

Constructions of contractibility in finite subset spaces

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Jānis Lazovskis
University of Latvia

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The topology of finite subset spaces

Setup: X is a path-connected topological space

↳ definitely Hausdorff

↳ later CW-complex

"all the ways to choose
a finite sample of X "



Def: $\text{Ran}(X) = \{P \subseteq X : 0 < |P| < \infty\}$ is the Ranspace of X .

↳ also "finite subset space"

↳ also "exponential of X " when $\mathbb{Q}$ considered (as the space $X^{[n]}$)

↳ may be constructed by gluing together (unordered) configuration spaces $\text{Conf}_n(X)$

The topology of finite subset spaces

$$\text{Pan}(X) = \{P \subseteq X : 0 < |P| < \infty\}$$

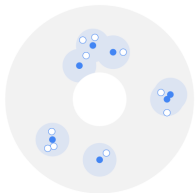
$$\text{Pan}_{\leq n}(X) = \{P \subseteq X : |P| \leq n\}, n \in \mathbb{Z}_{\geq 0}$$

Topology: metric topology induced by Hausdorff distance of subsets.

$\hookrightarrow \text{Conf}_n(X)$ is open in $\text{Pan}_{\leq n}(X)$

Ex 1: Two nearby elements

$$P, Q \in \text{Pan}_{\leq 10}(S' \times I)$$



Ex 2: A continuous map $S' \rightarrow \text{Pan}_{\leq 2}(S' \times I)$.

$$S' = [0, 1] / \sim 1$$

$t=0$: light blue

$t=1$: dark blue



The topology of finite subset spaces

(Co) multiplication: Natural operations on $\text{Ran}(X)$.

*: Based on $\text{Ran}^{\text{space}}(\text{Ran}(X), \{x, s\})$

$$\begin{array}{ccccc}
 \text{Ran}(X) & \xrightarrow[\text{comultiplication diagonal map}]{\Delta} & \text{Ran}(X) \times \text{Ran}(X) & \xrightarrow[\text{multiplication codiagonal map}]{\nabla} & \text{Ran}(X) \\
 p & \longmapsto & (p, p) & & \\
 & & (p, q) & \longmapsto & p \cup q
 \end{array}$$

Satisfies $\nabla \circ \Delta = \text{id}_{\text{Ran}(X)}$

Induced map on space of continuous functions $S' \rightarrow \text{Ran}(X)$:

$$\begin{array}{ccccc}
 \pi_1(\text{Ran}(X)) & \xrightarrow{\Delta_*} & \pi_1(\text{Ran}(X) \times \text{Ran}(X)) & \xrightarrow{\nabla_*} & \pi_1(\text{Ran}(X)) \\
 \gamma & \longmapsto & i_1(\gamma) \circ i_2(\gamma) & & \\
 & \text{inclusions to first and second factors} & & \text{projections onto first and second factors} & \\
 & & \gamma & \longmapsto & \text{proj}_1(\gamma) \cup \text{proj}_2(\gamma)
 \end{array}$$

Satisfies $\nabla_* \circ \Delta_*(\gamma) = \gamma \circ \gamma$

Restriction to $\text{Ran}_{\leq n}(X)$: Makes sense, but target of ∇ is $\text{Ran}_{\leq 2n}(X)$.

Satisfies $\nabla \circ \Delta = \text{id}_{\text{Ran}_{\leq n}(X) \hookrightarrow \text{Ran}_{\leq 2n}(X)}$

The topology of finite subset spaces

Known facts:

$$X \text{ path-connected top. space} \Rightarrow \pi_k(\text{Ran}(X)) = 0 \quad \forall k$$

(no later than Beilinson-Drinfeld 2004)

} non-constructive proof

$$X \text{ m-connected CW-cpx} \Rightarrow \pi_k(\text{Ran}_{\leq n}(X)) = 0 \quad \text{for } k=1, \dots, n+m-2$$

(for $m=0$ (X connected) Tuffley 2004)

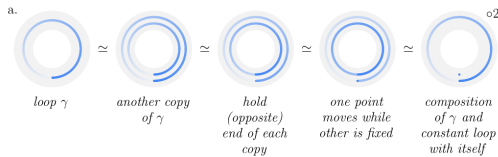
} proof by induction, explicit CW-complex for $X=S^1$, existence proof for general X .

