivalent if they are height equivalent and φ induces an isomorphism of nesting posets on all level sets.

Decomposition: Generating all functions on S^2

Cerf: Any two Morse functions on M are connected by a path in the space of all smooth functions on M, with finitely many non-Morse points along this path.

CCF+: For $M = S^2$, every such path can be considered as a sequence of one of three types of local changes to the Morse–Smale graph.

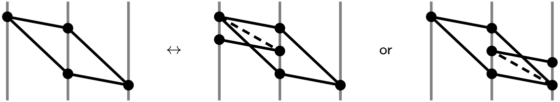


- Adding critical points must occur in pairs of adjacent indices (Euler char).
- Saddle can appear in face, on edge, or at existing vertex.
- Saddles always have degree 4. Faces always have same sequence of vertex types.
- Connections of new saddle determine type of move (vertex move in general).

Decomposition: In terms of other decompositions

Barannikov 1994: Each move adds / removes a pair of critical points. Known as "transformations (2a) and (2b)" of the abstract Morse complex.

Face-max,
face-min:

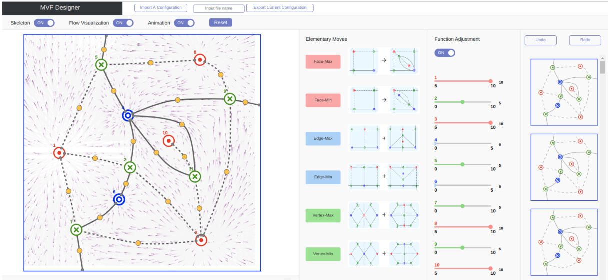


Di Fabio–Landi 2016: Each move adds / removes a branch from the Reeb graph of f. Known as "elementary deformations" of the Reeb graph.

Face-max, edge-max   Face-min, edge-min   Vertex-max   Vertex-min

Decomposition: Interactive sandbox

Youjia Zhou
University of Utah



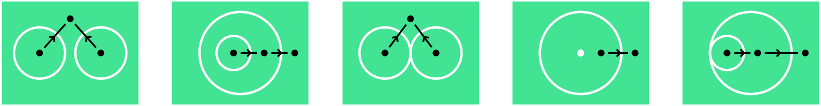
Playground for space of all Morse–Smale functions: `github.com/zhou325/VIS-MSVF`

- Explore combinatorics of Morse–Smale vector fields
- Extend face, edge, vertex moves with height changes
- Analyze associated barcode changes

Embeddings: The nesting poset for level sets

For $\iota\colon S^2 \to R^3$ an embedding and $z \in R$, the preimage $\iota(f^{-1}(z)) \subseteq \pi^{-1}(z) = R^2$ is:

- a union of circles for z regular, and
- a union of circles and $S^1 \vee S^1$ or $*$, for z critical.



Instead of an order on the circles in $\iota(f^{-1}(z))$, define an order on the connected components of $\pi^{-1}(z) - \iota(f^{-1}(z)) =: X_z$, that is, on $\pi_0(X_z)$.

1. Label Jordan curves $\gamma_1, \ldots, \gamma_n \in \iota(f^{-1}(z))$
2. Set $p_0 \in \pi_0(X_z)$ to be unbounded component
3. Set $p_i \in \pi_0(X_z)$ to be the component whose "exterior" boundary is γ_i

Definition: For $p_i, p_i \in \pi_0(X_z)$, let $p_i \leqslant p_j$ whenever

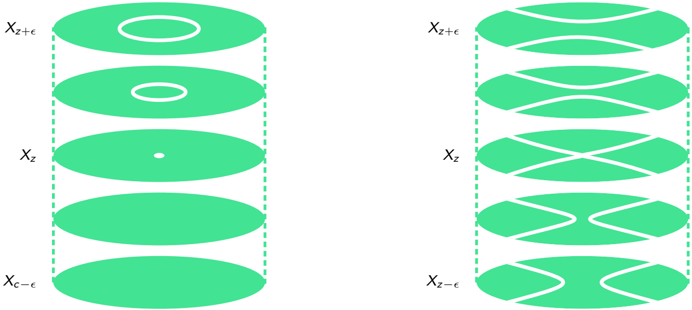
- $\text{int}(\gamma_i) \subseteq \text{int}(\gamma_j)$, or
- $R^2 \setminus \text{int}(\gamma_j)$ is unbounded.

This is a partial order on $\pi_0(X_z)$, so we call $(\pi_0(X_z), \leqslant)$ the nesting poset.

Embeddings: Relations among nesting posets

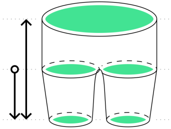
Motivation: How is the natural poset structure for z, z' related?
Intuition: Use topology to motivate maps between posets.

