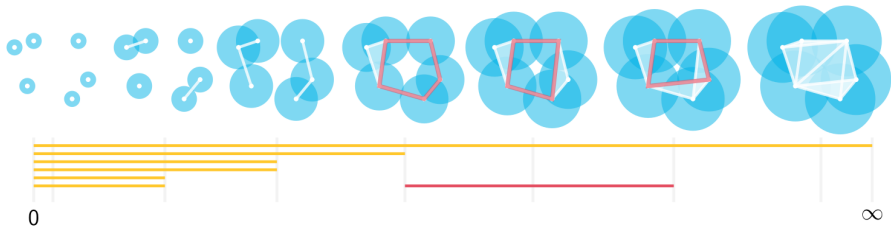


Topological structures and biological settings

Laboratoire Jean Alexandre Dieudonné
Université Côte d'Azur

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Slides at: jlazovskis.com/talks

Personal overview

- ▶ 2014 - 2019: University of Illinois at Chicago
- ▶ 2019 - 2020: University of Aberdeen
- ▶ 2020 - 2025: Riga Technical University
- ▶ 2023 - 2024: Printful
- ▶ 2025 - : University of Latvia

My work is...

- ▶ motivated by the deep structural tools that algebraic topology provides and which computational topology can realize
- ▶ often with teams and in interdisciplinary projects
- ▶ very frequently with a biological theme

Presentation overview

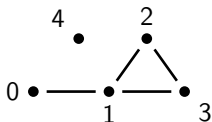
- ▶ Relevant mathematical structures
 - ▶ Graphs, simplicial complexes, invariants
- ▶ Neuro-topology: neural circuit structure and function through a topological lens
 - ▶ BlueBrain Project, digital reconstructions, simulations and classification
- ▶ Eco-topology: Ecological inferences for hypervolumes
 - ▶ Filtrations, the Vietoris–Rips complex, Persistent homology
 - ▶ Bounding distances between persistence diagrams
- ▶ Tracking dynamics: Topological changes at the foundations
 - ▶ Factoring the boundary matrix
 - ▶ Taking apart the machinery for computing persistence

Graphs

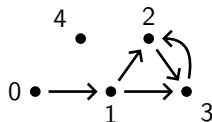
The basic mathematical concept throughout this talk will be a finite set.

$\{0, 1, 2, 3, 4\}$

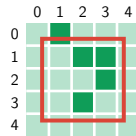
finite set V



undirected graph G



directed graph G'



adjacency matrix A

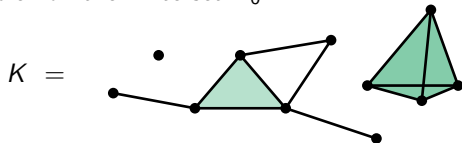
- The pair $G = (V, E)$ is an undirected graph, with $E = \{\{0, 1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.
- The pair $G' = (V, E')$ is a directed graph, with $E' = \{(0, 1), (1, 2), (1, 3), (2, 3), (3, 2)\}$.

We are interested in particular subgraphs of G' , with $V' = \{1, 2, 3\}$:

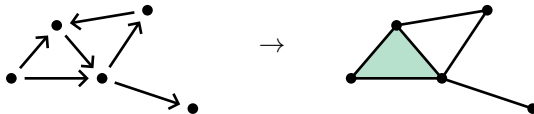
- V' and $(1, 2), (1, 3), (2, 3)$ form a *directed 3-clique*, as do $(1, 2), (1, 3), (3, 2)$
- V' and $(1, 2), (1, 3), (2, 3), (3, 2)$ form the *closed neighbourhood* of vertex 3 (or 2)

Simplicial complexes

Let K be a simplicial complex on the finite set K_0 .



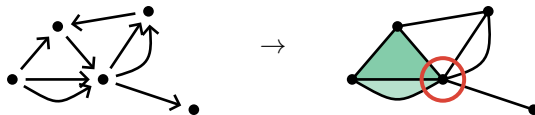
- ▶ A *0-simplex* is a set with a single element.
- ▶ An *n -simplex* is a set with $n + 1$ elements, for which every subset of size $n - 1$ is an $(n - 1)$ -simplex. The set of n -simplices is denoted K_n .
- ▶ A *simplicial complex* K is a set of simplices for which $\sigma \cap \tau \in K$, whenever $\sigma, \tau \in K$.



The simplicial complex associated to a directed graph is its *directed flag* (or *clique*) *complex*.