Motivation: how do loops actually disappear?

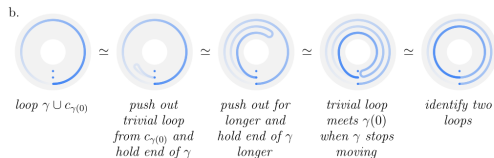
↳ need more pictures!

Contracting a loop through $\text{Ran}_{\leq 3}(S^1)$

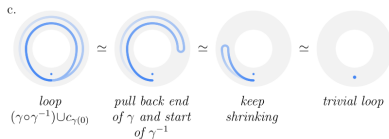
Step 1: Double a loop
to get $\gamma \simeq \nabla(r, c_0)^{\circ 2}$



Step 2: Push out a trivial loop following γ back & forth



Step 3: Contract as you would any $\gamma \circ \gamma^{-1}$



xvideo

Contracting a loop through $\text{Ran}_{\leq 3}(S^1)$

Result: Given any continuous $\gamma: S^1 \rightarrow S^1$, the map $S^1 \xrightarrow{\gamma} S^1 \rightarrow \text{Ran}_{\leq 3}(S^1)$ is homotopic to $c_0: S^1 \rightarrow \text{Ran}_{\leq 3}(S^1)$.

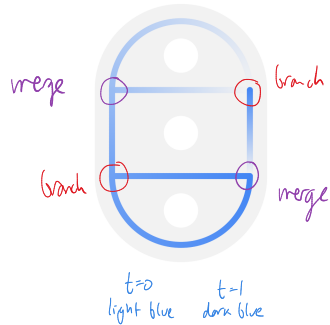
What about more general loops?

↳ a point does not "end up" where it started

↳ points "split" and "merge" in strange ways

Definition 4. Let $\sigma: S^1 \rightarrow \text{Ran}(X)$ be continuous, and $p \in X$, with $p \in \sigma(t)$ for some $t \in [0, 1]$.

- The point p is a **branch point** of σ at $t < 1$ if for every open neighborhood $U \subseteq X$ of p , there exists $\varepsilon > 0$ satisfying $|U \cap \sigma(t + \varepsilon)| > 1$.
- Analogously, p is a **merge point** of σ at $t > 0$ if for every open neighborhood $U \subseteq X$ of p , there exists $\varepsilon > 0$ satisfying $|U \cap \sigma(t - \varepsilon)| > 1$.
- Conversely, p is a **regular point** of σ at t if for every open neighborhood $U \subseteq X$ of p , there exists $\varepsilon > 0$ satisfying $|U \cap \sigma(t \pm \varepsilon)| = 1$, for every $0 \leq \varepsilon' < \varepsilon$.



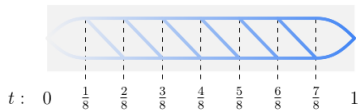
↳ all points in image of σ are one of these three if X is Hausdorff Ex: line w/ 2 origins

Counterexamples: strange loops

Reasonable hope: Describe arbitrary $\sigma: S' \rightarrow \text{Ran}_{\leq n}(X)$ as individual $\sigma_i: S' \rightarrow X$.

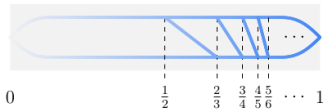
↳ How many σ_i do we need?

Ex: $\sigma: S' \rightarrow \text{Ran}_{\leq 3}(S' \times I)$ with k σ_i 's needed, for any k .



$k=9$
 $(t=0) \sim (t=1)$

Ex: $\sigma: S' \rightarrow \text{Ran}_{\leq 3}(S' \times I)$ with ∞ many σ_i 's needed.