Clear for regular values and max/min, but ambiguous for saddles:

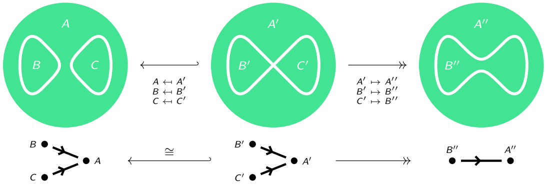


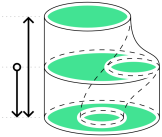
Resolution: Consider the larger picture at saddles.

CCF+: Canonical choices can be always be made based on the type of saddle.

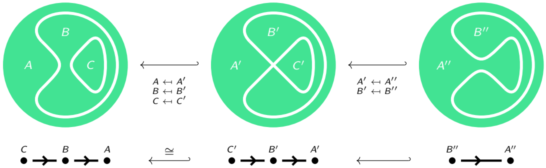


non-nesting saddle





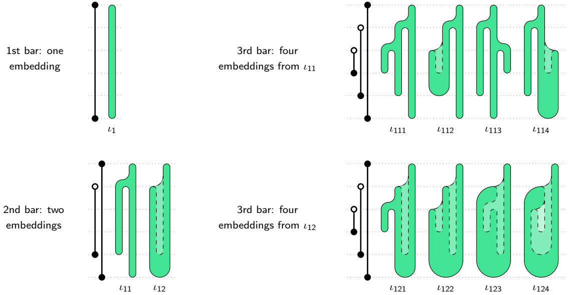
nesting saddle



Embeddings: Extensions

- **Enriched barcode:** We showed the barcode is a zigzag of posets. This can be generalized to a zigzag of algebras (the algebra of intervals of a poset)

- **Counting preimages of a barcode:** Every open end in a barcode / fork in a Reeb graph is a choice of nesting or non-nesting saddle.

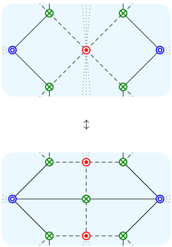


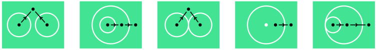
  - Are some choices forced / double counted by (non-)nested pairs of bars?
  - What happens when there is more than one maximum?

- **Setting change:** Heavy use was made of the nice properties of S^2.
  - How does our analysis work for surfaces with different Euler characteristic?
  - Non-orientable surfaces, other orientable manifolds?
  - Can 2-dim manfiolds be nested in preimages on M 3-dimensional?

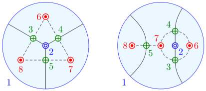
End

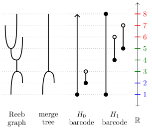
vertex (-max) move

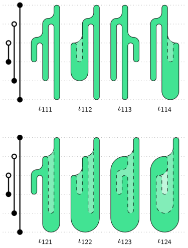




Thank you for your attention!







References.

- Barannikov, Sergei. *The Framed Morse complex and its invariants,* 1994.
- Catanzaro, Mike, Justin Curry, Brittany Fasy, Jānis Lazovskis, Greg Malen, Hans Riess, Bei Wang, Matt Zabka. *Moduli spaces of Morse functions for persistence,* 2020.
- Cerf, Jean. *La stratification naturelle des espaces de fonctions différentiables réelles et le théoréme de la pseudo-isotopie,* 1970.
- Di Fabio, Barbara and Claudia Landi. *The Edit Distance for Reeb Graphs of Surfaces,* 2016.
- Edelsbrunner, Herbert, John Harer, and Afra Zomorodian. *Hierarchical Morse Complexes for Piecewise Linear 2-Manifolds,* 2001.
- Nicolaescu, Liviu. *Counting Morse functions on the 2-sphere,* 2008.
- Zhou, Youjia, Jānis Lazovskis, Mike Catanzaro, Matt Zabka, Bei Wang. *MVF Designer: Design and Visualization of Morse Vector Fields,* 2019.

Acknowledgements.

The paper CCF+ grew out of a productive discussion during the special workshop "Bridging Statistics and Sheaves" at the Institute for Mathematics and Applications in May 2018. The authors would like to thank the organizers for putting together the workshop, the IMA for hosting the event, and Mikael Vejdemo-Johansson for insightful conversations at the onset of this collaboration. The authors also thank the reviewers for helpful comments and suggestions. JC is partially funded by NSF CCF-1850052. JL is partially funded by EP/P025072/1. BTF is partially funded by NSF CCF-1618605 and DMS-1664858. BW is partially funded by NSF IIS-1513616 and IIS-1910733.