Invariants

If G has a reciprocal edge or multiple edges, the associated topological space is a *cellular complex* K , which is defined by gluing copies of the *n -cell* D^n to K_{n-1} to form K_n .



From every graph or complex, we can compute values that represent some underlying feature.

Euler characteristic : $\chi(K) = \sum_n (-1)^n |K_n|$

transitive clustering coefficient : $\text{tcc}(K, v) = \frac{\text{number of 3-cliques containing } v}{\text{number of theoretically possible 3-cliques containing } v}$

adjacency spectral gap : difference between the two largest moduli of eigenvalues of A

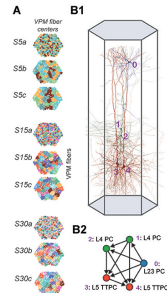
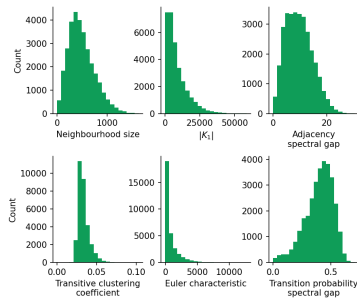
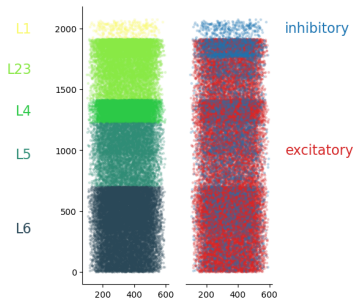
(*Bauer*) *Laplacian spectral gap* : Laplacian for directed graphs without strong connectivity

Main idea: **describe properties of closed neighbourhoods with numerical values.**

work continued by OBI

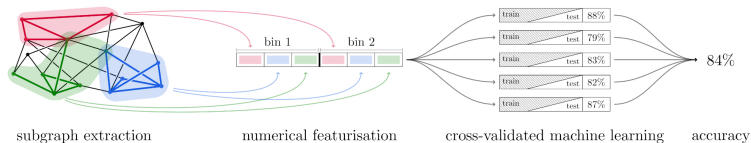
BBv5 connectome

- *Structure*: 31k neurons, 7.8m connections, 55 EM types
- *Topology*: Compared with random models, high dimensional cliques are overrepresented
- *Robustness*: Reciprocal connections preferentially appear in high dimensional cliques
- *Activity*: The number of active n -cliques can be used to classify simulations



Classifying simulations by neighbourhoods

- ▶ Compute parameter p_1 for every neighbourhood, select highest / lowest n
- ▶ For each selected neighbourhood, compute parameter p_2 of m active subgraphs
- ▶ Classify feature vectors of length $n \cdot m \cdot 8 \cdot 557$ with SVM



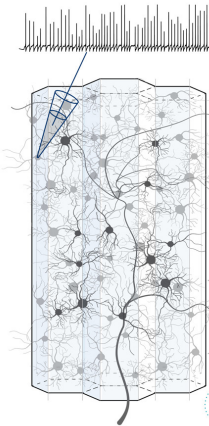
Validate: random selection, smaller regions, random regions, shuffled activity, NEST simulator

Main result (Conceição, Govc, Lazovskis, Levi, Riihimäki, Smith. 2022)

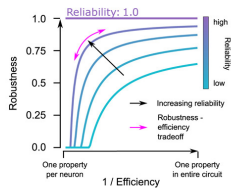
Simulations on neural circuits can be classified with high accuracy considering only the structure of and activity on a small number of neighbourhoods.

Reliability in neural circuits

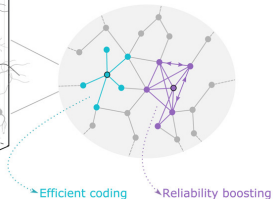
Biophysically detailed simulations



Tradeoff in the neural code



Heterogeneous solutions in the connectome



- Does there exist a random directed graph model with predetermined clique counts?
- How can firing rate be decoupled from every other parameter in active subgraphs?