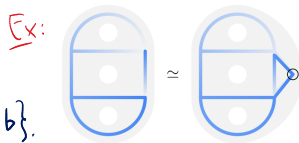


There is no factoring for arbitrary loops σ in the Ran space.

$$\begin{array}{ccc} S' & \xrightarrow{\sigma} & \text{Ran}_{\leq n}(X) \\ & \searrow \times \downarrow X^k & \nearrow \sigma_i \\ & & \end{array}$$

Counterexamples: strange loops

Assumptions we can still make:



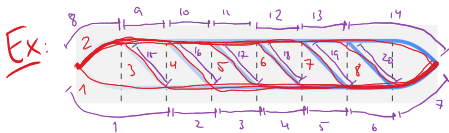
1. Every $\sigma: S' \rightarrow \text{Ran}_{\leq n}(X)$ is homotopic to $\hat{\sigma}$ with $\hat{\sigma}(0) = \hat{\sigma}(1) = \{b\}$.

↳ since X is path connected, precompose & postcompose σ with path from b to $\sigma(0)$.

2. If $\sigma: S' \rightarrow \text{Ran}_{\leq n}(X)$ has finitely many branch pts & merge pts, there exists $k \in \mathbb{Z}_{>0}$ s.t. σ factors through X^k .

↳ upper bound on k by equiv. rel. on regular points of σ (related by same maximal interval of reg. pts.)

↳ feels like Morse theory ???



Generalization 1: contracting a loop through $\text{Ran}_{\leq 2n+1}(X)$

Practical hope: Pass explicit S' contraction as argument to arbitrary $S' \rightarrow \text{Ran}_{\leq n}(X)$.

↳ given $\gamma: S' \rightarrow S'$ and $\sigma: S' \rightarrow \text{Ran}_{\leq n}(X)$, there is an induced map

$$\begin{aligned} S' &\longrightarrow \text{Ran}_{\leq n}(X) \\ t &\longmapsto \{\sigma(t') : t' \in \gamma(t)\} \end{aligned}$$

Proposition 10. For every continuous $\sigma: S^1 \rightarrow \text{Ran}_{\leq n}(X)$, the composition $S^1 \xrightarrow{\sigma} \text{Ran}_{\leq n}(X) \xrightarrow{i} \text{Ran}_{\leq 2n+1}(X)$ is homotopic to the trivial map, for all positive integers n .

↳ Homotopy in argument becomes homotopy in image.

↳ Corollary: $\pi_1(\text{Ran}_{\leq n}(X)) \xrightarrow{i_*} \pi_1(\text{Ran}_{\leq 2n+1}(X))$ is trivial

index $2n+1$ comes from explicit constr.:

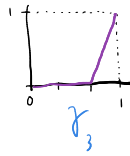
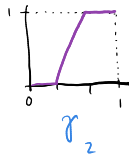
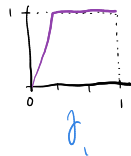
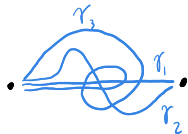
- 2 copies of γ moving
- 1 trivial loop fixed at 0

↳ index matches similar result of Handel (2000)

Generalization 2: contracting a loop through $\text{Ran}_{\leq n+2}(X)$ (finite case)

Revisiting initial vision: Push index $2n+1$ down much closer to n by "peeling off" copies of S^1 .

↳ Reparametrize as in step 1:



↳ *Only one "factor" γ_i moving at any $t \in [0, 1]$
 *Apply explicit construction to that factor, iterate.

Proposition 11. For every continuous $\sigma: S^1 \rightarrow \text{Ran}_{\leq n}(X)$ with finitely many branch points and merge points, the composition $S^1 \xrightarrow{\sigma} \text{Ran}_{\leq n}(X) \xrightarrow{i} \text{Ran}_{\leq n+2}(X)$ is homotopic to the trivial map, for all positive integers n .

*Can't get better than this:

$$\begin{aligned}\text{Ran}_{\leq 1}(S^1) &= S^1 \\ \text{Ran}_{\leq 2}(S^1) &= M_6 \\ \text{Ran}_{\leq 3}(S^1) &= S^3\end{aligned}$$

Generalization 2: contracting a loop through $\text{Ran}_{\leq n+2}(X)$ (infinite case)

Problem: A loop $\sigma: S^1 \rightarrow \text{Ran}_{\leq n}(X)$ may have only many branch & merge points

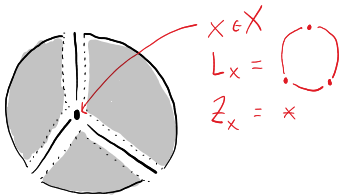
↳ Consider $\text{Ran}_{\leq n}(X)$ as a stratified space: only many crossings from one stratum to another

Def: A continuous map $X \xrightarrow{f} (A, \leq)$ is a stratification.

open sets: "upsets" $A_{\geq a} = \{b \in A : b \geq a\}$

A stratification is conical if every $x \in X$ has a conical neighborhood $Z_x \times C(L_x)$

Eg:
conical
nbhd of x

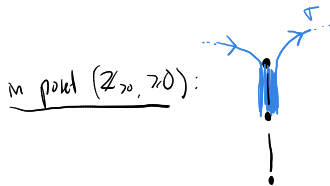
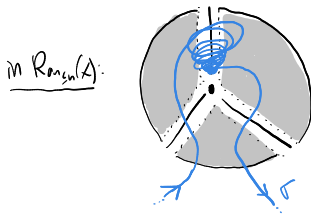


$\text{Ran}_{\leq n}(X) \rightarrow \mathbb{Z}_{\geq 0}$ is continuous
and conical

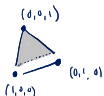
Generalization 2: contracting a loop through $\text{Ran}_{\leq n+2}(X)$ (infinite case)

Problem, rephrased: Can we get σ to stay within one stratum, locally?

↳ Careful: We already know every loop is homotopic to constant loop. Use homotopies as little/as local as possible.



Use exit paths & entry paths: conically stratified geometric realization of n -simplex



"exiting" lower strata



"entering" lower strata

- Lurie 2009
- Barwick-Glasman-Haure 2018

Allow us to precisely say when a stratum change happens

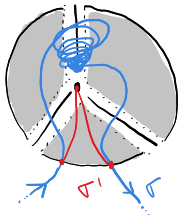
Generalization 2: contracting a loop through $\text{Ran}_{\leq n+2}(X)$ (infinite case)

Local setup: Given $x \in X$ and y in the boundary of a conical nbhd of x , we always have:

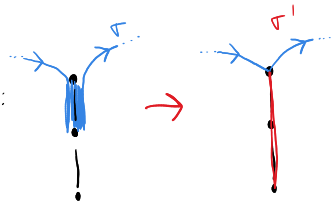
- an entrance path
- an exit path



in $\text{Ran}_{\leq n}(X)$:



in point $(\mathbb{Z}_{\geq 0}, \geq 0)$:



Observation: There exists a homotopy between σ and σ' , as they differ only in a contractible neighborhood of x .
 ↳ (a subset of) a conical nbhd of x

Proposition 15. For every continuous $\sigma: S^1 \rightarrow \text{Ran}_{\leq n}(X)$, there exists a continuous $\tilde{\sigma}: S^1 \rightarrow \text{Ran}_{\leq n}(X)$ homotopic to σ with finitely many branch points and merge points.

Corollary: induced map $\pi_1(\text{Ran}_{\leq n}(X)) \xrightarrow{\iota_X} \pi_1(\text{Ran}_{\leq n+2}(X))$ is trivial

Questions????

Hints at more ideas:

1. Every $p \in \text{Ran}_{\leq n}(X)$ has a "signature", its persistence diagram. Paths in $\text{Ran}_{\leq n}(X)$ have vineyards associated to them.

↳ How far away is a vineyard from a trivial vineyard? Our construction gives one such path.

2. When $\sigma: S' \rightarrow \text{Ran}_{\leq n}(X)$ has finitely many branch & merge points, we had a coarse upper bound on k for the factoring $S' \rightarrow X^k \xrightarrow{\sigma_i} \text{Ran}_{\leq n}(X)$

↳ Can we define k precisely? Maybe "pair" branch and merge points like PH & Morse theory (including multiplicity)?

Thank you

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