The level of tradeoff between efficiency and robustness depends on the *reliability* of the system

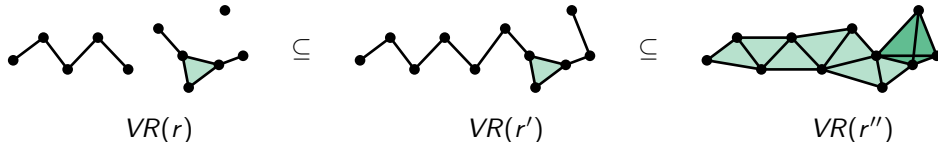
- The brain finds the sweet spot by having *heterogeneous connectivity*
- This is quantified by neighbourhoods with high dimensional simplices that are very robust, but very inefficient

Distances and filtrations

The *Vietoris–Rips complex* of a finite set is determined by pairwise distances among elements.

$$X = \text{[Diagram of a point cloud with green circles around points]} \quad VR(r)_n = \left\{ \sigma \subseteq X : \begin{array}{l} |\sigma| = n + 1 \\ d(x, y) \leq r \, \forall \, x, y \in \sigma \end{array} \right\}$$

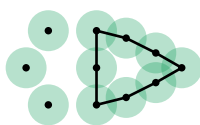
Increasing the threshold $r < r' < r''$ induces a *filtration*, or a sequence of nested spaces.



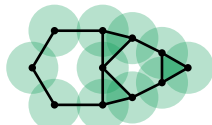
The *simplex-wise* filtration of a simplicial complex has one simplex added at each step.

Persistent homology

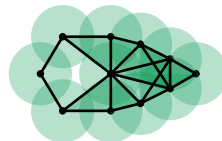
At each step r of the filtration \mathcal{F} , compute the homology $H_*(\mathcal{F}_r)$ of the space \mathcal{F}_r .



$$H_0(\mathcal{F}_r) = \mathbf{Z}^4, \quad H_1(\mathcal{F}_r) = \mathbf{Z}$$

 \subseteq


$$H_0(\mathcal{F}_{r'}) = \mathbf{Z}, \quad H_1(\mathcal{F}_{r'}) = \mathbf{Z}^2$$

 \subseteq


$$H_0(\mathcal{F}_{r''}) = \mathbf{Z}, \quad H_1(\mathcal{F}_{r''}) = \mathbf{Z}$$

Compose these assignments as $\mathbf{R} \xrightarrow{\mathcal{F}} \text{Top} \xrightarrow{H_*} \text{Vect}$ to get the *persistent homology* of \mathcal{F} .

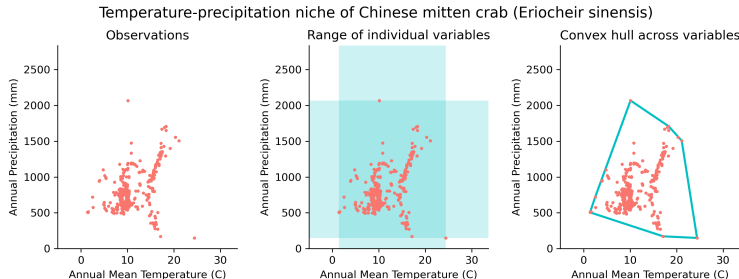
$$\begin{array}{ccccc}
 \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} \\
 r & & r' & & r''
 \end{array}$$

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} \\
 \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\
 r & & r' & & r''
 \end{array}$$

The *birth* and *death* times of each interval are determined by unique module decomposition.

The ecological niche and hypervolume

Environmental parameters (temperature, rain, sunlight, ...) are key to every species' survival.



Given a particular species:

- ▶ its *niche* is bounded by the range of environmental parameters in which it can exist
- ▶ its *realized niche* is the subset in which the species is observed
- ▶ its *hypervolume* is an approximation of the realized niche as a subset of Euclidean space

Constructing a biologically accurate hypervolume

- ▶ The size of a climatic niche is hypothesized to drive species *diversification rates*
- ▶ The similarity of species' environmental or functional trait hypervolumes measures niche divergence or packing, which may influence *species coexistence* and richness patterns
- ▶ Niche similarity also helps compare individuals within a species, assessing *climate change* impacts and niche shifts during *invasions*

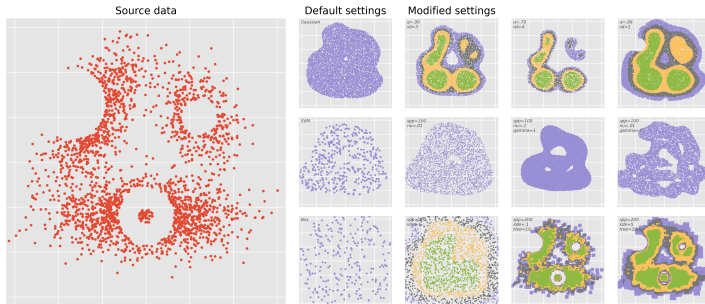
Inherent problem: **observations are heavily skewed to humans (with database access)**



```
> data.frame(x=0, y=0, z=0)
# Species observations
# Species: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 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Current and proposed hypervolume methods

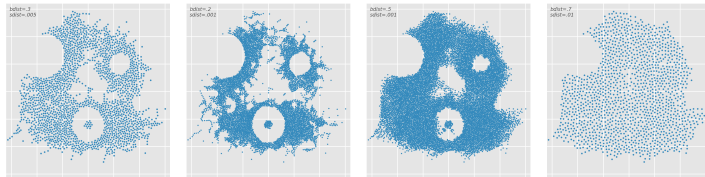
Current



► (Gaussian) KDE infers new data nearby each true observation

► Topology infers new data between collections of true observations

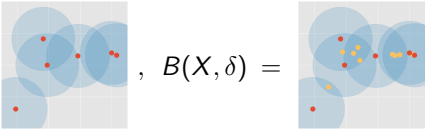
Proposed



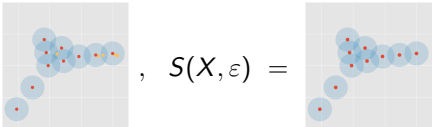
► Both methods infer more samples and uniformize with respect to density

A complementary approach

Barycentric subdivision: Add the average of $k + 1$ elements ($k = 1, 2$) within a distance $\delta > 0$:


$$X = \text{[Diagram]}, \quad B(X, \delta) = \text{[Diagram]} = X \cup \bigcup_k \left\{ \frac{\sum \sigma_i}{k+1} : \sigma = \{\sigma_0, \dots, \sigma_k\} \in VR_\delta(X)_k \right\}$$

Sparsification: Remove all other elements within a distance $\varepsilon > 0$ of each element:


$$X = \text{[Diagram]}, \quad S(X, \varepsilon) = \text{[Diagram]} = \{x \in X : d(x, x') > \varepsilon \forall x' \in X, x' \neq x\}$$

Topological guarantees

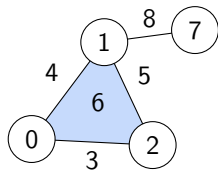
Main result (Lazovskis, Levi, Morimoto. 202x)

Let $X \subseteq \mathbf{R}^N$ be a finite set and D_X its persistence diagram of the Vietoris–Rips filtration in degree 0. Then $d_W(D_X, D_{B(X,\delta)}) \leq \frac{\delta}{4}$ and $d_W(D_X, D_{S(X,\varepsilon)}) \leq \frac{\varepsilon}{2}$ for every $\delta, \varepsilon > 0$.

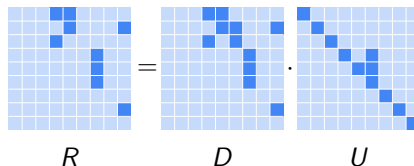
$$\begin{array}{ccc}
 S(X, \varepsilon) & \xrightarrow[\simeq]{\text{erule}_{S(X, \varepsilon)}} & D_{S(X, \varepsilon)} \\
 \downarrow \iota_S & & \downarrow \varphi_S \\
 X & \xrightarrow[\simeq]{\text{erule}_X} & D_X \\
 \downarrow \iota_B & & \downarrow \varphi_B \\
 B(X, \delta) & \xrightarrow[\simeq]{\text{erule}_{B(X, \delta)}} & D_{B(X, \delta)}
 \end{array}$$

- ▶ Constructed relation between persistent homology computation and hardware limitations
- ▶ Respects original data and maintains uniform density in hypervolume
- ▶ Next: Align to grid, take complement, use duality for top-dimensional holes. Track over time.

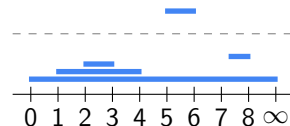
Persistence diagrams



filtered simplicial complex



bounday matrix factorization



barcode from pivots of R

The standard barcode algorithm performs Gaussian elimination on D to produce the barcode.

Algorithm 1: SBA - STANDARD BARCODE ALGORITHM

Input: Boundary matrix D , identity matrix I

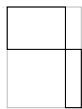
Output: Reduced matrix R , operations matrix U

```

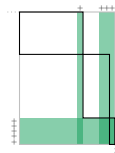
1  $R \leftarrow D$ 
2  $U \leftarrow I$ 
3 for  $j = 1, \dots, n$ 
4   while there exists  $j' < j$  for which  $\text{piv}(R[j']) = \text{piv}(R[j]) \neq 0$ 
5     add  $R[j']$  to  $R[j]$ 
6     add  $U[j']$  to  $U[j]$ 
7 return  $R, U$ 
```

Dynamic data

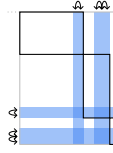
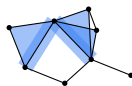
How does the boundary matrix of the Vietoris–Rips filtration change in dynamic settings?



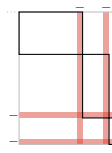
initial state



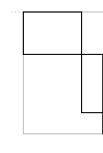
*insert into
filtration*



swap order



*remove from
filtration*

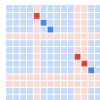
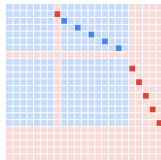
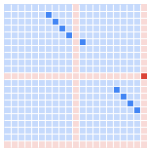
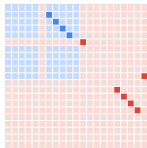
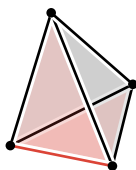
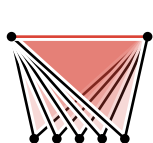
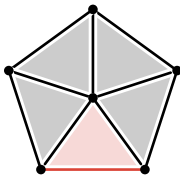
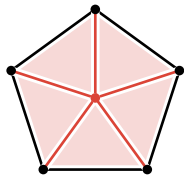


final state

- ▶ The number of simplices n is exponential in the number of points
- ▶ Insertion is reindexing, swapping is known to be $O(n)$

Removing a simplex

Removing a simplex $\sigma \in K$ means removing all cofaces of σ , described by the *star* $\text{st}(\sigma)$ of σ .



The SiRUP algorithm:

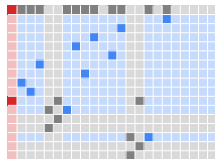
- ▶ For each simplex $\tau \in \text{st}(\sigma)$, collect the off-diagonal entries in the column of τ in U
- ▶ Adjust the columns in R and U for each affected simplex to respect *class representatives*

Minimality

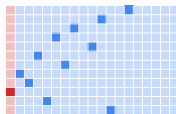
Main result (Giunti, Lazovskis. 2025)

Given the reduced boundary matrix of \mathcal{F} as input, SiRUP will output the reduced boundary matrix of $\mathcal{F} \setminus L$ with $O(mn^2)$ operations, where $m = |L|$ and $n = |\mathcal{F}|$.

Combine repeated additions to involve each affected column exactly once in an operation.



*reduced boundary
matrix*



*relevant
submatrix*



*add back
column of σ*



*apply standard
barcode algorithm*

Thank you

- ▶ F. Bauer. *Normalized graph laplacians for directed graphs*. Linear Algebra and its Applications, 2012.
- ▶ D. Cohen-Steiner, H. Edelsbrunner, D. Morozov. *Vines and vineyards by updating persistence in linear time*. Proceedings of the Twenty-Second Annual Symposium on Computational Geometry, 2006.
- ▶ P. Conceição, D. Govc, J. Lazovskis, R. Levi, H. Riihimäki, J.P. Smith. *An application of neighbourhoods in digraphs to the classification of binary dynamics*. Network Neuroscience, 2022.
- ▶ W. Crawley-Boevey. *Locally finitely presented additive categories*. Communications in Algebra, 1994.
- ▶ G. Fagiolo. *Clustering in complex directed networks*. Physical Review E, 2007.
- ▶ B. Giunti, J.Lazovskis. *Pruning vineyards: Pruning vineyards: updating barcodes by removing simplices*. Manuscript in preparation, 2025.
- ▶ G.E. Hutchinson. *Population studies-animal ecology and demography - concluding remarks*. Cold Spring Harbor symposia on quantitative biology, 1957.
- ▶ J. Jordan et al. *Nest 2.18.0*. Zenodo, 2019.
- ▶ H. Markram et al. *Reconstruction and Simulation of Neocortical Microcircuitry*. Cell, 2015.
- ▶ D.E. Santander, C. Pokorny, A. Ecker, J. Lazovskis, M. Santoro, J.P. Smith, K. Hess, R. Levi, M.W. Reimann. *Heterogeneous and higher-order cortical connectivity undergirds efficient, robust, and reliable neural codes*. iScience, 2025.